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Solution of time-fractional Cahn–Hilliard equation with reaction term using homotopy analysis method

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Abstract

In this article, the approximate analytical solution of the time-fractional Cahn–Hilliard equation with quadratic form of the source/sink term is obtained using the powerful homotopy analysis method, which permits us to select a convergence control parameter that minimizes residual errors. The concerned method is more general in theory and widely valid in practice to solve nonlinear problems even for fractional order systems as it provides a convenient way to guarantee the convergence of the approximate series. The results have been given to show the effect of the reaction term on the solution profile in both fractional and standard order cases for different particular cases. The main feature of this study is the authentication that only a few iterations are required to obtain the accurate approximate solution of the present mathematical model. This is justified through error analysis for both fractional and standard order cases. This striking feature of savings in time is exhibited through graphical presentations of the numerical values when the system passes from standard order to fractional order in the presence or absence of the reaction term.

Keywords

Cahn–Hilliard equation, reaction term, fractional order derivative, homotopy analysis method, convergence analysis

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Introduction

Interactions between convection and diffusion and also between diffusion and reaction cause many physical phenomena. From a physical perspective, the convection–diffusion and the reaction–diffusion processes describe a wide variety of problems arising in many branches of science and engineering. During the modelling of such processes, the nonlinear partial differential equations (PDEs) obtained provide new ideas regarding interactions of nonlinearity and diffusion. The importance of the analytical and numerical solutions of such nonlinear diffusion problems with reaction in mathematical physics can be found in soliton theory.

The theory of fractional calculus is an old mathematical subject with a history as long as integer order calculus. Fractional differential equation has recently proved to be an important tool for the modelling of many phenomena. Comparing integer and fractional orders,

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fractional order gives us a wonderful instrument for the description of memory and hereditary properties of various materials and processes. There are many methods, namely, Adomian decomposition method,[1] modified decomposition method,[2,3] homotopy perturbation method,[4,5] variational iteration method,[6–8] differential transformation method,[9,10] collocation methods[11–13] and Galerkin finite element method[14] for the analytical or numerical solution of fractional differential equations. One of the strongest analytical methods for solving nonlinear problems is the homotopy analysis method (HAM), which was first introduced by the Chinese mathematician S. J. Liao using the basic ideas of homotopy in topology. Later, he improved the method greatly in stages and finally it has become an important and efficient tool for solving nonlinear problems. The advantage of the concerned method over the other existing analytical methods is that it provides us great flexibility to choose the auxiliary operator and initial guess. The main advantage of the method is that the convergence control parameter used provides a convenient way to guarantee the convergence of the approximate series solution.

In 1958, the classical Cahn–Hilliard equation (C-H equation) introduced by American scientists JW Cahn and J Hilliard[15] is one of the most studied models of mathematical physics. The equation is related to a number of physical phenomena like the spinodal decomposition, phase separation and phase ordering dynamics. This equation of mathematical physics describes the process of phase separation by which the two components of a binary fluid are spontaneously separated. The essential property of the equation is that the interface between two phases is not sharp but has a finite thickness in which the composition changes gradually. Thus, it is said that the equation describes the temporal evolution of conserved fields. A generalization of the mathematical model capable of describing a phase separation in the C-H theory can be found in Berti and Bochicchio[16] where this is achieved without loss of generality by taking the potential function as $V(u) = (1/4)(1 - u^2)^2$ and combining the function V with the boundary penalty to form the energy functional of a particular configuration as

$$F(u(x,t)) = \frac{(1-u^2)^2}{4} - \frac{\gamma}{2}|\nabla u|^2 \tag{1}$$

where γ is a constant that penalizes phase boundaries.

Fick's first law states that the flux of particles in a system is proportional to the gradient of the chemical potential and thus

$$J = -D\nabla \frac{\partial F}{\partial u} \tag{2}$$

where D is the diffusion coefficient.

