Chapter 6

Study and analysis of nonlinear (2+1)-dimensional space-time fractional reaction-advection-diffusion equation

6.1 Introduction

The linear/nonlinear (2+1)-dimensional fractional-order problems do not have a precise analytic solution. Especially it is hard to get for nonlinear equations in fractional-order systems. Approximate analytical methods and numerical methods are very useful for solving these types of equations. Achieving computationally efficient solutions of these evolutions for different particular cases are very challenging jobs. Due to physical relevance and important applications to explore nonlinear space-time FRADE subject to different types of initial and boundary conditions

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viz., Dirichlet boundary conditions, Neumann boundary conditions and Robin-type boundary condition, mainly in (2+1)-dimensional, which have motivated the author to propose a number of mathematical models.

Many researchers have contributed to obtain efficient and reliable techniques for solution of linear/nonlinear differential equations viz., finite difference method [153], Chebyshev orthogonal collocation technique [154] and Adomian decomposition method [155], etc. Fractional-order partial differential equations are usually difficult to solve by the analytical method and thus various numerical methods are applied to obtain approximate solutions of the equation given in [145, 156, 157, 158, 159, 160, 161, 162, 163].

Study is mainly focused on nonlinear (2+1)-dimensional space-time FRAD solute transport model in $([0,l]\times[0,l]\times[0,\tau])$ given as follows:

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = D_1(x,y,t)\frac{\partial^\beta u(x,y,t)}{\partial x^\beta} + D_2(x,y,t)\frac{\partial^\beta u(x,y,t)}{\partial y^\beta} - V_1(x,y,t)\,\frac{\partial u(x,y,t)}{\partial x} - V_2(x,y,t)\,\frac{\partial u(x,y,t)}{\partial y} + \lambda\, R(u), \tag{6.1}$$

where $0<\alpha\leq 1, 1<\beta\leq 2$,

with initial condition

$$u(x,y,0) = \xi_1(x,y), 0\leq x\leq l, 0\leq y\leq l, \tag{6.2}$$

and boundaries conditions

$$a_1 u(0,y,t) + a_2\frac{\partial u(0,y,t)}{\partial x} = \xi_2(y,t), 0\leq y\leq l, 0\leq t\leq \tau, \tag{6.3}$$

$$a_3 u(l,y,t) + a_4\frac{\partial u(l,y,t)}{\partial x} = \xi_3(y,t), 0\leq y\leq l, 0\leq t\leq \tau, \tag{6.4}$$

$$a_5 u(x,0,t) + a_6 \frac{\partial u(x,0,t)}{\partial y} = \xi_4(x,t), 0 \leq x \leq l, 0 \leq t \leq \tau, \quad (6.5)$$

$$a_7 u(x,l,t) + a_8 \frac{\partial u(x,l,t)}{\partial y} = \xi_5(x,t), 0 \leq x \leq l, 0 \leq t \leq \tau, \quad (6.6)$$

where $\partial^\alpha u(x,y,t)/\partial t^\alpha$, $\partial^\beta u(x,y,t)/\partial x^\beta$ and $\partial^\beta u(x,y,t)/\partial y^\beta$ are fractional derivatives in Caputo sense, $a_i, i = 1,2,3,\cdots,8$ are constants, $\xi_j, j = 1,2,\cdots,5$, $R(u)$, $V_1(x,y,t)$, $V_2(x,y,t)$, $D_1(x,y,t)$ and $D_2(x,y,t)$ are known and $u(x,y,t)$ is the unknown function.

In general it is hard to obtain the exact solution of nonlinear FPDEs, so numerical and approximate techniques are needed to solve the equations. Legendre collocation method with operational matrix is reliable to obtained the solution of nonlinear FPDEs due to the fact that the Legendre polynomials satisfy orthogonality condition.

