Chapter 4

Study and analysis of one-dimensional nonlinear space-time fractional-order reaction-advection-diffusion equation

4.1 Introduction

Nonlinear fractional-order partial differential equations are generally hard to solve by the analytic method and thus various numerical methods are applied to get the approximate solutions of the equations. There are many researchers who have contributed to get reliable and efficient techniques for the solution of fractional-order differential equation using Finite difference method [133, 134], Sinc-Legendre collocation method [135], Chebyshev method [136], wavelet method [137], homotopy analysis method [122, 138, 139]

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In the present chapter, the study is mainly focused on the following spatial-time nonlinear fractional-order reaction-advection-diffusion equation given by

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + V(x,t)\,\frac{\partial u(x,t)}{\partial x} = D(x,t)\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} + \lambda\, R(u), \tag{4.1}$$

where $\partial^{\alpha} u(x,t)/\partial t^{\alpha}$ and $\partial^{\beta} u(x,t)/\partial x^{\beta}$ are fractional derivatives in Caputo sense, $R(u,x,t)$,$V(x,t)$ and $D(x,t)$ are known and $u(x,t)$ is the unknown.

Many researchers have studied the linear time fractional or space fractional diffusion equations. But to the best of author's knowledge, the above-mentioned space-time PDE in fractional-order system in nonlinear form has not yet been considered by any researcher. Due to important applications of this model, the study of the solutions of different forms of the above model for the standard order case $(\alpha = 1, \beta = 2)$ has been carried out during last few decades and still there is plenty of scope to develop better numerical methods to approximate the solutions specially for the above type of porous media equation in fractional-order system.

As already discussed that the numerical solutions using various tools are useful to deal with the nonlinear problems, Legendre collocation method (LCM) using operational matrix is very much reliable as Legendre polynomials are satisfying orthogonality condition and has received a considerable attention in dealing with various problems. A truncated orthogonal series is used in the method to solve differential equations. Saadatmandi and Dehghan [28] have generalized Legendre operational matrix to fractional calculus. The reason behind the approach for using the technique is that differential equation is reduced into a system of algebraic equations which just simplifies the problem.

The chapter aims to validate the proposed numerical method during the solution of the considered model with the existing analytical solution of Fisher equation towards confirmation of accuracy and efficiency of the method and then to apply the proposed method to solve the considered non-conservative nonlinear space-time FRADE.

In the present chapter, a drive has been taken to apply the Legendre polynomial approximation and an operational matrix of fractional-order derivatives to solve nonlinear FRADE. Applying the Legendre polynomial and operational matrix transform, the nonlinear PDE is reduced to a system of nonlinear algebraic equations. The equations thus obtained are solved using MATHEMATICA software (version 11.0). The main focus is concerned with the effect of advection term and also the effect of time and spatial order parameters on the solution profiles for different particular cases.

4.2 Solution of the problem

Let us consider the one-dimensional space-time fractional-order solute transport model in a bounded space domain $[0, 1]$ as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D\frac{\partial^\beta u(x,t)}{\partial x^\beta} - V\,\frac{\partial u(x,t)}{\partial x} + \lambda\, u(x,t)(1-u(x,t)),$$
$$0 < \alpha \leq 1, 1 < \beta \leq 2, \tag{4.2}$$

with initial and boundary conditions given by

$$u(x,0) = 0, \qquad 0 \leq x \leq 1, \tag{4.3}$$

$$u(0,t) = u_0(t) = t, \qquad \frac{\partial u(1,t)}{\partial x} = 0, \qquad t > 0, \tag{4.4}$$

where $u(x,t)$ is the concentration of species which have to trace at a constant velocity V of the groundwater which is assumed to be uniform. D is the diffusivity constant of the system. λ is the reaction rate coefficient. If $\lambda = 0$, the system is conservative otherwise non-conservative. The time and space derivatives α and β are to be considered in the Caputo sense.

In the case of $\lambda = 0$, equation (4.2) represents the space-time fractional-order advection-diffusion equation. When $V = 0$, the equation will be reduced to the space-time fractional-order reaction-diffusion equation which is obtained from the classical reaction-diffusion equation by replacing the first order time and second order space derivative with fractional derivatives $\alpha \in (0,1]$ and $\beta \in (1,2]$ respectively.

