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A robust domain decomposition method for solving singularly perturbed parabolic reaction-diffusion problems with time delay

We consider the following model problem

$$Lu(x,t) + b(x,t)u(x,t-\tau) = f(x,t), \quad (x,t) \in D \tag{2.0.1}$$

with

$$\begin{cases} u(x,t) = \phi_\ell(t), & (x,t) \in \Gamma_\ell = \{0\} \times (0,T], \\ u(x,t) = \phi_r(t), & (x,t) \in \Gamma_r = \{1\} \times (0,T], \\ u(x,t) = \phi_b(x,t), & (x,t) \in \Gamma_b = [0,1] \times [-\tau, 0], \end{cases} \tag{2.0.2}$$

where

$$Lu(x,t) := u_t(x,t) - \varepsilon u_{xx}(x,t) + a(x,t)u(x,t)$$

and

$$D := \Omega \times (0,T] = (0,1) \times (0,T].$$

We suppose $0 < \varepsilon \leq 1,\ \tau > 0$ and $b \leq 0,\ a + b \geq \alpha > 0$ on $\overline{D}$. Here, the assumption $a + b \geq \alpha > 0$ is important, but not necessary, as the same can be achieved by using the transformation $u = \exp(ct)\tilde{u}$ with some properly chosen constant $c > 0$. For simplicity of the presentation we consider $T = n\tau$ for some positive integer n. The results of this chapter are true even if this relation does not hold. We introduce the notation $\Gamma = \Gamma_\ell \cup \Gamma_r \cup \Gamma_b$ and $\widetilde{\Gamma} = \Gamma_\ell \cup \Gamma_r \cup \Gamma_0$, where $\Gamma_0 = [0,1] \times \{0\}$. We also suppose that the problem data is sufficiently smooth and appropriate compatibility conditions at $(0,0), (1,0),\ (0,-\tau)$ and $(1,-\tau)$ holds; cf. [23, 24]. The solution of problem (2.0.1)-(2.0.2) has boundary layers near Γ_ℓ and Γ_r.

Robust numerical methods for solving singularly perturbed delay *ordinary differential equations* have been developed extensively in literature (see [56–63] and the references therein). However, robust numerical methods for solving singularly perturbed delay *partial differential equations* are not so much developed. In particular, for problem (2.0.1)-(2.0.2), some work can be found in [23, 24] based on the fitted mesh approach and in [26] based on the fitted operator approach. We note that the existing methods [23, 24] for problem (2.0.1)-(2.0.2) are analyzed using the method of steps and the maximum principle for the discrete operator corresponding to the continuous operator L. Further, the method in [26] is analyzed using the method of steps and the matrix method (where matrix is formed from the discretization corresponding to L).

To the best of our knowledge no work is available in literature on domain decomposition for problem (2.0.1)-(2.0.2). Thus, the aim of this chapter is three fold:

- We design an efficient domain decomposition method of SWR type for solving problem (2.0.1)-(2.0.2). To achieve this we perform the decomposition and formulate the iterative process directly at the PDE level itself, unlike the classical approaches where the domain decomposition is applied after the semidiscretization process either in time or in space for parabolic problems.

- We provide an error analysis framework for the proposed domain decomposition method. The main idea in our error analysis is based on a new discrete maximum principle which we establish in Lemma 2.3.2 and the use of some auxiliary problems which are motivated from our earlier work in [48].

- We prove that the proposed method yields robust numerical approximations of almost second order in space and first order in time, and more importantly only one iteration is sufficient for small values of the perturbation parameter.

Note that our error analysis framework based on the new discrete maximum principle can also be used to analyze the numerical methods presented in [23,24,26] for problem (2.0.1)-(2.0.2), while if we apply the idea of analysis presented in [23, 24, 26] (which is based on method of steps as mentioned above) to analyze the proposed domain decomposition method, the error constant will not be independent of the iteration parameter and the delay parameter. At the end, some numerical results are given in support of the theory.

2.1 A priori bounds

We first introduce the continuous operator $\mathcal{L}$ defined by $\mathcal{L}u(x,t) := Lu(x,t) + b(x,t)u(x,t-\tau)$, $(x,t) \in D$, and establish the maximum principle for it. Then, using the maximum principle for the operator $\mathcal{L}$, we derive some a priori bounds for the solution derivatives of problem (2.0.1)-(2.0.2). Similar bounds are derived in [23] using the method of steps and the maximum principle for the operator L.

Lemma 2.1.1. *Suppose $z(x,t) \geq 0$ for $(x,t) \in \Gamma_b$ and $z(0,t) \geq 0$, $z(1,t) \geq 0$ for $t \in (0,T]$. Then $Lz \geq 0$ in D implies that $z \geq 0$ in $\overline{D}$.*

Proof. The proof follows using arguments in [64]. □

Lemma 2.1.2. *Suppose $u(x,t) \geq 0$ for $(x,t) \in \Gamma_b$ and $u(0,t) \geq 0$, $u(1,t) \geq 0$ for $t \in (0,T]$. If $\mathcal{L}u \geq 0$ in D then $u(x,t) \geq 0$ in $\overline{D}$.*

Proof. Supposing $z = u$ in $[0,1] \times [-\tau,\tau]$, we note that

$$z(x,t) \geq 0 \text{ for } (x,t) \in \Gamma_b \text{ and } z(0,t) \geq 0,\ z(1,t) \geq 0 \text{ for } t \in (0,\tau].$$

Also

$$Lz(x,t) \geq -b(x,t)z(x,t-\tau) \geq 0 \text{ for } (x,t) \in (0,1) \times (0,\tau],$$

as $b \leq 0$ and $z \geq 0$ in $[0,1] \times [-\tau,0]$. Hence, Lemma 2.1.1 gives $u = z \geq 0$ in $[0,1] \times [0,\tau]$. Now one can establish that $u \geq 0$ in $[0,1] \times [j\tau,(j+1)\tau], j \geq 1$, using the fact that $u \geq 0$ in $[0,1] \times [(j-1)\tau, j\tau]$, and repeating the previous arguments. □

Lemma 2.1.3. *The solution u of problem (2.0.1)-(2.0.2) satisfies*

$$\left\| \frac{\partial^{s+p} u}{\partial x^s \partial t^p} \right\|_{\overline{D}} \leq C\varepsilon^{-s/2} \quad \textit{for } 0 \leq s+2p \leq 4.$$

Proof. See [23, Theorem 3]. □

In the next lemma, using the maximum principle for $\mathcal{L}$ given in Lemma 2.1.3, we derive sharper bounds on the solution derivatives.

Lemma 2.1.4. *The solution u of problem (2.0.1)-(2.0.2) satisfies*

$$|\partial_x^s u(x,t)| \leqslant C(1 + \varepsilon^{-s/2}(e^{-x\sqrt{\alpha/\varepsilon}} + e^{-(1-x)\sqrt{\alpha/\varepsilon}})) \tag{2.1.1}$$

for $(x,t) \in \overline{D}$ and $s = 0,\dots,4$.