The purpose of the chapter is to extend the SLCM for solving fractional-order nonlinear partial differential equations. Applying shifted Legendre approximation, shifted Legendre operational matrix (SLOM) for fractional derivative and Kronecker product for the multi-dimensional space problems. An attempt has been taken to obtain the approximate solution of the nonlinear (2+1)-dimensional space-time FRADE with prescribed initial and boundary conditions by using SLCM. The solution of the integer-order form of the above-considered model has been carried out during last few decades due to important application of the model and still there is a penalty of scopes to develop efficient numerical methods to approximate solution for above type of fractional-order problem. The considered problem is first converted into system of nonlinear algebraic equations using shifted Legendre polynomial approximation and operational matrix for derivative. The algebraic equations thus obtained are solved using Newton iteration method through MATHEMATICA software (version 11.0). The results obtained are displayed graphically for different particular cases in different fractional-order systems.

In the present endeavor, author has confirmed the accuracy and efficiency of the proposed method through comparison of the results obtained for a particular form of the model with existing analytical solutions of the (2+1)-dimensional Fisher equation. After its validation, the proposed method is applied to solve the considered nonlinear (2+1)-dimensional space-time FRADE with the help of specified initial and boundary conditions. The main focus is concerned with the effect of the reaction term, advection term and also fractional-order parameters on the solution profiles for different particular cases.

6.2 Implementation of the method

To find the solution of nonlinear (2+1)-dimensional space-time FRADE with initial and boundaries conditions the shifted Legendre approximation and its derivatives as given by equations (1.41),(1.50) and (1.51) are applied on the equations (6.1)-(6.2) which give rise to

$$
\begin{aligned}
&(\phi_{m,\tau}(t))^T (D^{(\alpha)})^T.U.(\phi_{m,l}(x)\otimes\phi_{m,l}(y)) - D_1(x,y,t)(\phi_{m,\tau}(t))^T.U.(D^{(\beta)}\otimes I).(\phi_{m,l}(x)\\
&\otimes\phi_{m,l}(y)) - D_2(x,y,t)(\phi_{m,\tau}(t))^T.U.(I\otimes D^{(\beta)}).(\phi_{m,l}(x)\otimes\phi_{m,l}(y)) - V_1(x,y,t)\\
&(\phi_{m,\tau}(t))^T.U.(D^{(\alpha)}\otimes I).(\phi_{m,l}(x)\otimes\phi_{m,l}(y)) - V_2(x,y,t)(\phi_{m,\tau}(t))^T.U.(I\otimes D^{(1)})\\
&.(\phi_{m,l}(x)\otimes\phi_{m,l}(y)) - \lambda R\Big((\phi_{m,\tau}(t))^T.U.(\phi_{m,l}(x)\otimes\phi_{m,l}(y))\Big) + (\phi_{m,\tau}(0))^T.U\\
&.(\phi_{m,l}(x)\otimes\phi_{m,l}(y)) - \xi_1(x,y) = 0, \qquad (6.7)
\end{aligned}
$$

and the boundary conditions given by equations (6.3)-(6.6) are reduced to

$$a_1(\phi_{m,\tau}(t))^T.U.(\phi_{m,l}(0)\otimes\phi_{m,l}(y)) + a_2(\phi_{m,\tau}(t))^T.U.(D^{(1)}\otimes I).$$
$$(\phi_{m,l}(0)\otimes\phi_{m,l}(y)) - \xi_2(y,t) = 0, \tag{6.8}$$

$$a_3(\phi_{m,\tau}(t))^T.U.(\phi_{m,l}(l)\otimes\phi_{m,l}(y)) + a_4(\phi_{m,\tau}(t))^T.U.(D^{(1)}\otimes I).$$
$$(\phi_{m,l}(l)\otimes\phi_{m,l}(y)) - \xi_3(y,t) = 0, \tag{6.9}$$

$$a_5(\phi_{m,\tau}(t))^T.U.(\phi_{m,l}(x)\otimes\phi_{m,l}(0)) + a_6(\phi_{m,\tau}(t))^T.U.(I\otimes D^{(1)}).$$
$$(\phi_{m,l}(x)\otimes\phi_{m,l}(0)) - \xi_4(x,t) = 0, \tag{6.10}$$

$$a_7(\phi_{m,\tau}(t))^T.U.(\phi_{m,l}(x)\otimes\phi_{m,l}(l)) + a_8(\phi_{m,\tau}(t))^T.U.(I\otimes D^{(1)}).$$
$$(\phi_{m,l}(x)\otimes\phi_{m,l}(l)) - \xi_5(x,t) = 0. \tag{6.11}$$

