Chapter 5

Study of one-dimensional space-time fractional-order Burgers-Fisher and Burgers-Huxley fluid models

5.1 Introduction

Nonlinear RADEs are encountered in several fields of science and engineering, in which Burgers-Fisher equation (BFE) and Burgers-Huxley equation (BHE) are of high importance for describing different mechanisms. The BFE and BHE are termed as mixed hyperbolic-parabolic systems of partial differential equation. These are found in the field of gas dynamics, applied mathematics, financial mathematics and traffic flow. The BFE and BHE are prototypical models to explain the interaction between the reaction mechanisms, advection effect and diffusion transport.

The contents of this chapter have been accepted in **Mathematical Methods in the Applied Sciences**

In recent years, there is intensive study on fractional calculus due to its major applications in various fields viz., chemical, physical, biological, geological and financial systems. For example, modeling and analysis of reactive solute transport in deformable channels with wall adsorption-desorption [142], the mathematical model on fractional-order diffusion that describes nondiffusive transport in plasma turbulence [46] and a non-linear fractional diffusion model for capillary flow through porous media [47]. Fractional calculus gives more accurate models of systems under consideration [143]. Using fractional derivative as a mathematical tool to get the development of more robust mathematical model in particular areas of reservoir engineering, is gaining attention in both industry and academia. The realistic mathematical model of any physical phenomena depending on present and previous time history is achieved when fractional-order derivative is used in place of integer-order derivative. In particular, the microscopic behaviours of mass transportation in porous media are complex and the physical phenomena show strange kinetics which cannot be modeled by classical diffusion equation whereas fractional diffusion equation explains their microscopic dynamics. The fractional-order form of the law of conservation of mass is described in the research article [60] during fluid flow. The mathematical model which describes the solute transport in groundwater are presented in the articles [79, 144, 145]. The fractional-order form of groundwater flow problem can be seen in [61, 62] in which authors have generalized the classical Darcy law by taking the water flow as a function of a non-integer derivative of the Piezometric head. Benson *et al.* [63, 64] have explained that the fractional-order form of advection-diffusion equation is useful for contaminant flow in heterogeneous porous media and earth surfaces such as natural rivers. The fractional-order transport equations within Liouville equations have been considered to solve the fractional-order transport equation in disordered semi-conductors [52]. The fractional-order transport equations are also reported in [56] based on Levy stable processes. Many researchers have contributed to get reliable and efficient techniques for the solution of fractional differential equation [10, 53, 55, 111, 133, 136, 146, 147, 148, 149, 150, 151].

In the present chapter, the study is mainly focused on the following one-dimensional nonlinear spatial-time FPDE with initial and boundary conditions are considered as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^\beta u(x,t)}{\partial x^\beta} - \xi\, u(x,t)\, \frac{\partial u(x,t)}{\partial x} + \lambda\, R(u,x,t), \quad 0 < \alpha \leq 1, 1 < \beta \leq 2, \tag{5.1}$$

with initial condition

$$u(x,0) = \psi_1(x), \quad 0 \leq x \leq 1, \tag{5.2}$$

and boundary conditions

$$u(0,t) = \psi_2(t), \quad 0 < t \leq 1, \tag{5.3}$$

$$u(1,t) = \psi_3(t), \quad 0 < t \leq 1. \tag{5.4}$$

In general it is hard to obtain the exact solution of nonlinear FPDEs, so numerical and approximate techniques are needed to solve the equations. As already discussed that the numerical solutions using various tools are useful to deal with the nonlinear problems. Legendre collocation method with operational matrix is reliable to obtained the solution of nonlinear FPDEs due to the fact that the Legendre polynomials satisfy the orthogonality condition. A truncated orthogonal series is used in the method during solutions of differential equations. Saadatmandi and Dehghan [28] have generalized Legendre operational matrix to fractional-order derivative.

In the present endeavor, an attempt has been taken to obtain the approximate solution of the one-dimensional nonlinear space-time FRADE with prescribed initial and boundary conditions by using shifted Legendre collocation method. The considered problem is first converted into system of nonlinear algebraic equations using shifted Legendre polynomial approximation and operational matrix for fractional-order derivative which are solved using an iteration method. The results obtained are displayed graphically for different particular cases in different fractional-order systems. To validate the accuracy and efficiency of the considered method, it has been applied to two particular existing problems having analytical solutions for the integer-order case. After comparison of the fractional-order derivative obtained by Caputo derivative with the shifted Legendre operational matrix and validation through graph, the author has been motivated to apply the proposed method to solve the considered space-time fractional-order BFE and BHE.