Proof. Setting $P_s(x) = 1+\varepsilon^{-s/2}(e^{-x\sqrt{\alpha/\varepsilon}}+e^{-(1-x)\sqrt{\alpha/\varepsilon}})$, we prove the result by using mathematical induction on s. The bound (2.1.1) for $s = 0$ follows from Lemma 2.1.3. Assuming that (2.1.1) holds for $s = 0,\dots,\kappa-1$, $1 \leq \kappa \leq 4$, we shall prove (2.1.1) for $s = \kappa$. Letting $z = \partial_x^\kappa u$, we have

$$\begin{cases} \mathcal{L}z(x,t) := \partial_t z(x,t) - \varepsilon\partial_x^2 z(x,t) + az(x,t) + bz(x,t-\tau) \\ = \partial_x^\kappa f(x,t) - \sum_{l=0}^{\kappa-1} \begin{pmatrix} \kappa \\ l \end{pmatrix} a^{(\kappa-l)}\partial_x^l u(x,t) - \sum_{l=0}^{\kappa-1} \begin{pmatrix} \kappa \\ l \end{pmatrix} b^{(\kappa-l)}\partial_x^l u(x,t-\tau) \\ := \Psi_\kappa \quad \text{in } D = (0,1)\times(0,T], \end{cases}$$

and

$$\begin{cases} |z(x,t)| \le C\varepsilon^{-\kappa/2} \quad \text{in } \Gamma_b, \\ |z(0,t)| \le C\varepsilon^{-\kappa/2}, \quad |z(1,t)| \le C\varepsilon^{-\kappa/2} \quad \text{in } (0,T]. \end{cases}$$

Here, boundary and initial conditions bounds are deduced from Lemma 2.1.3. From the inductive hypothesis, it is clear that $|\Psi_\kappa(x,t)| \le CP_{\kappa-1}(x)$. Then, the maximum principle for $\mathcal{L}$ with the barrier function $CP_\kappa(x)$ gives the required bound. □

Lemma 2.1.5. *The solution u of problem (2.0.1)-(2.0.2) is decomposed into two parts as $u = v + w$, which satisfy*

$$||\partial_x^s v||_{\overline{D}} \le C(1 + \varepsilon^{(2-s)/2}), \tag{2.1.2}$$

$$|\partial_x^s w(x,t)| \le C\varepsilon^{-s/2}\left(\exp(-x\sqrt{\alpha/\varepsilon}) + \exp(-(1-x)\sqrt{\alpha/\varepsilon})\right), \tag{2.1.3}$$

for $(x,t) \in \overline{D}$, $s = 0, \ldots, 4$.

Proof. The decomposition follows using the idea in [65]. □

2.2 Domain decomposition method

The a priori bounds in Section 2.1 show the presence of boundary layers near Γ_ℓ and Γ_r in the solution of problem (2.0.1)-(2.0.2). Thus, to set up the method, we divide the domain into three subdomains that are overlapping, and the two layer regions are localized in two outer subdomains (see figure 2.1 on page 17). More precisely, using the Shishkin transition parameter σ (cf. [66]), the domain D is decomposed into $D_p = \Omega_p \times (0,T]$, $p = \ell, m, r$, where $\Omega_\ell = (0, 2\sigma)$, $\Omega_m = (\sigma, 1-\sigma)$, $\Omega_r = (1-2\sigma, 1)$ with

$$\sigma = \min\left\{\frac{1}{4},\ 2\sqrt{\frac{\varepsilon}{\alpha}}\ln N\right\}. \tag{2.2.1}$$

Here, N is the discretization parameter in the spatial direction. We define $\Gamma_{b,p} = \overline{\Omega}_p \times \overline{\omega}_0$, where $\overline{\omega}_0 = [-\tau, 0]$. On each $D_p = \Omega_p \times (0,T] = (c,d)\times(0,T]$, we consider a rectangular mesh that is uniform in both spatial and time directions. On $\overline{\Omega}_p$, we define $\overline{\Omega}_p^N = \{x_i\}_{i=0}^N$, where $x_0 = c, x_N = d, x_{i+1} = x_i + h_p, h_p = (d-c)/N$. Taking $M = nm_\tau$, the intervals $[-\tau, 0]$ and $[0, T]$, respectively, are divided into m_τ and M subintervals of equal length Δt. Suppose the meshes on $[-\tau, 0]$ and $[0, T]$ are denoted by $\overline{\omega}_0^{m_\tau}$ and $\overline{\omega}^M$, respectively. Introducing $\Omega_p^N = \overline{\Omega}_p^N \cap \Omega_p$, and $\omega^M = \overline{\omega}^M \cap (0, T]$, the mesh $D_p^{N,M}$ on D_p and $\Gamma_{b,p}^{N,m_\tau}$ on $\Gamma_{b,p}$ are defined by the tensor product

$$D_p^{N,M} = \Omega_p^N \times \omega^M \text{ and } \Gamma_{b,p}^{N,m_\tau} = \overline{\Omega}_p^N \times \overline{\omega}_0^{m_\tau}.$$

For each subdomain $D_p^{N,M}$, $p = \ell, m, r$, the discretization is

$$[L_p^{N,M} U_p]_{i,j} + b_{i,j} U_{p;i,j-m_\tau} = f_{i,j}, \quad (2.2.2)$$

where

$$[L_p^{N,M} U_p]_{i,j} = [\delta_t U_p]_{i,j} - \varepsilon[\delta_x^2 U_p]_{i,j} + a_{i,j} U_{p;i,j}, \quad (2.2.3)$$

$$[\delta_t Y]_{i,j} := (Y_{i,j} - Y_{i,j-1})/\Delta t,$$

$$[\delta_x^2 Y]_{i,j} = (Y_{i,j-1} - 2Y_{i,j} + Y_{i,j+1})/h_p^2.$$

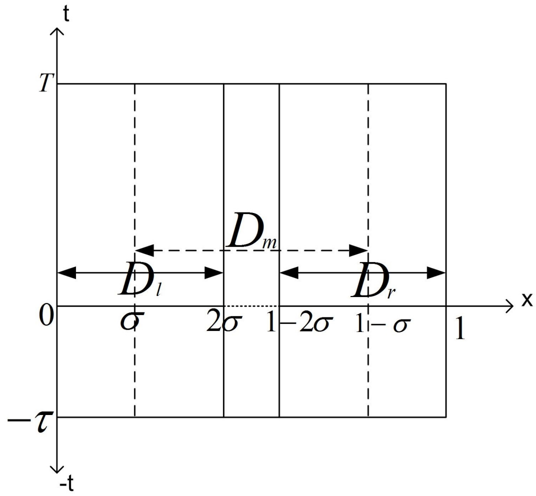

Figure 2.1: Decomposition of the computational domain.

After defining the discretization on each subdomain, the complete iterative process is given as follows.

Step 1. We start with the following initial approximation

$$U^{[0]}(x_i, t_j) = \begin{cases} 0, & 0 < x_i < 1,\ 0 < t_j \le T, \\ u(x_i, t_j), & 0 \le x_i \le 1,\ -\tau \le t_j \le 0, \\ u(0, t_j), & (x_i, t_j) \in \{0\} \times \omega^M, \\ u(1, t_j), & (x_i, t_j) \in \{1\} \times \omega^M. \end{cases} \quad (2.2.4)$$

Step 2. For each $k \ge 1$, we solve the following problems for $U_p^{[k]}$, $p = \ell, m, r$,

$$\begin{cases} [L_\ell^{N,M} U_\ell^{[k]}]_{i,j} + b_{i,j} U_{\ell;i,j-m_\tau}^{[k]} = f_{i,j} & \text{for } (x_i, t_j) \in D_\ell^{N,M}, \\ U_\ell^{[k]}(x_i, t_j) = \phi_b(x_i, t_j) & \text{for } (x_i, t_j) \in \Gamma_{b,\ell}^{N,m_\tau}, \\ U_\ell^{[k]}(0, t_j) = \phi_\ell(t_j), U_\ell^{[k]}(2\sigma, t_j) = \mathcal{I}_j U^{[k-1]}(2\sigma, t_j) & \text{for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [L_r^{N,M} U_r^{[k]}]_{i,j} + b_{i,j} U_{r;i,j-m_\tau}^{[k]} = f_{i,j} & \text{for } (x_i, t_j) \in D_r^{N,M}, \\ U_r^{[k]}(x_i, t_j) = \phi_b(x_i, t_j) & \text{for } (x_i, t_j) \in \Gamma_{b,r}^{N,m_\tau}, \\ U_r^{[k]}(1-2\sigma, t_j) = \mathcal{I}_j U^{[k-1]}(1-2\sigma, t_j),\ U_r^{[k]}(1, t_j) = \phi_r(t_j) & \text{for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [L_m^{N,M}U_m^{[k]}]_{i,j} + b_{i,j}U_{m;i,j-m_\tau}^{[k]} = f_{i,j} & \text{for } (x_i,t_j) \in D_m^{N,M}, \\ U_m^{[k]}(x_i,t_j) = \phi_b(x_i,t_j) & \text{for } (x_i,t_j) \in \Gamma_{b,m}^{N,m_\tau}, \\ U_m^{[k]}(\sigma,t_j) = \mathcal{I}_j U_\ell^{[k]}(\sigma,t_j), U_m^{[k]}(1-\sigma,t_j) = \mathcal{I}_j U_r^{[k]}(1-\sigma,t_j) & \text{for } t_j \in \omega^M, \end{cases}$$

where $\mathcal{I}_j V$ denotes the piecewise linear interpolant of mesh function V at time level t_j.