The energy changes when particles change position, that is, the chemical potential of the system is given by

$$\frac{\partial F}{\partial u} = u^3 - u - \gamma \frac{\partial^2 u}{\partial x^2} \tag{3}$$

Now, the flux must obey a continuity equation

$$\frac{\partial u}{\partial t} = D\nabla^2 \frac{\partial F}{\partial u} \tag{4}$$

Finally, we get

$$\frac{\partial u}{\partial t} = D\frac{\partial^2}{\partial x^2}\left(u^3 - u - \gamma \frac{\partial^2 u}{\partial x^2}\right) \tag{5}$$

The mathematical model gave a near accurate description of system dynamics during initial time range and provided an acceptable physical interpretation of system behaviour in intermediate time duration. The physical behaviour of the system in long time duration is studied because of its slow nature of evolution. Since the phenomena of downhill and uphill diffusion have their mathematical roots in C-H equation, it needs to be considered while development of models for binary mixtures. The equation stems from diverse phenomena like phase transition and moving process of river basin and finds applications in a variety of fields ranging from soft matter to complex fields. Recently, a few researchers have been involved in coupling the phase separation part of the equation to the Navier–Stokes equation of fluid flow. If $u(x,t)$ is the concentration of the fluid with $u(x,t) = \pm 1$ in an indicated domain, then the equation is written as

$$\frac{\partial u}{\partial t} = D\nabla^2(u^3 - u - \gamma\nabla^2 u) + \beta\frac{\partial u}{\partial x} + ku(1-u) \tag{6}$$

where the second term of the right-hand part is known as advection term and the last term is the reaction term. Modelling of nonlinear systems has gained tremendous popularity among scientists and engineers during the last few years as the nonlinearity phenomena are exhibited by most of the systems in nature. Moreover, if the models are of fractional order nonlinear problems, they are in a different dimension due to their stochastic nature. After the advent of powerful computers, various computational techniques and approximate methods or numerical methods employing effective software are used during the investigation of such types of problems. Generally, fractional order diffusion equations are obtained from the classical diffusion equations by replacing the first-order time derivative by a fractional order α satisfying $0<\alpha<1$, taking into the account the fact that these are of non-Markovian nature, generate the fractional Brownian motion and also have memory effect. In overcoming lot of difficulties while confronting the nonlinear problems in fractional order systems, researchers from various parts of the world have been actively engaged to provide an excellent description of the memory and hereditary properties of the systems. In the case of fractional order equations, the analysis has some unique features. The nonlinear equation is considered first and then the theory is developed for nonlinear fractional order problem.[8,14,17–26] The advantage of treating the nonlinear equation with fractional order is that there is a possibility that the analysis will have some unique features which will provide useful additional insight.

Taking into account of the above facts, here we want to introduce and study the nonlocal fractional order C-H equations with advection and reaction terms, which is described as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D\frac{\partial^2}{\partial x^2}\left(u^3 - u - \gamma\frac{\partial^2}{\partial x^2}u\right) + \beta\frac{\partial u}{\partial x} + k\,u\,(1-u) \tag{7}$$

In the next phase, an endeavour has been taken to solve this model with initial condition $u(x,0) = x$ using HAM. To the best of the authors' knowledge, the fractional order C-H equation has not yet been considered by any researcher. In this article, the authors have made a sincere attempt to find the approximate analytical solution of the equation for different particular cases which have been depicted through figures.

Basics of fractional calculus

The definitions and properties related to fractional calculus given by B Riemann and J Liouville and also by M Caputo are as follows.[27–31]

Definition 1. The Riemann–Liouville fractional integral operator of order $q>0$ of a function $f(x)$ is[27,32]

$$J_x^q f(x) = \frac{1}{\Gamma(q)}\int_0^x (x-\xi)^{q-1} f(\xi)\,d\xi,\quad q>0,\quad x>0$$

$$J_x^0 f(x) = f(x)$$

Definition 2. The Riemann–Liouville fractional derivative operator of order $q>0$ of a function $f(x)$ is defined by Oldham and Spanier[33]

$$D_x^q f(x) = \frac{d^n}{dx^n}J_x^{n-q}f(x),\quad n-1<q\le n,\quad n\in \mathrm{N}$$

where J_x^q for $f\in C_\mu,\ \mu\ge -1,\ \gamma\ge -1$ satisfies the following properties

1. $J_x^p J_x^q f(x) = J_x^{p+q} f(x) = J_x^q J_x^p f(x)$
2. $J_x^p x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(p+\gamma+1)}x^{p+\gamma}$