5.2 Estimation of the error

Let $u(x,t) \in ([0,1] \times [0,1])$ can be approximated by $u_{m,m}(x,t)$ as follows:

$$u(x,t) \approx u_{m,m}(x,t) = \sum_{i=0}^{m} \sum_{j=0}^{m} a_{i,j} P_i^1(t) P_j^1(x), \tag{5.5}$$

so that

$$u(x,t) - u_{m,m}(x,t) = \sum_{i=m+1}^{\infty} \sum_{j=m+1}^{\infty} a_{i,j} P_i^1(t) P_j^1(x). \tag{5.6}$$

Theorem 1. Let $u(x,t) \in L^2([0,1] \times [0,1])$, $u_{m,m}(x,t)$ obtained by using shifted Legendre polynomials is the approximation of $u(x,t)$ and $|\partial^4 u(x,t)/\partial x^2 \partial y^2| \leq M$, then

$$\| u(x,t) - u_{m,m}(x,t) \|_E \leq \frac{M}{16}\Big(\frac{\Gamma^{'}(m-0.5)}{\Gamma(m-0.5)}\Big)^{'''}, \tag{5.7}$$

where $\| u(x,t) \|_E^2 = \int_0^1 \int_0^1 u^2(x,t)dxdt$.

Proof.

$$\begin{aligned}
\| u(x,t) - u_{m,m}(x,t) \|_E^2 &= \int_0^1 \int_0^1 (u(x,t) - u_{m,m}(x,t))^2 dx\, dt \\
&= \int_0^1 \int_0^1 \sum_{i=m+1}^{\infty} \sum_{j=m+1}^{\infty} \Big(a_{i,j} P_i^1(t) P_j^1(x)\Big)^2 dx\, dt \\
&= \sum_{i=m+1}^{\infty} \sum_{j=m+1}^{\infty} a_{i,j}^2 \int_0^1 (P_i^1(x))^2 dx \int_0^1 (P_j^1(t))^2 dt.
\end{aligned}$$

Using orthogonality condition of the shifted Legendre sequence given in equation (1.37), I have

$$\| u(x,t) - u_{m,m}(x,t) \|_E^2 = \sum_{i=m+1}^{\infty} \sum_{j=m+1}^{\infty} a_{i,j}^2 \frac{1}{(2i+1)(2j+1)}. \tag{5.8}$$

The shifted Legendre polynomial coefficients $a_{i,j}$ of the function $u(x,t)$ are given by

$$a_{i,j} = (2i+1)(2j+1) \int_0^1 \int_0^1 u(x,t) P_i^1(x) P_j^1(t) dxdt.$$

Solving the above equation, we get

$$
\begin{aligned}
a_{i,j} &= (2j+1)\int_0^1 u(x,t)(P^1_{i+1}(x) - P^1_{i-1}(x))P^1_j(t)|_0^1 dt \\
&- (2j+1)\int_0^1\int_0^1 \frac{\partial u(x,t)}{\partial x}(P^1_{i+1}(x) - P^1_{i-1}(x))P^1_j(t)dxdt \\
&= -(2j+1)\int_0^1\int_0^1 \frac{\partial u(x,t)}{\partial x}(P^1_{i+1}(x) - P^1_{i-1}(x))P^1_j(t)dxdt \\
&= -(2j+1)\int_0^1 \frac{\partial u(x,t)}{\partial x}\Big(\frac{P^1_{i+2}(x) - P^1_i(x)}{2i+3} - \frac{P^1_i(x) - P^1_{i-2}(x)}{2i-1}\Big)P^1_j(t)|_0^1 dt \\
&+ (2j+1)\int_0^1\int_0^1 \frac{\partial^2 u(x,t)}{\partial x^2}\Big(\frac{P^1_{i+2}(x) - P^1_i(x)}{2i+3} - \frac{P^1_i(x) - P^1_{i-2}(x)}{2i-1}\Big)P^1_j(t)dxdt,
\end{aligned}
$$

$$
= (2j+1)\int_0^1\int_0^1 \frac{\partial^2 u(x,t)}{\partial x^2}\Big(\frac{P^1_{i+2}(x) - P^1_i(x)}{2i+3} - \frac{P^1_i(x) - P^1_{i-2}(x)}{2i-1}\Big)P^1_j(t)dxdt.
$$

Now considering $Q_i(x) = (2i-1)P^1_{i+2}(x) - 2(2i+1)P^1_i(x) + (2i+3)P^1_{i-2}(x)$, we get

$$
a_{i,j} = \frac{(2j+1)}{(2i-1)(2i+3)}\int_0^1\int_0^1 \frac{\partial^2 u(x,t)}{\partial x^2}Q_i(x)P^1_j(t)dxdt.
$$

Again solving the above equation, we have

$$
a_{i,j} = \frac{1}{(2i-1)(2i+3)(2j-1)(2j+3)}\int_0^1\int_0^1 \frac{\partial^4 u(x,t)}{\partial x^2 \partial y^2}Q_i(x)Q_j(t)dxdt.
$$