Step 3. After computing the solutions on each subdomain, we obtain an approximation to the solution of problem (2.0.1)-(2.0.2) as follows

$$U^{[k]}(x_i,t_j) = \begin{cases} U_\ell^{[k]}(x_i,t_j), & (x_i,t_j) \in \overline{D}_\ell^{N,M} \setminus \overline{D}_m, \\ U_m^{[k]}(x_i,t_j), & (x_i,t_j) \in \overline{D}_m^{N,M}, \\ U_r^{[k]}(x_i,t_j), & (x_i,t_j) \in \overline{D}_r^{N,M} \setminus \overline{D}_m. \end{cases} \tag{2.2.5}$$

Step 4. We stop, if

$$||U^{[k+1]} - U^{[k]}||_{\overline{D}^{N,M}} \leq tol$$

is satisfied; otherwise we go to Step 2 and repeat the process. Here, *tol* is a user-prescribed value for ensuring the accuracy of the solution.

2.3 Error analysis

In this section, we provide error analysis of the method proposed in Section 2.2. We introduce $[\mathcal{L}_p^{N,M}U_p]_{i,j} = [L_p^{N,M}U_p]_{i,j} + b_{i,j}U_{p;i,j-m_\tau}$ for $(x_i,t_j) \in D_p^{N,M}$. The operators $L_p^{N,M}$ and $\mathcal{L}_p^{N,M}$ satisfy the following discrete maximum principles.

Lemma 2.3.1. *Let Y be the mesh function such that $Y(x_i,t_j) \geq 0$ for $(x_i,t_j) \in \Gamma_{b,p}^{N,m_\tau}$ and $Y(x_0,t_j) \geq 0$, $Y(x_N,t_j) \geq 0$ for $t_j \in \omega^M$. Then $[L_p^{N,M}Y]_{i,j} \geq 0$ for $(x_i,t_j) \in D_p^{N,M}$ implies that $Y(x_i,t_j) \geq 0$ for $(x_i,t_j) \in \overline{D}_p^{N,M}$.*

Proof. The proof follows using arguments in [66]. □

Lemma 2.3.2. *Let Z be the mesh function such that $Z(x_i,t_j) \geq 0$ for $(x_i,t_j) \in \Gamma_{b,p}^{N,m_\tau}$ and $Z(x_0,t_j) \geq 0$, $Z(x_N,t_j) \geq 0$ for $t_j \in \omega^M$. Then $[\mathcal{L}_p^{N,M}Z]_{i,j} \geq 0$ for $(x_i,t_j) \in D_p^{N,M}$ implies that $Z(x_i,t_j) \geq 0$ for $(x_i,t_j) \in \overline{D}_p^{N,M}$.*

Proof. Let $p = \ell, m, r$. For $s = 0, 1 \ldots, n$, let $\overline{D}_{p,s}^{N,m_\tau} = \overline{\Omega}_p^N \times \overline{\omega}_s^{m_\tau}$, where $\overline{\omega}_s^{m_\tau}$ is obtained by dividing $[(s-1)\tau, s\tau]$ into m_τ equidistant intervals. We also introduce the notation $\overline{D}_{p;s,q}^{N,m_\tau} = \overline{\Omega}_p^N \times \overline{\omega}_{s,q}^{m_\tau}$, where $\overline{\omega}_{s,q}^{m_\tau}$ is obtained by dividing $[(s-1)\tau, q\tau]$ into $(q-s+1)m_\tau$ equidistant intervals. Suppose $Y(x_i,t_j) = Z(x_i,t_j)$ for $(x_i,t_j) \in \overline{D}_{p;0,1}^{N,m_\tau}$.

Thus, we have

$$Y(x_i,t_j)\geq 0 \ \text{ for } (x_i,t_j)\in\Gamma_{b,p}^{N,m_\tau} \ \text{ and } \ Y(x_0,t_j)\geq 0,\, Y(x_N,t_j)\geq 0 \ \text{ for } \ t_j\in\omega_1^{m_\tau}.$$

Also

$$[L_p^{N,M}Y]_{i,j}\geq -b_{i,j}Y(x_i,t_{j-m_\tau})\geq 0 \text{ for } (x_i,t_j)\in D_{p,1}^{N,m_\tau}.$$

Therefore, by using Lemma 2.3.1, we get $Z(x_i,t_j)=Y(x_i,t_j)\geq 0$ for $(x_i,t_j)\in\overline{D}_{p,1}^{N,m_\tau}$. The proof of $Z(x_i,t_j)\geq 0$ for $(x_i,t_j)\in\overline{D}_{p,s}^{N,m_\tau}$, $s\geq 2$, follows using $Z(x_i,t_j)\geq 0$ for $(x_i,t_j)\in\overline{D}_{p,s-1}^{N,m_\tau}$, and repeating the previous arguments. □

We shall prove the uniform convergence of our method with the help of the following auxiliary problems

$$\begin{cases}[L_\ell^{N,M}\widetilde{U}_\ell]_{i,j}+b_{i,j}\widetilde{U}_{\ell;i,j-m_\tau}=f_{i,j} & \text{for } (x_i,t_j)\in D_\ell^{N,M},\\ \widetilde{U}_\ell(x_i,t_j)=u(x_i,t_j) & \text{for } (x_i,t_j)\in\Gamma_{b,\ell}^{N,m_\tau},\\ \widetilde{U}_\ell(0,t_j)=u(0,t_j),\widetilde{U}_\ell(2\sigma,t_j)=u(2\sigma,t_j) & \text{for } t_j\in\omega^M,\end{cases}$$

$$\begin{cases}[L_m^{N,M}\widetilde{U}_m]_{i,j}+b_{i,j}\widetilde{U}_{r;i,j-m_\tau}=f_{i,j} & \text{for } (x_i,t_j)\in D_m^{N,M},\\ \widetilde{U}_m(x_i,t_j)=u(x_i,t_j) & \text{for } (x_i,t_j)\in\Gamma_{b,m}^{N,m_\tau},\\ \widetilde{U}_m(\sigma,t_j)=u(\sigma,t_j),\widetilde{U}_m(1-\sigma,t_j)=u(1-\sigma,t_j) & \text{for } t_j\in\omega^M,\end{cases}$$

$$\begin{cases}[L_r^{N,M}\widetilde{U}_r]_{i,j}+b_{i,j}\widetilde{U}_{m;i,j-m_\tau}=f_{i,j} & \text{for } (x_i,t_j)\in D_r^{N,M},\\ \widetilde{U}_r(x_i,t_j)=u(x_i,t_j) & \text{for } (x_i,t_j)\in\Gamma_{b,r}^{N,m_\tau},\\ \widetilde{U}_r(1-2\sigma,t_j)=u(1-2\sigma,t_j),\ \widetilde{U}_r(1,t_j)=u(1,t_j) & \text{for } t_j\in\omega^M.\end{cases}$$