Equation (6.7) is collocated at the points (x_i, y_i, t_j) for $(m-1)\times(m-1)\times(m+1)$ points. Equations (6.8) and (6.9) are collocated at the points (y_i, t_j) for $m\times(m+1)$ and $(m+1)\times(m+1)$ points respectively. Equations (6.10) and (6.11) are collocated at the points (x_i, t_j) for $(m-1)\times(m+1)$ and $m\times(m+1)$ points respectively. x_i , y_i are the shifted Legendre-Gauss-Lobatto (LGL) points of $P^l_{m-1}(x)$, $P^l_{m-1}(y)$ and t_j are the roots of $P^\tau_{n+1}(t)$. After collocation equations (6.7)-(6.11) are transformed into $(m+1)\times(m+1)^2$ nonlinear equations for the unknown coefficient vector U. The system of nonlinear algebraic equations can be solved using Newton iteration method by mathematical computation for unknown coefficient vector U. The approximate solution $u_{m,m,n}(x, y, t)$ can be found from equation (1.41) by substituting the value of the unknown matrix U.

6.3 Solution of solute transport model

Consider the following nonlinear solute transport model in (2+1)-dimensional space-time FRADE in domain $([0,1] \times [0,1] \times [0,1])$ as

$$\begin{aligned}\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} =& D\Big(\frac{\partial^\beta u(x,y,t)}{\partial x^\beta} + \frac{\partial^\beta u(x,y,t)}{\partial y^\beta}\Big) - V\,\frac{\partial u(x,y,t)}{\partial x} \\ &+ \lambda\, u(x,y,t)(1-u(x,y,t)),\end{aligned} \tag{6.12}$$

where $0 < \alpha \leq 1, 1 < \beta \leq 2$

with initial condition and first type source boundary conditions for finite soil column length of porous medium given as

$$u(x,y,0) = 0, 0 \leq x \leq 1, 0 \leq y \leq 1, \tag{6.13}$$

$$u(0,y,t) = u_0 = t, 0 \leq y \leq 1, 0 \leq t \leq 1, \tag{6.14}$$

$$\frac{\partial u(1,y,t)}{\partial x} = 0, 0 \leq y \leq 1, 0 \leq t \leq 1, \tag{6.15}$$

$$\frac{\partial u(x,0,t)}{\partial y} = 0, 0 \leq x \leq 1, 0 \leq t \leq 1, \tag{6.16}$$

$$\frac{\partial u(x,1,t)}{\partial y} = 0, 0 \leq x \leq 1, 0 \leq t \leq 1. \tag{6.17}$$

The considered nonlinear (2+1)-dimensional space-time FRADE with initial and boundary conditions have been solved with the method already described.

6.4 Numerical results and discussion

The values of $u(x, y, t)/u_0$, the normalized solute concentration, are found numerically for $m = 6$ at fixed time $t = 0.5$ for the considered mathematical models of space-time FRADE and space-time FRDE. The effects of reaction and advection terms for various values of α and β on the solution profile with finite column length are shown through Figures 6.2-6.7. During computation the parameters are considered as $D = 25$ and $\lambda = -0.5$.