Therefore

$$
|a_{i,j}| \leq \frac{1}{(2i-1)(2i+3)(2j-1)(2j+3)}\int_0^1\int_0^1 |\frac{\partial^4 u(x,t)}{\partial x^2 \partial y^2}|.|Q_i(x)|.|Q_j(t)|dxdt.
$$

Hence

$$
|a_{i,j}| \leq \frac{M}{(2i-1)(2i+3)(2j-1)(2j+3)}\int_0^1 |Q_i(x)|dx \int_0^1 |Q_j(t)|dt.
$$

As $\int_0^1 |Q_i(x)|dx = \frac{\sqrt{24}}{2}.\frac{(2i+3)}{\sqrt{2i-3}}$, the above inequality becomes

$$|a_{i,j}| \leq \frac{6M}{(2i-1)(2i+3)(2j-1)(2j+3)}.\frac{(2i+3)}{\sqrt{2i-3}}.\frac{(2j+3)}{\sqrt{2j-3}}.$$

Thus

$$|a_{i,j}|^2 \leq \frac{36M^2}{(2i-3)^3(2j-3)^3}.$$

Substituting the value of $a_{i,j}^2$ in the equation (5.8), we get

$$\begin{aligned}
\parallel u(x,t) - u_{m,m}(x,t) \parallel_E^2 &\leq \sum_{i=m+1}^{\infty} \sum_{j=m+1}^{\infty} \frac{36M^2}{(2i-3)^3(2j-3)^3}.\frac{1}{(2i+1)(2j+1)} \\
&\leq \sum_{i=m+1}^{\infty} \sum_{j=m+1}^{\infty} \frac{36M^2}{(2i-3)^4(2j-3)^4} \\
&\leq \Big(\sum_{i=m+1}^{\infty} \frac{6M}{(2i-3)^4} \Big)^2 = \frac{1}{4}\Big[\Big(\frac{M}{8} \frac{\Gamma'(m-0.5)}{\Gamma(m-0.5)} \Big)''' \Big]^2.
\end{aligned}$$

Therefore

$$\parallel u(x,t) - u_{m,m}(x,t) \parallel_E \leq \frac{M}{16} \Big(\frac{\Gamma'(m-0.5)}{\Gamma(m-0.5)} \Big)'''. \tag{5.9}$$

□

5.3 Solution of the fractional-order partial differential equations

To solve the nonlinear space-time FRADE (5.1) with initial and boundary conditions (5.2)-(5.4), the function $u(x,t) \in ([0,1] \times [0,1])$ is approximated by shifted Legendre polynomial as

$$u(x,t) \approx \phi_{m,1}^T(t).A.\phi_{m,1}(x). \tag{5.10}$$

Then derivatives are defined as

$$\begin{aligned}\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} &\approx \frac{\partial^\alpha(\phi_{m,1}^T(t).A.\phi_{m,1}(x))}{\partial t^\alpha} = \Big(\frac{\partial^\alpha \phi_{m,1}(t)}{\partial t^\alpha}\Big)^T.A.\phi_{m,1}(x) \\ &= \phi_{m,1}^T(t).(D^{(\alpha)})^T.A.\phi_{m,1}(x),\end{aligned} \tag{5.11}$$

$$\begin{aligned}\frac{\partial^\beta u(x,t)}{\partial x^\beta} &\approx \frac{\partial^\beta(\phi_{m,1}^T(t).A.\phi_{m,1}(x))}{\partial x^\beta} = \phi_{m,1}^T(t).A.\frac{\partial^\beta \phi_{m,1}(x)}{\partial x^\beta} \\ &= \phi_{m,1}^T(t).A.(D^{(\beta)}.\phi_{m,1}(x)),\end{aligned} \tag{5.12}$$

$$\begin{aligned}\frac{\partial u(x,t)}{\partial x} &\approx \frac{\partial(\phi_{m,1}^T(t).A.\phi_{m,1}(x))}{\partial x} = \phi_{m,1}^T(t).A.\frac{\partial \phi_{m,1}(x)}{\partial x} \\ &= \phi_{m,1}^T(t).A.(D^{(1)}.\phi_{m,1}(x)).\end{aligned} \tag{5.13}$$