To solving the considered FRADE with initial and boundary conditions, shifted Legendre collocation method is used. Let us approximate $u(x,t)$ by the shifted Legendre polynomials for $(m+1)$ and $(n+1)$ terms for time and space respectively as

$$u_{n,m}(x,t) = \phi^T_{n,\tau}(t) A \phi_{m,l}(x), \tag{4.5}$$

where A is $(n+1)\times(m+1)$ unknown matrix. The derivatives can be approximated as

$$\frac{\partial^\alpha u_{n,m}(x,t)}{\partial t^\alpha} = \phi^T_{n,\tau}(t)(D^{(\alpha)})^T A \phi_{m,l}(x), \tag{4.6}$$

$$\frac{\partial^\beta u_{n,m}(x,t)}{\partial x^\beta} = \phi^T_{n,\tau}(t) A (D^{(\beta)}) \phi_{m,l}(x), \tag{4.7}$$

$$\frac{\partial u_{n,m}(x,t)}{\partial x} = \phi^T_{n,\tau}(t) A D^{(1)} \phi_{m,l}(x). \tag{4.8}$$

With the help of equations (4.5)-(4.8), equations (4.2)-(4.4) can be approximated as

$$\begin{aligned} &\phi^T_{n,\tau}(t)(D^{(\alpha)})^T A\phi_{m,l}(x) - D.\phi^T_{n,\tau}(t)AD^{(\beta)}\phi_{m,l}(x) + V.\phi^T_{n,\tau}(t)AD^{(1)}\phi_{m,l}(x) \\ &- \lambda.\phi^T_{n,\tau}(t)A\phi_{m,l}(x)\Big(1 - \phi^T_{n,\tau}(t)A\phi_{m,l}(x)\Big) + \phi^T_{n,\tau}(0)A\phi_{m,l}(x) = 0, \end{aligned} \tag{4.9}$$

$$\phi^T_{n,\tau}(t)A\phi_{m,l}(0) - t = 0, \tag{4.10}$$

$$\phi^T_{n,\tau}(t)AD^{(1)}\phi_{m,l}(1) = 0, \tag{4.11}$$

Equation (4.9) is collocated at (x_i, t_j) points and equations (4.10) and (4.11) are collocated at t_j, where x_i, $i = 1, 2, \cdots, (m-1)$ are the roots of shifted Legendre polynomial of $P^l_{m-1}(x)$ and t_j , $j = 1, 2, 3, \cdots, (m+1)$ are the roots of $P^l_{n+1}(t)$. The number of unknown coefficients of can be obtained from equations (4.9)-(4.11). Consequently $u_{n,m}(x,t)$ given by equation (4.5) can be calculated with simple computation.

4.3 Numerical results and discussion

In the present section, the numerical values of normalized solute concentration are calculated for different fractional spatial and time derivatives for both RDE and RADE for different particular cases which have been depicted through Figures 4.2-4.5. During numerical computation it is assumed that the arbitrary transport parameters of the medium are $D = 0.6\, in^2/h$, $t = 0.5\, h$ and $\lambda = -0.5$.

To validate our purposed method first, it is applied to solve the well-known integer order reaction-diffusion equation called Fisher equation ($\alpha = 1, \beta = 2$ and $V = 0$) under the following initial and boundary conditions:

$$u(x,0) = \frac{1}{4}\Big(1 - tanh(\frac{x}{2\sqrt{6}})\Big)^2, \qquad 0 \leq x \leq 1, \tag{4.12}$$

$$u(0,t) = \frac{1}{4}\Big(1 - tanh\Big(-\frac{5t}{12}\Big)\Big)^2, u(1,t) = \frac{1}{4}\Big(1 - tanh\Big(\frac{1}{2\sqrt{6}}\Big(1 - \frac{5t}{\sqrt{6}}\Big)\Big)\Big)^2, \quad t > 0. \tag{4.13}$$

The absolute error $(ER_j(x,t))$ for discrete points are given as $ER_j(x,t) = |u_{exact}(x,t) - u_{j,j}(x,t)|$ between our obtained result and exact solution [140] $u_{exact}(x,t) = \frac{1}{4}\Big(1 - tanh\Big(\frac{1}{2\sqrt{6}}\Big(x - \frac{5t}{\sqrt{6}}\Big)\Big)\Big)^2$ is calculated and the results are displayed through Figure 4.1 and Table 4.1 for $t = 1$, which show that the absolute error decreases with the increase in values of m. It is seen that at $m = 7$, our numerical results are effective and reliable. This has motivated the author to use their proposed method to solve the considered model equation (4.2) for $m = 7$.