To validate our proposed method, it is first applied to solve (2+1)-dimensional Fishers equation ($\alpha = 1, \beta = 2, D = 1$ and $V = 0$) under following conditions:

$$u(x, y, 0) = \left(1 + \exp\left(\frac{1}{\sqrt{6}}\left(x - \frac{y}{\sqrt{2}}\right)\right)\right)^{-2},$$

$$u(0, y, t) = \left(1 + \exp\left(\frac{1}{\sqrt{6}}\left(-\frac{y}{\sqrt{2}} - \frac{5}{\sqrt{6}}t\right)\right)\right)^{-2},$$

$$u(1, y, t) = \left(1 + \exp\left(\frac{1}{\sqrt{6}}\left(1 - \frac{y}{\sqrt{2}}\right) - \frac{5}{\sqrt{6}}t\right)\right)^{-2},$$

$$u(x, 0, t) = \left(1 + \exp\left(\frac{1}{\sqrt{6}}\left(x - \left(\frac{5}{\sqrt{6}}t\right)\right)\right)\right)^{-2},$$

$$u(x, 1, t) = \left(1 + \exp\left(\frac{1}{\sqrt{6}}\left(x - \frac{1}{\sqrt{2}}\right) - \frac{5}{\sqrt{6}}t\right)\right)^{-2},$$

whose the exact solution is [164]

$$u(x,y,t) = \left(1 + \exp\left(\frac{1}{\sqrt{6}}\left(x - \frac{y}{\sqrt{2}}\right) - \frac{5}{\sqrt{6}}t\right)\right)^{-2}.$$

The absolute errors between the exact solution and the numerical solution obtained by using our proposed method are calculated and results are displayed through Figure 6.1 for $y = 1$ and $t = 1$. It is seen from Table 6.1 that the absolute error $ER_{m,n}(x,1,1) = \max\limits_{0\le x\le 1}|u(x,1,1) - u_{m,m,n}(x,1,1)|$ decreases as the values of m, n increase and for $m = n = 6$ our numerical results become reliable and effective. This has inspired the author to find the solution of considered model (6.12) using proposed method for $m = n = 6$.

As per the definition, $u_{m,m,n}(x,y,t)$ converges linearly to $u(x,y,t)$ if there exists a number $\mu \in (0,1)$ such that

$$\lim_{m\to\infty} \frac{|u_{m+1,m+1,n+1}(x,y,t) - u(x,y,t)|}{|u_{m,m,n}(x,y,t) - u(x,y,t)|} = \mu, \tag{6.18}$$

where μ is called the rate of convergence.

Let $\mu_1 = \dfrac{|u_{4,4,4}(x,y,t) - u(x,y,t)|}{|u_{2,2,2}(x,y,t) - u(x,y,t)|}$ and $\mu_2 = \dfrac{|u_{6,6,6}(x,y,t) - u(x,y,t)|}{|u_{4,4,4}(x,y,t) - u(x,y,t)|}, \mu_1, \mu_2 \in (0,1)$,

If $\mu_1 > \mu_2$, it may be concluded that μ_k varies from step to step and $\mu_k \to 0$ as $k \to \infty$. Therefore our aim is to perform the error to show the superlinearly rate of convergence of our proposed method.

In Figure 6.2, the variations of normalized solute concentration with the finite column length for different reaction rate coefficients $\lambda = -1, 0, 1$ for the case of integer-order model RADE ($\alpha = 1, \beta = 2$) are shown. It is seen from the figure that the concentration of solute is less for sink term ($\lambda = -1$) as compared to source term ($\lambda = 1$) as well as for the case of conservative system ($\lambda = 0$). In the case of the integer-order model, the solute concentration covers more column length as time increases which can be seen from Figure 6.3.

In the case of time-fractional RADE ($\beta = 2$), the movement of normalized concentration for $\alpha = 0.4, 0.6, 0.8$ and 1 with finite soil column length are shown in Figure 6.4. It is seen from the figure that the solute concentration in soil column decreases as α approaches from fractional-order system to integer-order of time derivative. The solute concentration is less in integer-order system as compared to fractional-order system. Figure 6.5 shows the variation of normalized solute concentration for time-fractional reaction-diffusion equation ($V = 0$) when $\alpha = 0.4, 0.6, 0.8$ and 1. The nature of the solution profile is just the same as the Figure 6.4, but due to the absence of advection term the concentration of solute is less in each case.