Substituting equations (5.10)-(5.13) into equations (5.1) and (5.2), we have

$$\begin{aligned}\phi_{m,1}^T(t).(D^{(\alpha)})^T.A.\phi_{m,1}(x) &= \phi_{m,1}^T(t).A.D^{(\beta)}.\phi_{m,1}(x) - \xi(\phi_{m,1}^T(t).A.\phi_{m,1}(x)) \\ &(\phi_{m,1}^T(t).A.D^{(1)}.\phi_{m,1}(x)) + \lambda\, R((\phi_{m,1}^T(t).A.\phi_{m,1}(x)), x, t),\end{aligned} \tag{5.14}$$

$$\phi_{m,1}^T(0).A.\phi_{m,1}(x) = \psi_1(x), \tag{5.15}$$

Equations (5.14) and (5.15) are rewritten as

$$\begin{aligned}H(x,t) = D^{(\alpha)}\phi_{m,1}^T(t).A.\phi_{m,1}(x) &- \phi_{m,1}^T(t).A.D^{(\beta)}.\phi_{m,1}(x) \\ &+ \xi\,(\phi_{m,1}^T(t).A.\phi_{m,1}(x))\,(\phi_{m,1}^T(t).A.D^{(1)}.\phi_{m,1}(x)) \\ &- \lambda R(\phi_{m,1}^T(t).A.\phi_{m,1}(x), x, t) + \phi_{m,1}^T(0).A.\phi_{m,1}(x) - \psi_1(x),\end{aligned} \tag{5.16}$$

and boundary conditions (5.3)-(5.4) become

$$\phi_{m,1}^T(t).A.\phi_{m,1}(0) = \psi_2(t), \tag{5.17}$$

$$\phi_{m,1}^T(t).A.\phi_{m,1}(1) = \psi_3(t). \tag{5.18}$$

Equation (5.16) is collocated at (x_i, t_j) for $(m-1)\times(m+1)$ points and equations (5.17) and (5.18) are collocated at t_j for $(m+1)$ points respectively where x_i's are the shifted Legendre-Gauss-Lobatto (SLGL) points of $P_{m-1}^l(x)$ and $t_j's$ are the roots of shifted Legendre polynomial $P_{m+1}^\tau(t)$. After collocation $(m+1)\times(m+1)$ nonlinear equations for $(m+1)\times(m+1)$ unknowns are obtained which are given as follows:

$$\begin{aligned} H(x_i,t_j) = {} & D^{(\alpha)}.\phi_{m,1}^T(t_j).A.\phi_{m,1}(x_i) - \phi_{m,1}^T(t_j).A.D^{(\beta)}.\phi_{m,1}(x_i) \\ & + \xi\,(\phi_{m,1}^T(t_j).A.\phi_{m,1}(x_i))\,(\phi_{m,1}^T(t_j).A.D^{(1)}.\phi_{m,1}(x_i)) \\ & - \lambda R(\phi_{m,1}^T(t_j).A.\phi_{m,1}(x_i), x_i, t_j) + \phi_{m,1}^T(0).A.\phi_{m,1}(x_i) - \psi_1(x_i) = 0, \end{aligned} \tag{5.19}$$

and

$$\phi_{m,1}^T(t_j).A.\phi_{m,1}(0) - \psi_2(t_j) = 0, \tag{5.20}$$

$$\phi_{m,1}^T(t_j).A.\phi_{m,1}(1) - \psi_3(t_j) = 0. \tag{5.21}$$

The system of nonlinear equations are solved using Newton iteration method for $(m+1)\times(m+1)$ unknown entries of unknown matrix A. Consequently approximate solution $u_{m,m}(x,t)$ given by equation (1.40) can be calculated with simple computation.

5.4 Mathematical models

In the present chapter, I have extended the well known BFE and BHE to space-time fractional-order BFE and BHE. Mathematical models of space-time fractional-order BFE and BHE are realistic mathematical models having physical phenomena of dependence not only at the time instant, but also the previous time history. The solutions of the considered problems predict the concentration $u(x,t)$ of solute (impurity) which contaminate the groundwater by transportation in soil column through porous media from land surface to groundwater level. Initial condition (at $t = 0$) is taken in such a way that the concentration of solute, initially in the soil column of finite length is zero. And the boundary conditions are such that, the concentration of solute at inlet boundary ($x = 0$) linearly increases with time and at outlet boundary ($x = 1$) is constant with respect to column length of the soil column.

5.4.1 Space-time fractional-order Burgers-Fisher equation

The considered one-dimensional space-time fractional-order BFE with initial and boundary conditions in a bounded space domain $[0, 1]$ is given as

$$\begin{aligned}&\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^\beta u(x,t)}{\partial x^\beta} + u(x,t)\frac{\partial u(x,t)}{\partial x} + \lambda u(x,t)(1-u(x,t)),\\ &0 < \alpha \le 1, 1 < \beta \le 2,\end{aligned} \tag{5.22}$$