In Figure 4.2, the movements of the normalized solution concentration with the column length are shown due to variations of $\beta = 1.6, 1.7, 1.8, 1.9$ and 2 for the space fractional-order reaction-advection-diffusion equation ($\alpha = 1$). It is seen from the figure that as the parameter β approaches from fractional-order to the standard order the normalized concentration decreases. For the case of standard order system ($\beta = 2$), the solute covers less length in the soil column compared to fractional-order system, which is physically justified.

Figure 4.3 shows the movements of solute concentration in the column length for reaction-diffusion equation when $V = 0$ for various $\beta = 1.6, 1.7, 1.8, 1.9$

when $\alpha = 1$. The nature of the graph is similar to previous one, but the movement of the solute concentration is less in each case due to the absence of advection term.

The movements of solute concentration with the column length for $\beta = 2$ and fractional time derivative as $\alpha = 0.6, 0.7, 0.8, 0.9$ and 1 for reaction-advection-diffusion equation are depicted through Figure 4.4. It is seen from the figure that as order of time derivative goes from integer-order to fractional-order, the solute covers more length in the soil column, which justifies the fact already stated by [141] that it is due to the power decay of $u(x, t)$ with the fractional-order time derivative α in contrast to the stretched exponential decay characteristic observed in fractional Brownian motion.

Figure 4.5 shows the movements of solute concentration with the column length for reaction-diffusion equation ($V = 0$) for $\beta = 2$ and $\alpha = 0.6, 0.7, 0.8, 0.9$ and 1. The nature of the graphs are similar to the previous one having only difference that the movement of the solute concentration is less due to the absence of advection.

Table 4.1 Maximum absolute error for Fisher equation for $u(x, 1)$ and $m = 3, 5$ and 7.

x	$u_{exact}(x,1)$	$ER_3(x,1)$	$ER_5(x,1)$	$ER_7(x,1)$
0	0.485852	9.00379×10^{-6}	1.23524×10^{-8}	6.63112×10^{-10}
0.1	0.473853	7.09506×10^{-6}	5.79846×10^{-7}	3.62284×10^{-8}
0.2	0.461782	2.27573×10^{-5}	1.08785×10^{-6}	6.85356×10^{-8}
0.3	0.449692	3.60265×10^{-5}	1.48490×10^{-6}	9.43549×10^{-8}
0.4	0.437597	4.52009×10^{-5}	1.73919×10^{-6}	1.11105×10^{-7}
0.5	0.425509	4.87837×10^{-5}	1.83084×10^{-6}	1.17073×10^{-7}
0.6	0.413442	4.54304×10^{-5}	1.74681×10^{-6}	1.11583×10^{-7}
0.7	0.401410	3.38946×10^{-5}	1.47815×10^{-6}	9.50567×10^{-8}
0.8	0.389427	1.29709×10^{-5}	1.01924×10^{-6}	6.89142×10^{-8}
0.9	0.377506	1.85629×10^{-5}	3.68376×10^{-7}	3.53387×10^{-8}
1	0.365661	6.20049×10^{-5}	4.70447×10^{-7}	3.07156×10^{-9}

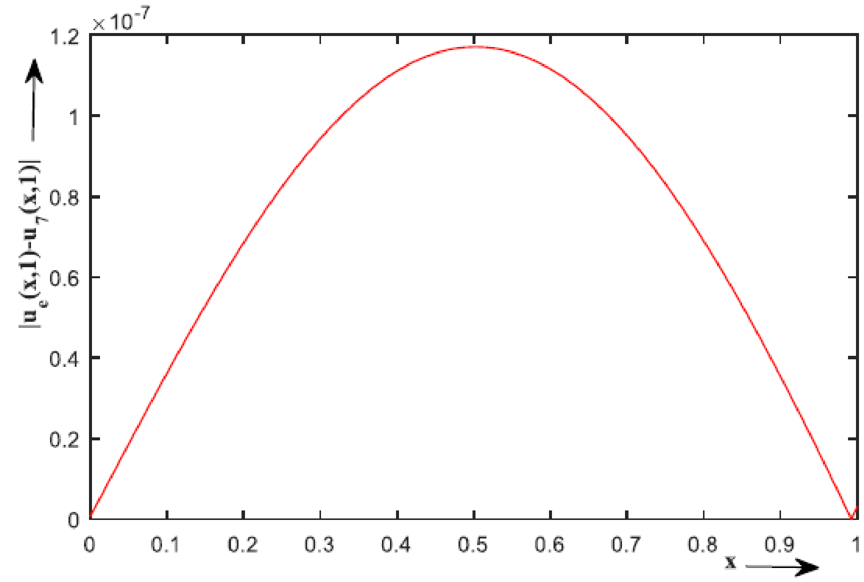

Figure 4.1: Plot of the error function $|u_{exact}(x,1) - u_{numerical}(x,1)|$ vs. x for Fisher equation.