For space-fractional RADE ($\alpha = 1$), the movement of normalized solute concentration for $\beta = 1.4, 1.6, 1.8$ and 2 with finite soil column length are shown in Figure 6.6. It is seen from the figure that the solute concentration in soil column decreases as β approaches from fractional-order system to integer-order of space derivative. Here also solute concentration is less in integer-order system as compared to fractional-order system. As the spatial order derivative approaches from integer-order to fractional-order, the diffusion term approaches to advection term, and as a result concentration of solute will cover more length in soil column length. Figure 6.7 shows the variation of normalized concentration for space-fractional reaction-diffusion equation when $\beta = 1.4, 1.6, 1.8$ and 2. The nature of the solution profile is just the same as Figure 6.6, but due to the absence of advection term the concentration of solute is less in each case.

Table 6.1 Maximum absolute error of Fishers equation with $y = 1, t = 1$ and different x for $m = 2, 4, 6$.

x	Analytic result $u(x,1,1)$	$ER_{2,2}(x,1,1)$	$ER_{4,4}(x,1,1)$	$ER_{6,6}(x,1,1)$
0	0.528014	9.41254×10^{-4}	6.34643×10^{-6}	3.56394×10^{-8}
0.1	0.516188	7.00066×10^{-4}	7.4644×10^{-5}	6.84355×10^{-5}
0.2	0.504286	5.43200×10^{-4}	1.09490×10^{-4}	1.03141×10^{-4}
0.3	0.492321	4.58923×10^{-4}	1.18222×10^{-4}	1.12466×10^{-4}
0.4	0.480305	4.35094×10^{-4}	1.07946×10^{-4}	1.03749×10^{-4}
0.5	0.468250	4.59209×10^{-4}	8.55011×10^{-5}	8.34631×10^{-5}
0.6	0.456168	5.18437×10^{-4}	5.74149×10^{-5}	5.74114×10^{-5}
0.7	0.444074	5.99667×10^{-4}	2.98595×10^{-5}	3.0932×10^{-5}
0.8	0.431980	6.89558×10^{-4}	8.6037×10^{-6}	9.11441×10^{-6}
0.9	0.419900	7.74585×10^{-4}	1.03816×10^{-6}	2.97165×10^{-6}
1	0.407848	8.41098×10^{-4}	5.74003×10^{-6}	3.66204×10^{-8}

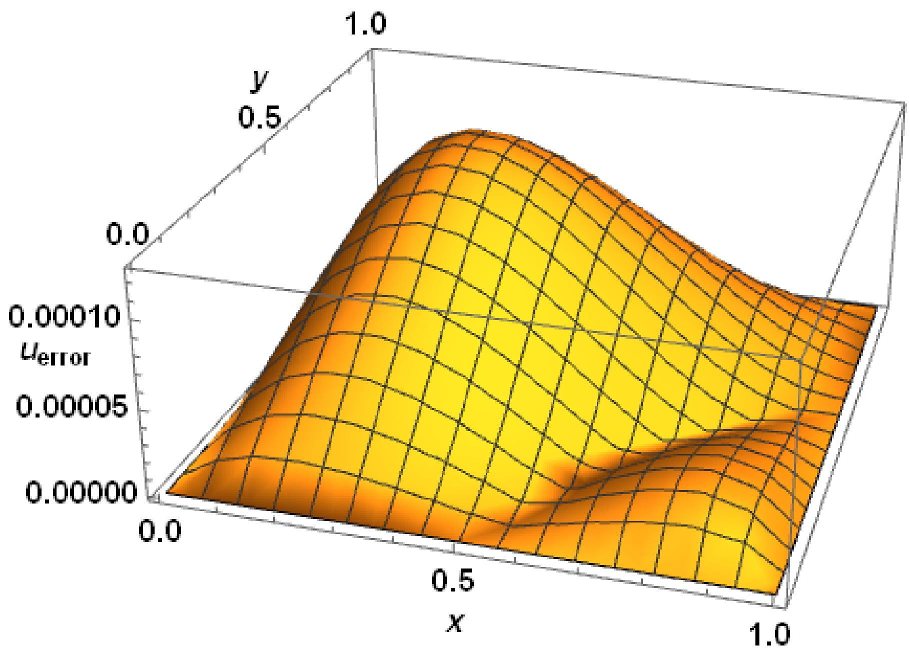

Figure 6.1: The plot of the error function $u_{exact}(x, y, 1) - u_{6,6,6}(x, y, 1)$ for the Fishers equation.