with the following initial and first type source boundary conditions as

$$u(x,0) = 0, \quad 0 \le x \le 1, \tag{5.23}$$

$$u(0,t) = u_0 = t, \quad t > 0, \tag{5.24}$$

$$\frac{\partial u(1,t)}{\partial x} = 0, \quad t > 0. \tag{5.25}$$

5.4.2 Space-time fractional-order Burgers-Huxley equation

The considered one-dimensional space-time fractional-order BHE with initial and boundary conditions in a boundary space domain $[0,1]$ is given as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^\beta u(x,t)}{\partial x^\beta} + u(x,t)\frac{\partial u(x,t)}{\partial x} + \lambda u(x,t)(1-u(x,t))(u(x,t)-1),$$
$$0 < \alpha \leq 1, 1 < \beta \leq 2, \tag{5.26}$$

with the following initial and first type source boundary conditions as

$$u(x,0) = 0, \quad 0 \leq x \leq 1, \tag{5.27}$$

$$u(0,t) = u_0 = t, \quad t > 0, \tag{5.28}$$

$$\frac{\partial u(1,t)}{\partial x} = 0, \quad t > 0. \tag{5.29}$$

The considered space-time fractional-order BFE and BHE with given initial and boundary conditions have been solved with the method described in section 5.3.

5.5 Numerical results and discussion

To illustrate the validity and applicability of the approach, we apply the method to solve the integer-order BFE and BHE ($\alpha = 1, \beta = 2, \lambda = 1$) which have exact solutions under the prescribed initial and boundary conditions. The integer-order

BFE under the following initial and boundary conditions

$$u(x,0) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{4}\right), \tag{5.30}$$

$$u(0,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{5}{8}t\right), \tag{5.31}$$

$$u(1,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{4}\left(1+\frac{5}{2}t\right)\right), \tag{5.32}$$

has the exact solution as [152]

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{4}\left(x+\frac{5}{2}t\right)\right). \tag{5.33}$$

Also the integer-order BHE under the suitable initial and boundary conditions as

$$u(x,0) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{4}x\right), \tag{5.34}$$

$$u(0,t) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{3}{8}t\right), \tag{5.35}$$

$$u(1,t) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{4}\left(1+\frac{3}{2}t\right)\right), \tag{5.36}$$

has the exact solution as [152]

$$u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{4}\left(x+\frac{3}{2}t\right)\right). \tag{5.37}$$

We have found the absolute errors ($ER_j(x,t)$) and orders of convergence (CO) for discrete points which are given as

$ER_j(x,t) = |u(x,t) - u_{j,j}(x,t)|$ and $\mathrm{CO}(j) = \dfrac{\log\Big(ER_j(x,t)/ER_{j+1}(x,t)\Big)}{\log\Big((j+1)/j\Big)}$,

and found between our obtained numerical results and the exact solutions for both the problems and the results are displayed through Tables 5.1-5.2. In the case of BFE, Table 5.1 shows that the order of convergence increases as the shifted Legendre polynomial approximation increases. Similarly, in the case of BHE, the order of convergence increases rapidly as the approximated degree of the polynomial increases as shown in Table 5.2. The test examples demonstrate the efficiency, versatility and accuracy of the proposed method. It confirms that the proposed numerical method gives better results for nonlinear PDEs. After the successful validation of the method for integer-order models, the author is motivated to apply the proposed method to solve the concerned models in time and space fractional-orders.

The numerical values of the normalized solute concentration $u(x,t)/u_0$ are calculated for $m = 12$ at fixed time $t = 0.5$ for the considered mathematical models of porous media with space-time fractional derivatives for both BFE and BHE equations with finite column length for various values of α and β and the results are depicted through Figures 5.1-5.5.

In Figure 5.1, the movements of the normalized solute concentration with the finite column length are shown for time-fractional derivative i.e., $\alpha = 0.6, 0.7, 0.8, 0.9, 1$ for time fractional-order BFE ($\beta = 2$). Similar behaviour can be seen for time fractional-order BHE ($\beta = 2$) in Figure 5.3. It is seen from the figures that as the parameter α approaches from fractional-order to the integer-order the normalized concentration decreases. It is also observed that the solute covers less length in soil column for integer-order case as compared to fractional-order case.

In Figure 5.2, the movements of the normalized solute concentration with the finite column length are shown with the variation of space-fractional derivative ($\beta = 1.6, 1.7, 1.8, 1.9, 2$) in the case of space fractional-order BFE ($\alpha = 1$). Similar behaviour can be seen for space fractional-order BHE in Figure 5.4. It is seen from the figures that as the parameter β approaches from fractional-order to the integer-order the normalized concentration decreases. For the case of integer-order system ($\alpha = 1, \beta = 2$), the solute covers less length in the soil column compared to fractional-order system. As the spatial order derivative decreases, the diffusive term approaches to advection term, and as a result the concentration of solute will cover more soil column length.