In what follows, we use the following notation.

$$\begin{aligned}\vartheta_\sigma &= \max\left\{\max_{t_j\in\omega^M}|(\widetilde{U}_m-\widetilde{U}_\ell)(\sigma,t_j)|,\max_{t_j\in\omega^M}|(\widetilde{U}_m-\widetilde{U}_r)(1-\sigma,t_j)|\right\},\\ \vartheta_{2\sigma} &= \max\left\{\max_{t_j\in\omega^M}|(\widetilde{U}_\ell-\widetilde{U}_m)(2\sigma,t_j)|,\max_{t_j\in\omega^M}|(\widetilde{U}_r-\widetilde{U}_m)(1-2\sigma,t_j)|\right\},\\ \vartheta^{[k]} &= \max\{\max_{t_j\in\omega^M}|(\widetilde{U}_\ell-\mathcal{I}_jU^{[k-1]})(2\sigma,t_j)|,\max_{t_j\in\omega^M}|(\widetilde{U}_r-\mathcal{I}_jU^{[k-1]})(1-2\sigma,t_j)|\},\\ \xi^{[k]} &= \max\{||\widetilde{U}_\ell-U^{[k]}||_{\overline{D}_\ell^{N,M}\setminus\overline{D}_m},||\widetilde{U}_m-U^{[k]}||_{\overline{D}_m^{N,M}},||\widetilde{U}_r-U^{[k]}||_{\overline{D}_r^{N,M}\setminus\overline{D}_m}\}.\end{aligned}$$

Lemma 2.3.3. *Let u be the solution of problem (2.0.1)-(2.0.2) and $\widetilde{U}_p$, $p=\ell,m,r$, be the solutions of the auxiliary problems defined in this section. Then*

$$||u-\widetilde{U}_p||_{\overline{D}_p^{N,M}}\leq C(\Delta t+N^{-2}\ln^2 N),\quad p=\ell,m,r. \tag{2.3.1}$$

Proof. For $(x_i, t_j) \in D_\ell^{N,M}$, we have

$$\begin{aligned}
[\mathcal{L}_\ell^{N,M}(u - \widetilde{U}_\ell)]_{i,j} &= [(\mathcal{L}_\ell^{N,M} - \mathcal{L})u]_{i,j} \\
&= \left(\delta_t - \frac{\partial}{\partial t}\right) u(x_i, t_j) - \varepsilon \left(\delta_x^2 - \frac{\partial^2}{\partial x^2}\right) u(x_i, t_j).
\end{aligned}$$

By Taylor expansions, Lemma 2.1.3 and $h_\ell \leq C\sqrt{\varepsilon} N^{-1} \ln N$, we have

$$\begin{aligned}
\left|[\mathcal{L}_\ell^{N,M}(u - \widetilde{U}_\ell)]_{i,j}\right| &\leq \frac{1}{2}(t_j - t_{j-1}) \left\| \frac{\partial^2 u(x_i, .)}{\partial t^2} \right\|_{[t_{j-1}, t_j]} + \frac{\varepsilon}{12} h_\ell^2 \left\| \frac{\partial^4 u(., t_j)}{\partial x^4} \right\|_{[x_{i-1}, x_{i+1}]} \\
&\leq C(\Delta t + N^{-2} \ln^2 N).
\end{aligned}$$

So, Lemma 2.3.2 with a constant barrier function gives

$$||u - \widetilde{U}_\ell||_{\overline{D}_\ell^{N,M}} \leq C(\Delta t + N^{-2} \ln^2 N).$$

Similarly, we have

$$||u - \widetilde{U}_r||_{\overline{D}_r^{N,M}} \leq C(\Delta t + N^{-2} \ln^2 N).$$

To bound $||u - \widetilde{U}_m||_{\overline{D}_m^{N,M}}$, we consider two different cases: $\sigma = 1/4$ and $\sigma = 2\sqrt{\frac{\varepsilon}{\alpha}} \ln N$. For the first case $h_m = 1/(2N)$ and $\varepsilon^{-1} \leq C \ln^2 N$. So, by Taylor expansions and Lemma 2.1.3 (as previously) we get

$$\left|[\mathcal{L}_m^{N,M}(u - \widetilde{U}_m)]_{i,j}\right| \leq C(\Delta t + N^{-2} \ln^2 N) \quad \text{for} \quad (x_i, t_j) \in D_m^{N,M}.$$

For the case $\sigma = 2\sqrt{\frac{\varepsilon}{\alpha}} \ln N$, we consider Taylor expansion and Lemma 2.1.3 to get

$$\left|\left(\delta_t - \frac{\partial}{\partial t}\right) u(x_i, t_j)\right| \leq C\Delta t \quad \text{for} \quad (x_i, t_j) \in D_m^{N,M}.$$

We use the solution decomposition $u = v + w$ and Taylor expansions to get

$$\begin{aligned}
\varepsilon \left|\left(\delta_x^2 - \frac{\partial^2}{\partial x^2}\right) u(x_i, t_j)\right| &\leq \varepsilon \left|\left(\delta_x^2 - \frac{\partial^2}{\partial x^2}\right) v(x_i, t_j)\right| + \varepsilon \left|\left(\delta_x^2 - \frac{\partial^2}{\partial x^2}\right) w(x_i, t_j)\right| \\
&\leq C\varepsilon h_m^2 \left\| \frac{\partial^4 v(., t_j)}{\partial x^4} \right\|_{[x_{i-1}, x_{i+1}]} + C\varepsilon \left\| \frac{\partial^2 w(., t_j)}{\partial x^2} \right\|_{[x_{i-1}, x_{i+1}]}
\end{aligned}$$

Now using Lemma 2.1.5 and $h_m \leq CN^{-1}$ we get

$$\begin{aligned}
\varepsilon \left|\left(\delta_x^2 - \frac{\partial^2}{\partial x^2}\right) u(x_i, t_j)\right| &\leq CN^{-2} + C \left\| \exp(-x\sqrt{\alpha/\varepsilon}) + \exp(-(1-x)\sqrt{\alpha/\varepsilon}) \right\|_{[x_{i-1}, x_{i+1}]} \\
&\leq CN^{-2},
\end{aligned}$$

since, for $(x_i, t_j) \in D_m^{N,M}$,

$$\left\|\exp(-x\sqrt{\alpha/\varepsilon}) + \exp(-(1-x)\sqrt{\alpha/\varepsilon}\right\|_{[x_{i-1},x_{i+1}]} \leq (e^{-\sigma\sqrt{\alpha/\varepsilon}} + e^{-(1-(1-\sigma))\sqrt{\alpha/\varepsilon}})$$

$$= 2e^{-\sigma\sqrt{\alpha/\varepsilon}}$$

$$= 2N^{-2}.$$

Thus, for $(x_i, t_j) \in D_m^{N,M}$, we have

$$\left|[\mathcal{L}_m^{N,M}(u - \widetilde{U}_m)]_{i,j}\right| \leq C(\Delta t + N^{-2}\ln^2 N).$$

So, using Lemma 2.3.2 with a constant barrier function we get $||u - \widetilde{U}_m||_{\overline{D}_m^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N)$. □

In the following theorem we prove that, when $\sigma = 2\sqrt{\frac{\varepsilon}{\alpha}}\ln N$, one iteration is sufficient to provide robust convergent approximation of almost second order in space and first order in time.