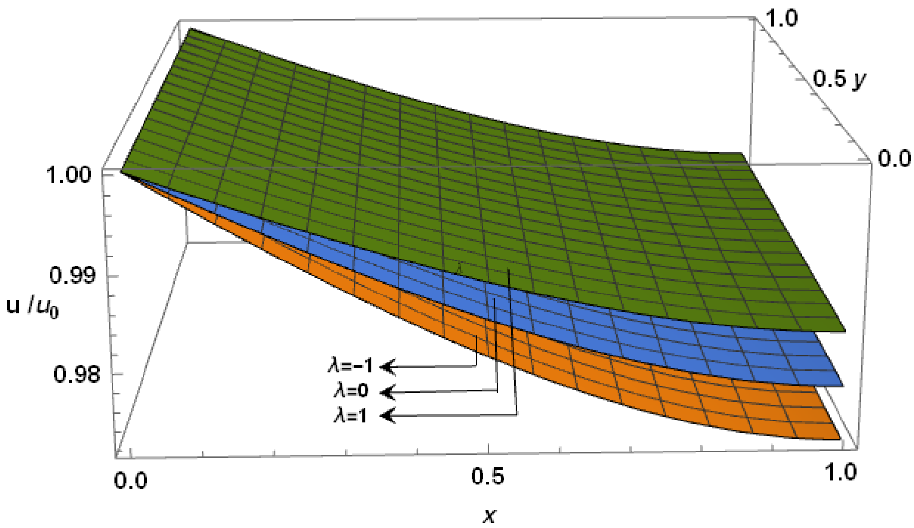

Figure 6.2: Plots of normalised concentration factor $u(x, y, 0.5)/u_0$ vs. column length for $\lambda = -1, 0, 1$ when $D = 25$, $V = 50$, $\alpha = 1$, $\beta = 2$.

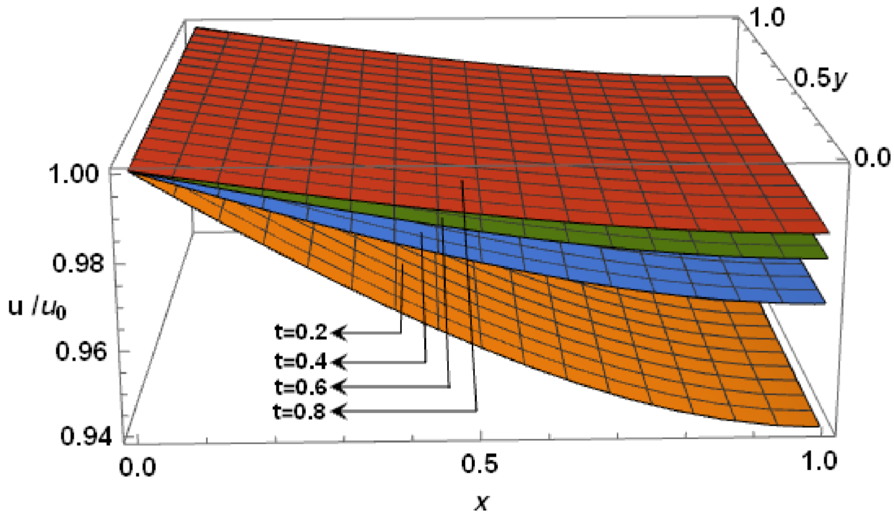

Figure 6.3: Plots of normalised concentration factor $u(x, y, t)/u_0$ vs. column length for time $t = 0.2, 0.4, 0.6, 0.8$ when $D = 25$, $V = 50$, $\alpha = 1$, $\beta = 2$.