The effect of the reaction term, on the solution profile is expressed through Figure 5.5. The movement of solute concentration is similar for conservative ($\lambda = 0$) and non-conservative systems (BFE, BHE). The normalized solute concentration covers more column length in the case of BFE compared to the conservative system due to the effect of source term ($u(x,t)(1-u(x,t)) > 0$) in BFE and it is less for the case of BHE compared to the conservative system due to the effect of sink term ($u(x,t)(1-u(x,t))(u(x,t)-1) < 0$) in BHE which are physically justified.

Table 5.1 Maximum absolute error and order of convergence of BFE with $t = 0.5$ and different x for $m = 3, 6$ and 12

x	$ER_3(x, 0.5)$	$ER_6(x, 0.5)$	$ER_{12}(x, 0.5)$	CO(3)	CO(6)
0.1	2.03493×10^{-4}	3.03610×10^{-6}	2.50293×10^{-9}	6.06661	10.2444
0.2	3.41039×10^{-4}	4.98004×10^{-6}	4.60862×10^{-9}	6.09764	10.0776
0.3	4.21791×10^{-4}	6.37079×10^{-6}	6.13964×10^{-9}	6.04891	10.0191
0.4	4.54119×10^{-4}	7.11294×10^{-6}	6.98383×10^{-9}	5.99648	9.99221
0.5	4.45601×10^{-4}	7.17704×10^{-6}	7.10239×10^{-9}	5.95622	9.98087
0.6	4.03019×10^{-4}	6.59908×10^{-6}	6.52979×10^{-9}	5.93244	9.98101
0.7	3.32359×10^{-4}	5.47414×10^{-6}	5.36617×10^{-9}	5.92397	9.99452
0.8	2.38818×10^{-4}	3.94469×10^{-6}	3.76369×10^{-9}	5.91986	10.0335
0.9	1.26813×10^{-4}	2.18381×10^{-6}	1.90838×10^{-9}	5.85971	10.1603

Table 5.2 Maximum absolute error and order of convergence of BHE with $t = 0.5$ and different x for $m = 3, 6$ and 12

x	$ER_3(x, 0.5)$	$ER_6(x, 0.5)$	$ER_{12}(x, 0.5)$	CO(3)	CO(6)
0.1	1.22006×10^{-4}	1.51746×10^{-7}	1.66030×10^{-11}	9.65108	13.1579
0.2	1.86267×10^{-4}	2.62709×10^{-7}	3.09349×10^{-11}	9.46969	13.0519
0.3	2.05453×10^{-4}	3.58042×10^{-7}	4.17091×10^{-11}	9.16446	13.0675
0.4	1.91603×10^{-4}	4.21728×10^{-7}	4.80435×10^{-11}	8.82759	13.0997
0.5	1.56083×10^{-4}	4.39185×10^{-7}	4.95181×10^{-11}	8.47327	13.1146
0.6	1.09548×10^{-4}	4.06287×10^{-7}	4.61907×10^{-11}	8.07485	13.0719
0.7	1.19127×10^{-4}	3.32075×10^{-7}	3.85655×10^{-11}	8.48678	13.0602
0.8	1.23278×10^{-4}	2.35105×10^{-7}	2.75255×10^{-11}	9.03439	13.0602
0.9	8.35988×10^{-5}	1.33479×10^{-7}	1.42311×10^{-11}	9.29907	13.1953

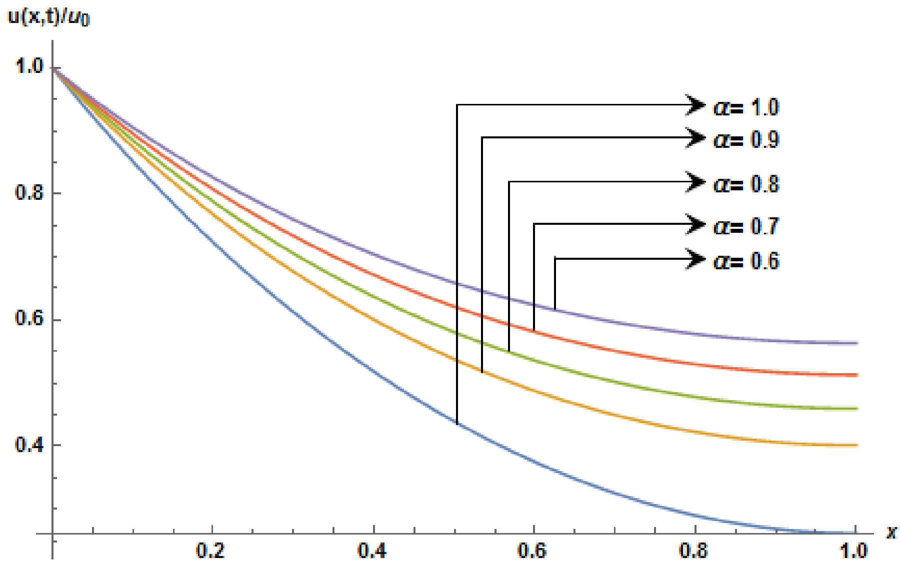

Figure 5.1: Plots of normalised concentration factor vs. column length with first type source boundary condition for space-time fractional-order BFE when $\alpha = 0.6, 0.7, 0.8, 0.9$ and 1 at fixed $\beta = 2$.