Theorem 2.3.4. *Let u be solution of problem (2.0.1)-(2.0.2) and $U^{[1]}$ be the first iterate of the proposed method. If $\sigma = 2\sqrt{\frac{\varepsilon}{\alpha}}\ln N$, then*

$$||u - U^{[1]}||_{\overline{D}^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

Proof. We define the mesh function

$$\Psi^{\pm}(x_i, t_j) = \psi_\ell(x_i, t_j) \pm (\widetilde{U}_\ell - U_\ell^{[1]})(x_i, t_j),$$

where ψ_ℓ satisfies

$$\begin{cases} \delta_t[\psi_\ell]_{i,j} - \varepsilon\delta_x^2[\psi_\ell]_{i,j} + \frac{\alpha}{2}[\psi_\ell]_{i,j} + \frac{\alpha}{2}[\psi_\ell]_{i,j-m_\tau} = 0 & \text{for } (x_i, t_j) \in D_\ell^{N,M}, \\ \psi_\ell(x_i, t_j) = \vartheta^{[1]}\frac{(\zeta_1+\zeta_2)^i - (\zeta_1-\zeta_2)^i}{(\zeta_1+\zeta_2)^N - (\zeta_1-\zeta_2)^N} & \text{for } (x_i, t_j) \in \Gamma_{b,\ell}^{N,m_\tau}, \\ \psi_\ell(0, t_j) = 0, \psi_\ell(2\sigma, t_j) = \vartheta^{[1]} & \text{for } t_j \in \omega^M, \end{cases} \tag{2.3.2}$$

with

$$\zeta_1 = 1 + \left(\frac{\sigma}{N}\sqrt{\frac{\alpha}{\varepsilon}}\right)^2, \; \zeta_2 = 2\left(\frac{\sigma}{N}\sqrt{\frac{\alpha}{\varepsilon}}\right)\sqrt{1 + \left(\frac{\sigma}{N}\sqrt{\frac{\alpha}{\varepsilon}}\right)^2}.$$

The solution ψ_ℓ is independent of t and is given as follows

$$\psi_\ell(x_i, t_j) = \vartheta^{[1]}\frac{(\zeta_1 + \zeta_2)^i - (\zeta_1 - \zeta_2)^i}{(\zeta_1 + \zeta_2)^N - (\zeta_1 - \zeta_2)^N}.$$

Noting that $\widetilde{U}_\ell - U_\ell^{[1]}$ satisfies

$$\begin{cases} [\mathcal{L}_\ell^{N,M}(\widetilde{U}_\ell - U_\ell^{[1]})]_{i,j} = 0 & \text{for } (x_i,t_j) \in D_\ell^{N,M}, \\ (\widetilde{U}_\ell - U_\ell^{[1]})(x_i,t_j) = 0 & \text{for } (x_i,t_j) \in \Gamma_{b,\ell}^{N,m_\tau}, \\ (\widetilde{U}_\ell - U_\ell^{[1]})(0,t_j) = 0, |(\widetilde{U}_\ell - U_\ell^{[1]})(2\sigma,t_j)| \le \vartheta^{[1]} & \text{for } t_j \in \omega^M, \end{cases}$$

we have

$$\Psi^{\pm}(x_i,t_j) \ge 0 \ \text{ for } (x_i,t_j) \in \Gamma_{b,\ell}^{N,m_\tau}, \qquad \Psi^{\pm}(0,t_j) = 0, \Psi^{\pm}(2\sigma,t_j) \ge 0 \ \text{ for } t_j \in \omega^M,$$

and for $(x_i,t_j) \in D_\ell^{N,M}$, $[\mathcal{L}_\ell^{N,M}\Psi^{\pm}]_{i,j} \ge 0$. Then, by using Lemma 2.3.2, we get

$$|(\widetilde{U}_\ell - U_\ell^{[1]})(x_i,t_j)| \le \psi_\ell(x_i,t_j) \quad \text{ for } (x_i,t_j) \in \overline{D}_\ell^{N,M}.$$

Now, for $(x_i,t_j) \in \overline{D}_\ell^{N,M} \backslash \overline{D}_m$, we obtain

$$\psi_\ell(x_i,t_j) \le \vartheta^{[1]} \frac{(\zeta_1+\zeta_2)^{N/2} - (\zeta_1-\zeta_2)^{N/2}}{(\zeta_1+\zeta_2)^{N} - (\zeta_1-\zeta_2)^{N}}$$

$$= \frac{\vartheta^{[1]}}{(\zeta_1+\zeta_2)^{N/2} + (\zeta_1-\zeta_2)^{N/2}} \le \frac{\vartheta^{[1]}}{(\zeta_1+\zeta_2)^{N/2}}.$$

Further, we have $\zeta_2 \ge 2\left(\frac{\sigma}{N}\sqrt{\frac{\alpha}{\varepsilon}}\right)$ and

$$\left(1+\frac{\sigma}{N}\sqrt{\frac{\alpha}{\varepsilon}}\right)^{-N} = \left(1+2\frac{\ln N}{N}\right)^{-N} \le 4N^{-2}, \quad N \ge 1 \ \text{ for } \sigma = 2\sqrt{\varepsilon}\ln N/\sqrt{\alpha},$$

where the last inequality can be proved following the arguments in [4, Lemma 5.1]. Hence, we get $\psi_\ell(x_i,t_j) \le 4\vartheta^{[1]}N^{-2}$ for $(x_i,t_j) \in \overline{D}_\ell^{N,M} \backslash \overline{D}_m$. Thus

$$||\widetilde{U}_\ell - U_\ell^{[1]}||_{\overline{D}_\ell^{N,M}\backslash\overline{D}_m} \le 4\vartheta^{[1]}N^{-2}. \tag{2.3.3}$$

Similarly

$$||\widetilde{U}_r - U_r^{[1]}||_{\overline{D}_r^{N,M}\backslash\overline{D}_m} \le 4\vartheta^{[1]}N^{-2}. \tag{2.3.4}$$

Next

$$[\mathcal{L}_m^{N,M}(\widetilde{U}_m - U_m^{[1]})]_{i,j} = 0 \quad \text{for } (x_i,t_j) \in D_m^{N,M},$$

$$(\widetilde{U}_m - U_m^{[1]})(x_i,t_j) = 0 \quad \text{for } (x_i,t_j) \in \Gamma_{b,m}^{N,m_\tau},$$

$$|(\widetilde{U}_m - U_m^{[1]})(\sigma,t_j)| = |(\widetilde{U}_m - \mathcal{I}_j U_\ell^{[1]})(\sigma,t_j)| \le |(\widetilde{U}_m - \widetilde{U}_\ell)(\sigma,t_j)| + |(\widetilde{U}_\ell - U_\ell^{[1]})(\sigma,t_j)|$$

$$\le \vartheta_\sigma + 4\vartheta^{[1]}N^{-2} \quad \text{for } t_j \in \omega^M,$$

and

$$\begin{aligned} |(\widetilde{U}_m - U_m^{[1]})(1-\sigma, t_j)| &= |(\widetilde{U}_m - \mathcal{I}_j U_r^{[1]})(1-\sigma, t_j)| \\ &\leq |(\widetilde{U}_m - \widetilde{U}_r)(1-\sigma, t_j)| + |(\widetilde{U}_r - U_r^{[1]})(1-\sigma, t_j)| \\ &\leq \vartheta_\sigma + 4\vartheta^{[1]} N^{-2} \quad \text{for } t_j \in \omega^M, \end{aligned}$$

as $(\sigma, t_j) \in \overline{D}_\ell^{N,M}$ and $(1-\sigma, t_j) \in \overline{D}_r^{N,M}$.

Thus, by Lemma 2.3.2, we have

$$||\widetilde{U}_m - U_m^{[1]}||_{\overline{D}_m^{N,M}} \leq \vartheta_\sigma + 4\vartheta^{[1]} N^{-2}. \tag{2.3.5}$$

Hence

$$\xi^{[1]} \leq \vartheta_\sigma + 4\vartheta^{[1]} N^{-2}. \tag{2.3.6}$$

Now using Lemma 2.3.3, $\vartheta_\sigma \leq C(\Delta t + N^{-2}\ln^2 N)$, as $(\sigma, t_j) \in \overline{D}_\ell^{N,M}$ and $(1-\sigma, t_j) \in \overline{D}_r^{N,M}$. Since $\vartheta^{[1]} \leq C$, the desired result can be obtained by combining (2.3.6) and Lemma 2.3.3. □

In the following theorem, we establish uniform convergence of the method for $\sigma = 1/4$.