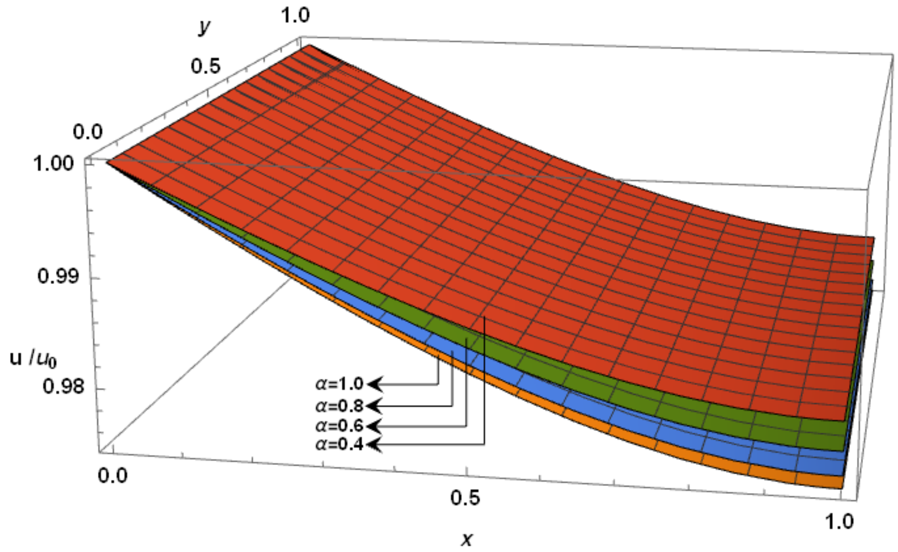

Figure 6.4: Plots of normalised concentration factor $u(x, y, 0.5)/u_0$ vs. column length for different $\alpha = 0.4, 0.6, 0.8$ and 1 when $\beta = 2$, $V = 50$.

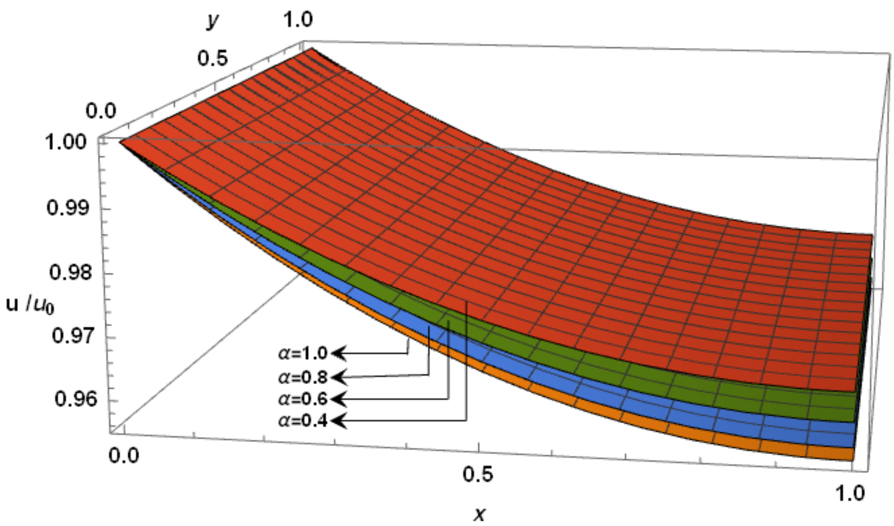

Figure 6.5: Plots of normalised concentration factor $u(x, y, 0.5)/u_0$ vs. column length for different $\alpha = 0.4, 0.6, 0.8$ and 1 when $\beta = 2$, $V = 0$.