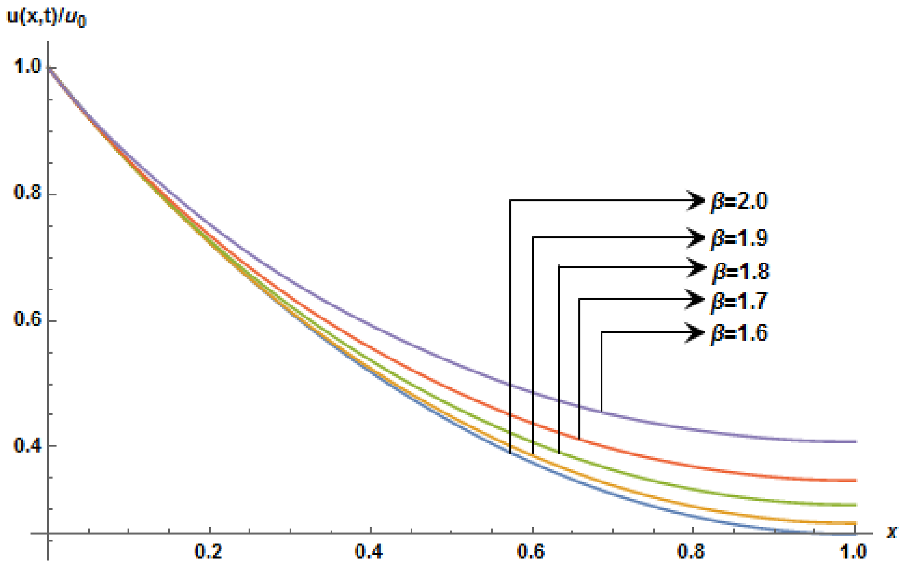

Figure 5.2: Plots of normalised concentration factor vs. column length with first type source boundary condition for space-time fractional-order BFE when $\beta = 1.6, 1.7, 1.8, 1.9$ and 2 at fixed $\alpha = 1$.

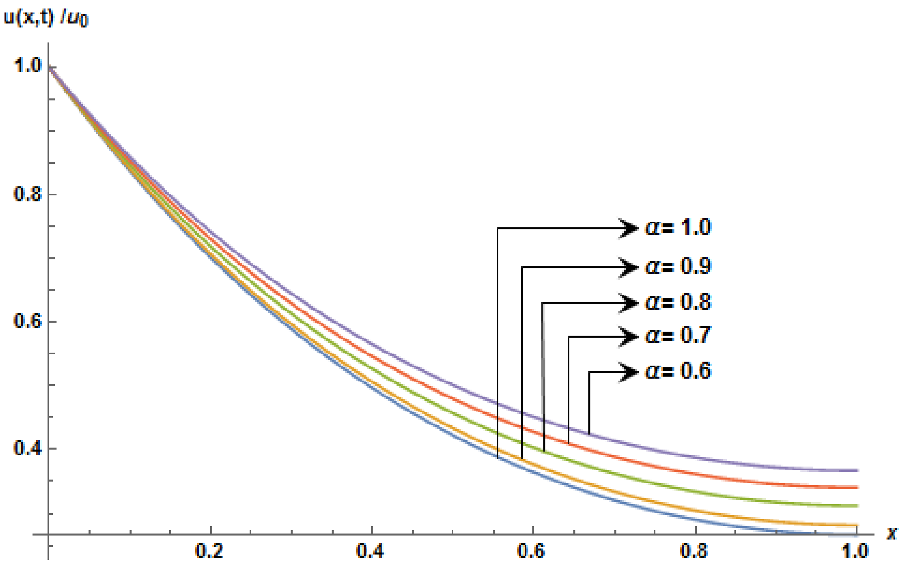

Figure 5.3: Plots of normalised concentration factor vs. column length with first type source boundary condition for space-time fractional-order BHE when $\alpha = 0.6, 0.7, 0.8, 0.9$ and 1 at fixed $\beta = 2$.