Theorem 2.3.5. *Let u be the solution of (2.0.1)-(2.0.2) and $U^{[k]}$ be the k^{th} approximation of the proposed method. If $\sigma = 1/4$, then*

$$||u - U^{[k]}||_{\overline{D}^{N,M}} \leq C2^{-k} + C(\Delta t + N^{-2}\ln^2 N). \tag{2.3.7}$$

Proof. Introducing the mesh function

$$\Psi^{\pm}(x_i, t_j) = \frac{x_i}{2\sigma}\vartheta^{[1]} \pm (\widetilde{U}_\ell - U_\ell^{[1]})(x_i, t_j),$$

where $\widetilde{U}_\ell - U_\ell^{[1]}$ satisfies

$$\begin{cases} [\mathcal{L}_\ell^{N,M}(\widetilde{U}_\ell - U_\ell^{[1]})]_{i,j} = 0 & \text{for } (x_i, t_j) \in D_\ell^{N,M}, \\ (\widetilde{U}_\ell - U_\ell^{[1]})(x_i, t_j) = 0 & \text{for } (x_i, t_j) \in \Gamma_{b,\ell}^{N,m_\tau}, \\ (\widetilde{U}_\ell - U_\ell^{[1]})(0, t_j) = 0, |(\widetilde{U}_\ell - U_\ell^{[1]})(2\sigma, t_j)| \leq \vartheta^{[1]} & \text{for } t_j \in \omega^M, \end{cases}$$

we have

$$\begin{cases} \Psi^{\pm}(x_i, t_j) \geq 0 & \text{for } (x_i, t_j) \in \Gamma_{b,\ell}^{N,m_\tau}, \\ \Psi^{\pm}(0, t_j) = 0, \Psi^{\pm}(2\sigma, t_j) \geq 0 & \text{for } t_j \in \omega^M, \end{cases}$$

and for $(x_i, t_j) \in D_\ell^{N,M}$,

$$[\mathcal{L}_\ell^{N,M}\Psi^\pm]_{i,j} \;=\; (a_{i,j}+b_{i,j})\frac{x_i}{2\sigma}\vartheta^{[1]} \pm 0 \geq 0.$$

Then, by using Lemma 2.3.2, we get

$$|(\widetilde{U}_\ell - U_\ell^{[1]})(x_i,t_j)| \leq \frac{x_i}{2\sigma}\vartheta^{[1]} \quad \text{for } (x_i,t_j) \in \overline{D}_\ell^{N,M}.$$

Hence

$$||\widetilde{U}_\ell - U_\ell^{[1]}||_{\overline{D}_\ell^{N,M}\setminus\overline{D}_m} \leq \frac{\vartheta^{[1]}}{2}, \quad \text{as } x_i \leq \sigma. \tag{2.3.8}$$

Similarly we can also show that

$$||\widetilde{U}_r - U_r^{[1]}||_{\overline{D}_r^{N,M}\setminus\overline{D}_m} \leq \frac{\vartheta^{[1]}}{2}. \tag{2.3.9}$$

Next, we find an estimate for $||\widetilde{U}_m - U_m^{[1]}||_{\overline{D}_m^{N,M}}$. We have

$$\begin{aligned}
[\mathcal{L}_m^{N,M}(\widetilde{U}_m - U_m^{[1]})]_{i,j} &= 0 \quad \text{for } (x_i,t_j)\in D_m^{N,M},\\
(\widetilde{U}_m - U_m^{[1]})(x_i,t_j) &= 0 \quad \text{for } (x_i,t_j)\in \Gamma_{b,m}^{N,m_\tau}.\\
\text{Also } |(\widetilde{U}_m - U_m^{[1]})(\sigma,t_j)| &= |(\widetilde{U}_m - \mathcal{I}_j U_\ell^{[1]})(\sigma,t_j)|\\
&\leq \vartheta_\sigma + \frac{\vartheta^{[1]}}{2}, \quad \text{for } t_j \in \omega^M,\\
|(\widetilde{U}_m - U_m^{[1]})(1-\sigma,t_j)| &= |(\widetilde{U}_m - \mathcal{I}_j U_r^{[1]})(1-\sigma,t_j)|\\
&\leq \vartheta_\sigma + \frac{\vartheta^{[1]}}{2}, \quad \text{for } t_j \in \omega^M,
\end{aligned}$$

as $(\sigma,t_j) \in \overline{D}_\ell^{N,M}$ and $(1-\sigma,t_j)\in\overline{D}_r^{N,M}$. So, using Lemma 2.3.2, we get

$$||\widetilde{U}_m - U_m^{[1]}||_{\overline{D}_m^{N,M}} \leq \vartheta_\sigma + \frac{\vartheta^{[1]}}{2}. \tag{2.3.10}$$

For estimating $\xi^{[2]}$ we will require a bound on $\vartheta^{[2]}$. For $\sigma = 1/4$, $(2\sigma,t_j),(1-2\sigma,t_j)\in \overline{D}_m^{N,M}$. Thus

$$\begin{aligned}
|(\widetilde{U}_\ell - \mathcal{I}_j U^{[1]})(2\sigma,t_j)| &\leq \vartheta_{2\sigma}+\vartheta_\sigma+\frac{\vartheta^{[1]}}{2}\\
\text{and } |(\widetilde{U}_r - \mathcal{I}_j U^{[1]})(1-2\sigma,t_j)| &\leq \vartheta_{2\sigma}+\vartheta_\sigma+\frac{\vartheta^{[1]}}{2}.
\end{aligned}$$

Therefore $\vartheta^{[2]} \leq \vartheta_{2\sigma}+\vartheta_\sigma+\frac{\vartheta^{[1]}}{2}$ for $t_j\in\omega^M$. Hence

$$\max\{\xi^{[1]},\vartheta^{[2]}\} \leq \lambda + \frac{\vartheta^{[1]}}{2}, \quad \lambda = \vartheta_{2\sigma}+\vartheta_\sigma.$$

We repeat previous arguments to get

$$\max\{\xi^{[k]}, \vartheta^{[k+1]}\} \leq \lambda + \frac{\vartheta^{[k]}}{2}.$$

Simplifying this we get

$$\xi^{[k]} \leq 2\lambda + 2^{-k}\vartheta^{[1]}.$$

Hence

$$\vartheta^{[k]} \leq 2\lambda + 2^{-(k-1)}\vartheta^{[1]}. \tag{2.3.11}$$

From Lemma 2.3.3, $\lambda \leq C(\Delta t + N^{-2}\ln^2 N)$, as $(2\sigma, t_j)$, $(1-2\sigma, t_j) \in \overline{D}_m^{N,M}$, and $(\sigma, t_j) \in \overline{D}_\ell^{N,M}$ and $(1-\sigma, t_j) \in \overline{D}_r^{N,M}$. Note that $\vartheta^{[1]} \leq C$. Hence, by combining (2.3.11) and Lemma 2.3.3, we have the result. □

2.4 Numerical results

To verify the theoretical results established in the previous section we consider two test problems similar to [23, 24].

Example 2.4.1. Consider the following test problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon\frac{\partial^2 u(x,t)}{\partial x^2} + (1.1 + x^2)u(x,t) - u(x,t-1) = t^3 & (x,t) \in D := \Omega \times (0,2], \\ u(x,t) = 0 & (x,t) \in [0,1] \times [-1,0], \\ u(0,t) = 0, u(1,t) = 0 & t \in (0,2], \end{cases}$$

whose exact solution is not known.