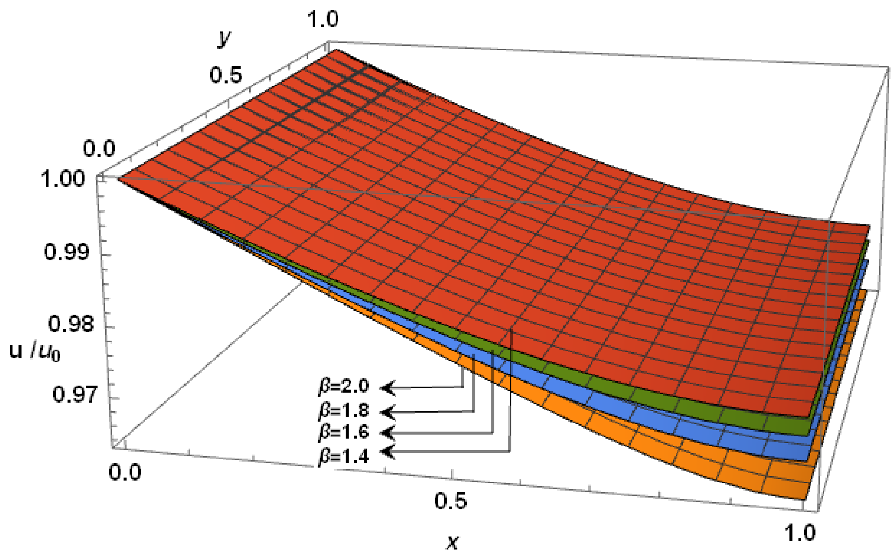

Figure 6.6: Plots of normalised concentration factor $u(x, y, 0.5)/u_0$ vs. column length for different $\beta = 1.4, 1.6, 1.8$ and 2 when $\alpha = 1$, $V = 50$.

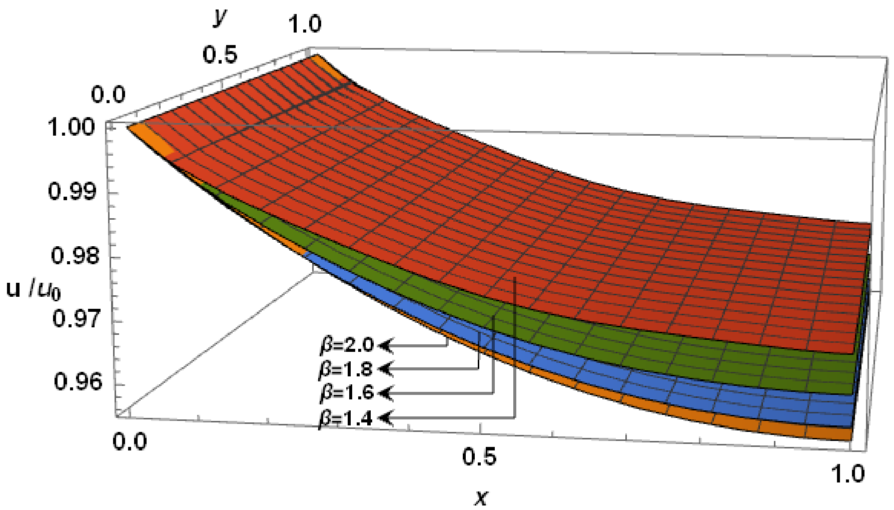

Figure 6.7: Plots of normalised concentration factor $u(x, y, 0.5)/u_0$ vs. column length for different $\beta = 1.4, 1.6, 1.8$ and 2 when $\alpha = 1$, $V = 0$.

6.5 Conclusion

In this chapter, a numerical method called the SLCM using operational matrix for derivatives is extended for solving the nonlinear (2+1)-dimensional space-time FRADE. The effectiveness and efficiency of the method are validated through applying it to a existing nonlinear PDE models of integer-order system having exact solution and showing the superlinearly convergence rate of the proposed method through error analysis. The effects of advection term and reaction term on the solution profile for various space and time fractional-order derivatives are graphically shown for different particular cases. The beauty of the chapter is the explanation of the decay of the solute concentration when the system approaches to fractional-order from the integer-order. The authentication of the fact that the concentration of solute will cover more soil column length as diffusion term approaches to advection term with the decrease in the spatial order derivative is the most important part of the study.

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