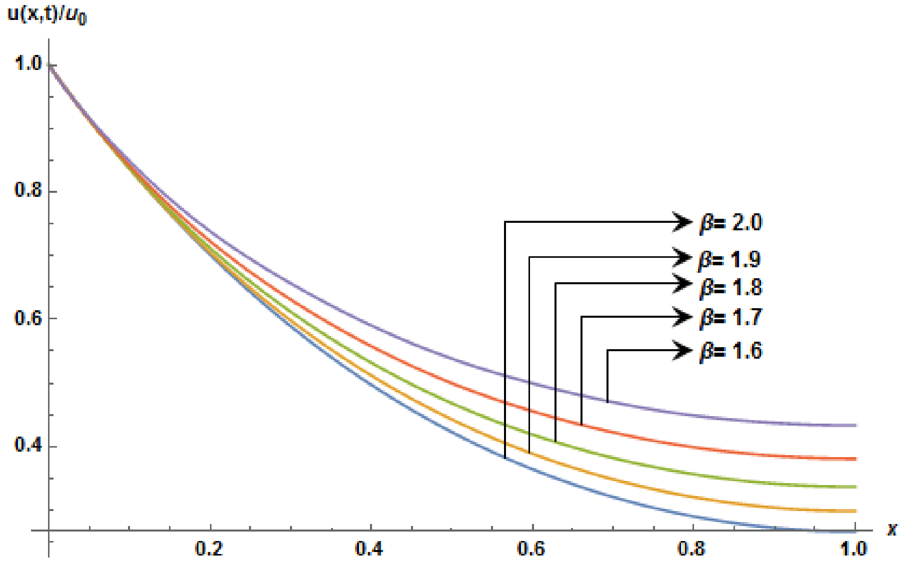

Figure 5.4: Plots of normalised concentration factor vs. column length with first type source boundary condition for space-time fractional-order BHE when $\beta = 1.6, 1.7, 1.8, 1.9$ and 2 at fixed $\alpha = 1$.

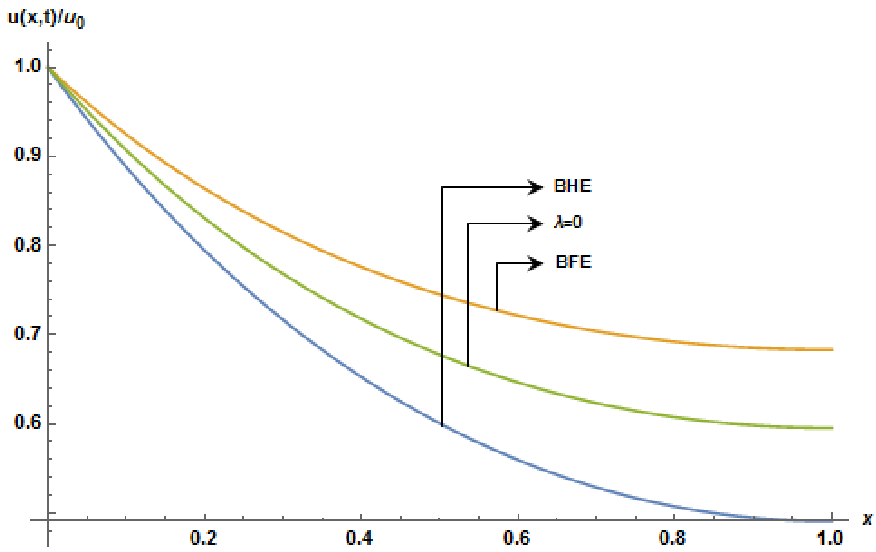

Figure 5.5: Plots of normalised concentration factor $u(x, 1)$ vs. column length with first type source boundary condition for BFE, BHE and $\lambda = 0$ when $\alpha = 1$, $\beta = 2$.

5.6 Conclusion

In this chapter, a drive has been taken to solve particular types of nonlinear space-time FRADE by using shifted Legendre collocation method with the aid of operational matrix. The method easily reduces the corresponding nonlinear FPDE to a system of nonlinear algebraic equations which is easily solvable. The method provides a highly accurate solution as the order of convergence increases by increasing the degree of approximation of shifted Legendre polynomials. Particular cases of the space-time fractional-order BFE and BHE with initial and boundary conditions have been solved. The microscopic behaviour of mass transportation in porous media equation is shown through applications in space-time fractional BFE and BHE. For this purpose the variations of normalized solute concentrations are presented graphically. The main contribution of the present research work is the pictorial presentations of the possibility of covering more soil length by the solute concentration with the decrease in spatial derivative for space fractional-order BFE and BHE due to the reason that the diffusion term approaches towards advection term. The variations of solution profiles for time fractional-order BFE and BHE are also discussed. Another important contribution of the chapter is the graphical showcasing of the effects of reaction terms on the solution profiles for the considered problems for both conservative and nonconservative cases. The author believes that the proposed numerical method will be useful for solving various types of nonlinear PDEs in fractional-order as well as integer-order systems.

****