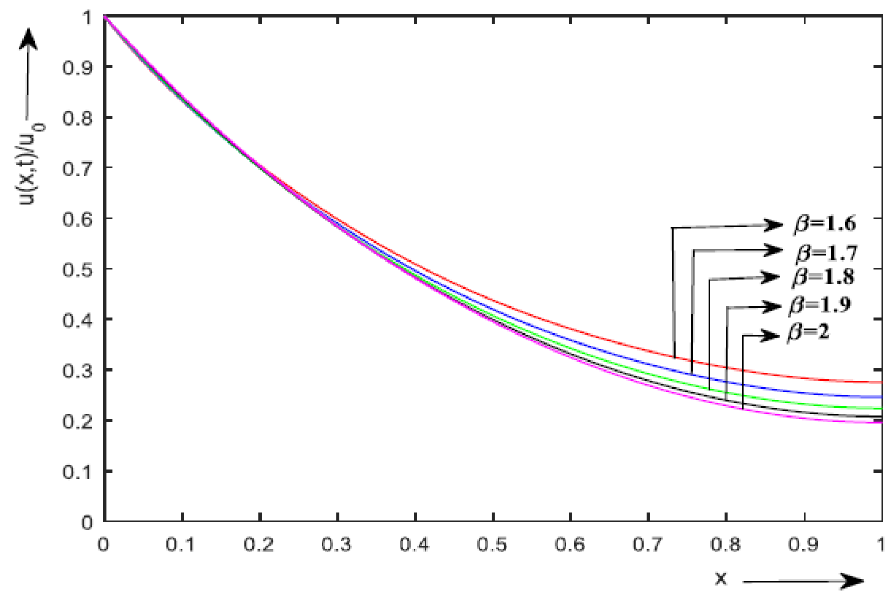

Figure 4.2: Plots of normalised concentration factor vs. column length with first type source boundary condition when $V = 0.6$ in/h and $\beta = 1.6, 1.7, 1.8, 1.9$ and 2 at fixed $\alpha = 1$.

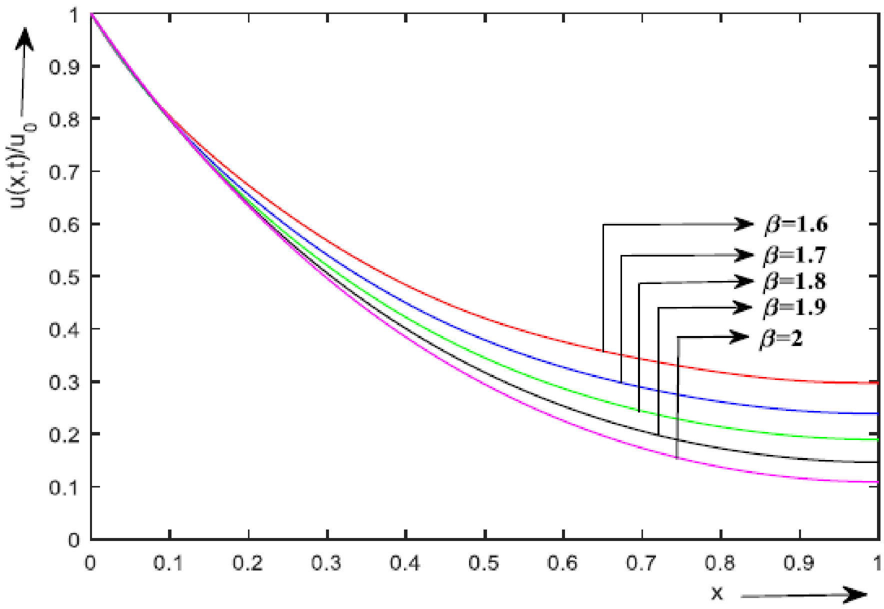

Figure 4.3: Plots of normalised concentration factor vs. column length with first type source boundary condition when $V = 0$ in/h and $\beta = 1.6, 1.7, 1.8, 1.9$ and 2 at fixed $\alpha = 1$.