For stopping the iterative process, we consider $tol = N^{-2}$. After stopping the iterative process, the resulting approximate solution is denoted by $U_\varepsilon^{N,\Delta t}$. Since we do not know the solution of the above test problem, we compute the maximum pointwise error and uniform error in the following way

$$E_\varepsilon^{N,\Delta t} = ||U_\varepsilon^{N,\Delta t} - U_\varepsilon^{2N,\Delta t/2^q}||_{\overline{D}^{N,M}} \quad \text{and} \quad E^{N,\Delta t} = \max_\varepsilon E_\varepsilon^{N,\Delta t}, \tag{2.4.1}$$

where $U_\varepsilon^{2N,\Delta t/2^q}$, $q = 1, 2$, is the approximate solution of the test problem obtained with time step size $\Delta t/2^q$ and $2N+1$ mesh points in spatial direction in each subdomain but with subdomain parameter σ considered for computing $U_\varepsilon^{N,\Delta t}$.

After that we compute uniform rate of convergence by

$$\rho^{N,\Delta t} = \log_2\left(\frac{E^{N,\Delta t}}{E^{2N,\Delta t/2^q}}\right).$$

Table 2.1: Maximum errors $E_\varepsilon^{N,\Delta t}$, $E^{N,\Delta t}$ and uniform rate of convergence $\rho^{N,\Delta t}$ for Example 2.4.1.

$\varepsilon = 10^{-p}$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$
	$\Delta t = 0.25$	$\Delta t = 0.25/4$	$\Delta t = 0.25/4^2$	$\Delta t = 0.25/4^3$	$\Delta t = 0.25/4^4$
$p = 0$	1.65E-02	5.26E-03	1.73E-03	4.66E-04	1.19E-04
1	3.31E-01	8.62E-02	2.18E-02	5.46E-03	1.37E-03
2	6.78E-01	1.75E-01	4.40E-02	1.10E-02	2.76E-03
3	7.71E-01	1.98E-01	4.97E-02	1.24E-02	3.11E-03
4	7.92E-01	2.03E-01	5.09E-02	1.27E-02	3.19E-03
5	7.95E-01	2.03E-01	5.11E-02	1.28E-02	3.20E-03
6	7.95E-01	2.03E-01	5.12E-02	1.28E-02	3.20E-03
7	7.95E-01	2.03E-01	5.12E-02	1.28E-02	3.20E-03
8	7.95E-01	2.03E-01	5.12E-02	1.28E-02	3.20E-03
$E^{N,\Delta t}$	7.95E-01	2.03E-01	5.12E-02	1.28E-02	3.20E-03
$\rho^{N,\Delta t}$	1.97	1.99	2.00	2.00	

Table 2.2: Maximum errors $E_\varepsilon^{N,\Delta t}$, $E^{N,\Delta t}$ and uniform rate of convergence $\rho^{N,\Delta t}$ for Example 2.4.1.

$\varepsilon = 10^{-p}$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$
	$\Delta t = 0.25$	$\Delta t = 0.25/2$	$\Delta t = 0.25/2^2$	$\Delta t = 0.25/2^3$	$\Delta t = 0.25/2^4$
$p = 0$	6.55E-03	3.42E-03	1.79E-03	9.01E-04	4.55E-04
1	1.02E-01	5.28E-02	2.69E-02	1.36E-02	6.82E-03
2	1.77E-01	9.05E-02	4.57E-02	2.30E-02	1.15E-02
3	1.94E-01	9.88E-02	4.98E-02	2.50E-02	1.25E-02
4	1.98E-01	1.00E-01	5.06E-02	2.54E-02	1.27E-02
5	1.99E-01	1.01E-01	5.07E-02	2.54E-02	1.27E-02
6	1.99E-01	1.01E-01	5.07E-02	2.54E-02	1.27E-02
7	1.99E-01	1.01E-01	5.07E-02	2.54E-02	1.27E-02
8	1.99E-01	1.01E-01	5.07E-02	2.54E-02	1.27E-02
$E^{N,\Delta t}$	1.99E-01	1.01E-01	5.07E-02	2.54E-02	1.27E-02
$\rho^{N,\Delta t}$	0.98	0.99	1.00	1.00	

Table 2.3: Number of iterations required by the method for Example 2.4.1.

$\varepsilon = 10^{-p}$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$
	$\Delta t = 0.25$	$\Delta t = 0.25/4$	$\Delta t = 0.25/4^2$	$\Delta t = 0.25/4^3$	$\Delta t = 0.25/4^4$
$p = 0$	10	11	13	14	16
1	5	6	6	6	7
2	2	2	2	2	3
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1
6	1	1	1	1	1
7	1	1	1	1	1
8	1	1	1	1	1

For $q = 2$, Table 2.1 displays the errors $E_\varepsilon^{N,\Delta t}$, $E^{N,\Delta t}$ and rate of convergence $\rho^{N,\Delta t}$ computed using the proposed method for this test problem. The last row of this table corresponds to the rate of convergence $\rho^{N,\Delta t}$, which clearly verify our theoretical results proved in Section 2.3. To see the convergence corresponding to the time discretization error, we divided the time step size by two, as displayed in Table 2.2. From it, we observe the first order uniform convergence. Table 2.3 displays the iterations needed to get the approximate solution. One can observe that when ε is small, only one iteration is required to obtained the desired result, as has been proved in Theorem 2.3.4.

Example 2.4.2. Consider the following test problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + 4u(x,t) - 2e^{-1}u(x,t-1) = 0 & (x,t) \in D := \Omega \times (0,2], \\ u(x,t) = e^{-(t+x/\sqrt{\varepsilon})} & (x,t) \in [0,1] \times [-1,0], \\ u(0,t) = e^{-t}, u(1,t) = e^{-(t+1/\sqrt{\varepsilon})} & t \in (0,2], \end{cases}$$

whose exact solution is $u(x,t) = e^{-(t+x/\sqrt{\varepsilon})}$.

Table 2.4: Maximum errors $E_\varepsilon^{N,\Delta t}$, $E^{N,\Delta t}$ and uniform rate of convergence $\rho^{N,\Delta t}$ for Example 2.4.2.

$\varepsilon = 10^{-p}$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$
	$\Delta t = 0.25$	$\Delta t = 0.25/4$	$\Delta t = 0.25/4^2$	$\Delta t = 0.25/4^3$	$\Delta t = 0.25/4^4$
$p = 0$	4.36E-03	1.31E-03	3.45E-04	8.73E-05	2.19E-05
1	6.32E-03	1.75E-03	4.53E-04	1.14E-04	2.86E-05
2	6.37E-03	1.77E-03	4.60E-04	1.16E-04	2.91E-05
3	6.53E-03	1.88E-03	5.07E-04	1.32E-04	3.32E-05
4	6.53E-03	1.88E-03	5.07E-04	1.32E-04	3.32E-05
5	6.53E-03	1.88E-03	5.07E-04	1.32E-04	3.44E-05
6	6.53E-03	1.88E-03	5.07E-04	1.32E-04	3.32E-05
7	6.53E-03	1.88E-03	5.07E-04	1.32E-04	3.32E-05
8	6.53E-03	1.88E-03	5.07E-04	1.32E-04	3.32E-05
$E^{N,\Delta t}$	6.53E-03	1.88E-03	5.07E-04	1.32E-04	3.32E-05
$\rho^{N,\Delta t}$	1.80	1.89	1.94	1.94	

We compute the maximum pointwise error and uniform error in the following way

$$E_\varepsilon^{N,\Delta t} = ||u - U_\varepsilon^{N,\Delta t}||_{\overline{D}^{N,M}} \quad \text{and} \quad E^{N,\Delta t} = \max_\varepsilon E_\varepsilon^{N,\Delta t}. \tag{2.4.2}$$

After that uniform convergence rate is computed as follows

$$\rho^{N,\Delta t} = \log_2 \left(\frac{E^{N,\Delta t}}{E^{2N,\Delta t/4}} \right).$$

Table 2.5: Maximum errors $E_{\varepsilon}^{N,\Delta t}$, $E^{N,\Delta t}$ and uniform rate of convergence $\rho^{N,\Delta t}$ for Example 2.4.2.