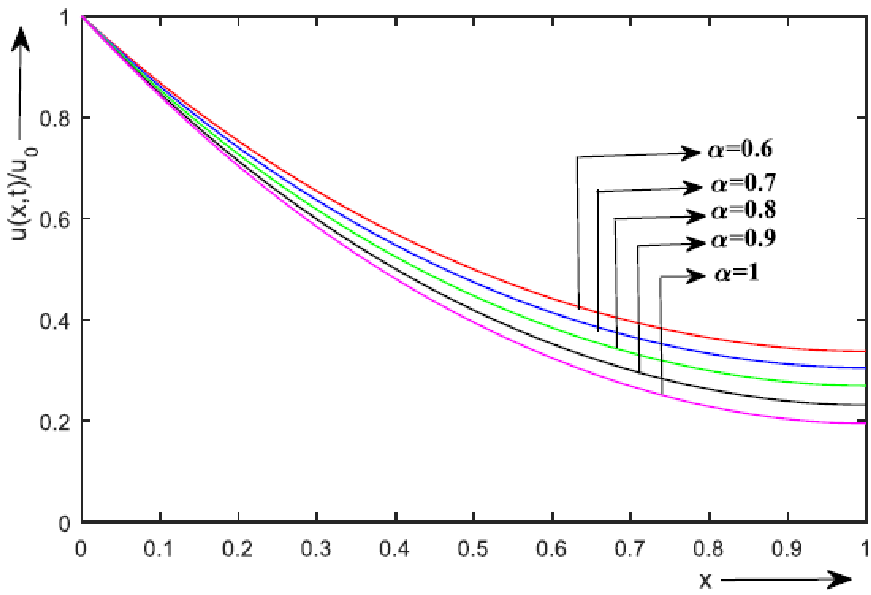

Figure 4.4: Plots of normalised concentration factor vs. column length with first type source boundary condition when $V = 0.6$ in/h and $\alpha = 0.6, 0.7, 0.8, 0.9$ and 1 at fixed $\beta = 2$.

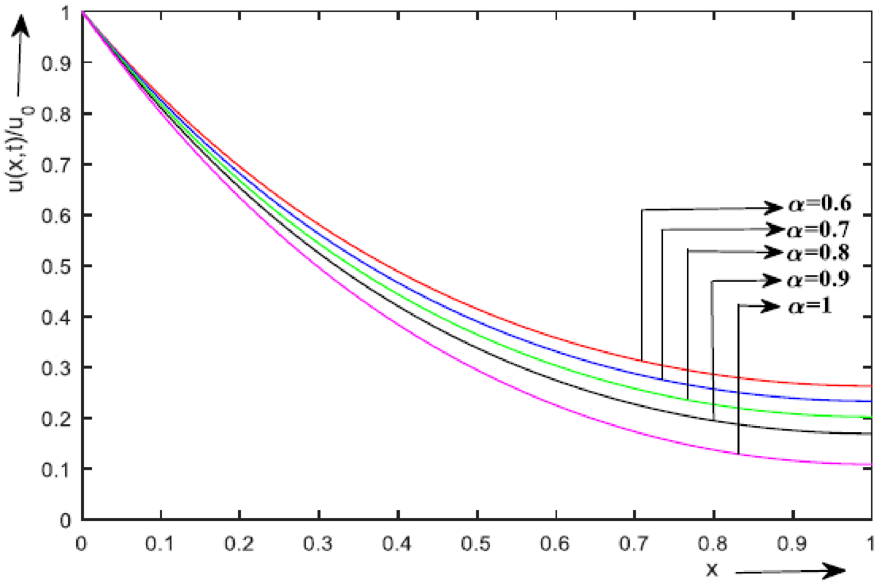

Figure 4.5: Plots of normalised concentration factor vs. column length with first type source boundary condition when $V = 0$ in/h and $\alpha = 0.6, 0.7, 0.8, 0.9$ and 1 at fixed $\beta = 2$.

4.4 Conclusion

In the present scientific contribution, a numerical method called the shifted Legendre collocation method using operational matrix for derivatives is used to solve the nonlinear space-time fractional-order reaction-advection-diffusion equation. The efficiency and effectiveness of the method are validated comparing the results obtained by using the present method with an existing analytical result through error analysis. The effect of advection term on the solution profile for variations of space and time fractional-orders are graphically presented for different particular cases. Another point of the study is the explanation of the decay of the solution profile when the fractional time derivative of the system approaches to fractional-order from the standard order. The most important point of the study is that as the spatial order derivative decreases the diffusive term approaches to advection term, and as a result the concentration of solute will cover more soil column length, which is clearly observed from the pictorial representations of our obtained results.

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