$\varepsilon = 10^{-p}$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$
	$\Delta t = 0.25$	$\Delta t = 0.25/2$	$\Delta t = 0.25/2^2$	$\Delta t = 0.25/2^3$	$\Delta t = 0.25/2^4$
$p = 0$	4.36E-03	2.49E-03	1.32E-03	6.78E-04	3.45E-04
1	6.32E-03	3.36E-03	1.75E-03	8.95E-04	4.53E-04
2	6.37E-03	3.38E-03	1.76E-03	8.96E-04	4.53E-04
3	6.53E-03	3.45E-03	1.80E-03	9.11E-04	4.57E-04
4	6.53E-03	3.45E-03	1.80E-03	9.11E-04	4.58E-04
5	6.53E-03	3.45E-03	1.80E-03	9.11E-04	4.58E-04
6	6.53E-03	3.45E-03	1.80E-03	9.11E-04	4.58E-04
7	6.53E-03	3.45E-03	1.80E-03	9.11E-04	4.58E-04
8	6.53E-03	3.45E-03	1.80E-03	9.11E-04	4.58E-04
$E^{N,\Delta t}$	6.53E-03	3.45E-03	1.80E-03	9.11e-04	4.58E-04
$\rho^{N,\Delta t}$	0.92	0.94	0.98	0.99	

Table 2.6: Number of iterations required by the method for Example 2.4.2.

$\varepsilon = 10^{-p}$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$
	$\Delta t = 0.25$	$\Delta t = 0.25/4$	$\Delta t = 0.25/4^2$	$\Delta t = 0.25/4^3$	$\Delta t = 0.25/4^4$
$p = 0$	6	8	10	11	13
1	3	3	4	4	5
2	1	1	1	2	2
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1
6	1	1	1	1	1
7	1	1	1	1	1
8	1	1	1	1	1

For $q = 2$, the maximum pointwise error $E_{\varepsilon}^{N,\Delta t}$, uniform error $E^{N,\Delta t}$ and uniform rate of convergence $\rho^{N,\Delta t}$ computed using the proposed method are given in Table 2.4. Taking $q = 1$, results are given in Table 2.5, where we observe the first order convergence corresponding to the time discretization. Table 2.6 gives the number of iterations required by the iterative process for computing the approximate solution. One can observe that the numerical results are well in accordance with our theoretical findings.

The numerical solution plots for Examples 2.4.1 and 2.4.2 taking $\varepsilon = 10^{-6}$, $N = 32$, $M = 64$ are given in Figure 2.2. In Figure 2.3, we show loglog plot of the maximum pointwise errors vs N for both the examples. The slopes of these plots also validate the theoretically obtained convergence result.

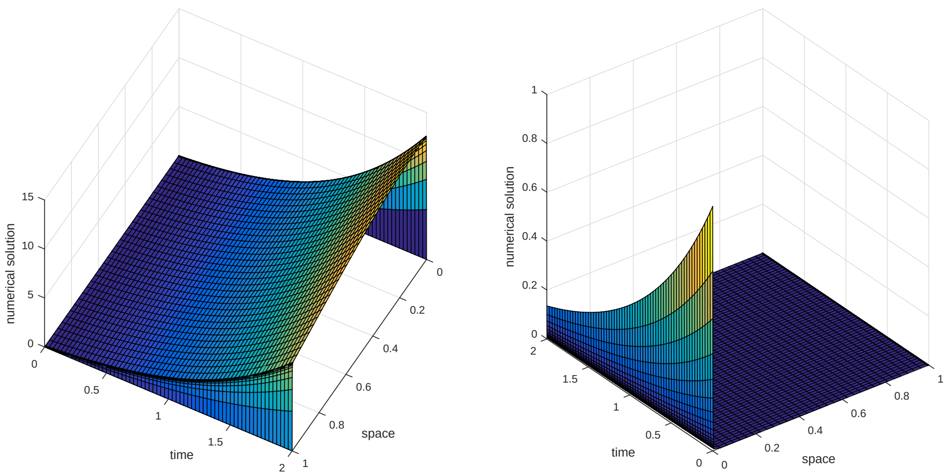

Figure 2.2: Numerical solutions of Examples 2.4.1 and 2.4.2 for $\varepsilon = 10^{-6}$ with $N = 32,\ M = 64$ are depicted in the left and right figures, respectively.

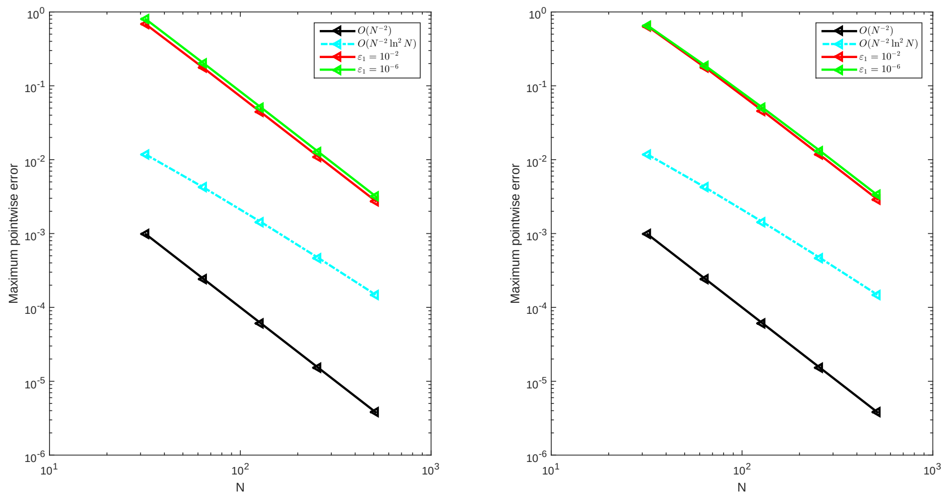

Figure 2.3: Loglog plot of the maximum pointwise errors for Examples 2.4.1 and 2.4.2 are depicted in the left and right subfigures, respectively.

Table 2.7: The used CPU time in seconds for both the examples with $\varepsilon = 10^{-5}$.

	$N = 2^5$ $\Delta t = 0.25$	$N = 2^6$ $\Delta t = 0.25/4$	$N = 2^7$ $\Delta t = 0.25/4^2$	$N = 2^8$ $\Delta t = 0.25/4^3$
Example 2.4.1	0.006750	0.040082	0.304704	2.951105
Example 2.4.2	0.007936	0.057514	0.390472	3.176893

The proposed method is implemented in MATLAB R2011b (TheMathworks,Inc.), on a 64 bit Windows7 machine, with Intel(R) Core(TM) i5-2430M processor running at 2.4GHz and 4.00Gb RAM. The used CPU time in seconds for the proposed method for Examples 2.4.1 and 2.4.1 is given in Table 2.7.

2.5 Conclusions

In this chapter, we have developed a domain decomposition method of SWR type for solving singularly perturbed parabolic reaction-diffusion problems with time delay. To set up the method the original domain is decomposed into three overlapping subdomains and on each subdomain the problem is discretized by the backward Euler scheme in the time direction and the central difference scheme in the spatial direction. After that an iterative process is given with Dirichlet type boundary conditions passed from the previous iterate. The error analysis is given with the help of some auxiliary problems. It is proved that the method is uniformly convergent, which is enabled by the discrete maximum principle established in Lemma 2.3.2 which is the discrete equivalent of the continuous maximum principle given in Lemma 2.1.2. In addition, much faster convergence of the algorithm for small values of the perturbation parameter is also established. Finally, some numerical results are given in support of the theoretical error estimates.