Definition 3. The Caputo order fractional derivative of a function $f(x)$ is[27,32]

$$D_x^q f(x) = \frac{1}{\Gamma(n-q)}\int_0^x (x-\xi)^{n-q-1} f^n(\xi)\,d\xi,\quad n-1<q<n,\quad n\in \mathrm{N}$$

$$D_x^q f(x) = \frac{d^n f(x)}{dx^n},\quad q=n$$

where $D_x^q f(x)$ satisfies the following basic property

$$\left(J_x^q D_x^q\right) f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0+)\frac{x^k}{k!},$$

$$x\ge 0,\ n-1<q<n,\ n\in\mathrm{N}\ \text{and}\ f\in C_\mu^n,\ \mu\ge -1$$

Solution of the problem by homotopy analysis method

To solve the C-H equation by HAM,[34–40] we choose the auxiliary linear operator $L[\varphi(x,t;q)] = \varphi(x,t;q)$, where $\varphi(x,t;q)$ is an unknown function and the nonlinear operator is defined as

$$\begin{aligned} N[\varphi(x,t;q)] &= -\frac{\partial^\alpha}{\partial t^\alpha}\varphi(x,t;q) - \frac{\partial^4}{\partial x^4}\varphi(x,t;q) \\ &+ 6\,\varphi(x,t;q)\left(\frac{\partial\varphi(x,t;q)}{\partial x}\right)^2 + 3\,(\varphi(x,t;q))^2\frac{\partial^2}{\partial x^2}\varphi(x,t;q) \\ &- \frac{\partial^2}{\partial x^2}\varphi(x,t;q) + \beta\frac{\partial\varphi(x,t;q)}{\partial x} + k\left(\varphi(x,t;q) - (\varphi(x,t;q))^2\right) \end{aligned} \tag{8}$$

Let us construct the zeroth order deformation equation as

$$(1-q)L\left[\varphi(x,t;q) - u_0(x,t)\right] = q\hbar H(x,t)N[\varphi(x,t;q)] \tag{9}$$

where $q\in[0,1]$ denotes the embedding parameter, L is the auxiliary linear operator, $H(x,t)\neq 0$ is the auxiliary function, $\hbar\neq 0$ is the auxiliary parameter, and $u_0(x,t)$ represents the initial approximation of field variable. If we substitute $q=0$ in equation (9), then we simply obtain $\varphi(x,t;0) = u_0(x,t)$ and for $q=1$ we easily get $\varphi(x,t;1) = u(x,t)$.

Now expanding the function $\varphi(x,t;q)$ in Taylor series form with respect to the parameter q, we get

$$\varphi[x,t;q] = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t)q^n \quad (10)$$

where

$$u_n(x,t) = \frac{1}{n!}\left[\frac{\partial^n \varphi(x,t;q)}{\partial q^n}\right]_{q=0} \quad (11)$$

If the auxiliary linear operator, the initial guess and the convergence control parameter are properly chosen, the above series converges for $q = 1$ as

$$\varphi[(x,t;1)] = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) \quad (12)$$

which must be one of the solutions of the original equation as proved in Liao.[38]

Differentiating equation (9) n times with respect to the embedding parameter q and then setting $q = 0$, and dividing by $n!$, we get the nth order deformation equation as

$$L[u_n(x,t) - \chi_n u_{n-1}(x,t)] = h\,H(x,t)\,R_n(\overline{u_{n-1}}(x,t)) \quad (13)$$

with initial condition $u_n(x,0) = 0$
where

$$\chi_n = 0,\ \ n \le 1\ ;\ 1\ ,\ \ n>1 \quad (14)$$

Here, $\hbar$ is a non-zero auxiliary linear operator, $\overline{u_{n-1}}(x,t) = \{u_0(x,t), u_1(x,t),\ \ldots,\ u_{n-1}(x,t)\}$ and

$$\begin{aligned} R_n(\overline{u_{n-1}}(x,t)) &= \frac{1}{(n-1)!}\left\{\frac{\partial^{n-1}}{\partial q^{n-1}} N[\varphi(x,t;q)]\right\} \\ &= -\frac{\partial^\alpha u_{n-1}}{\partial t^\alpha} - \frac{\partial u_{n-1}}{\partial x^4} \\ &\quad + 6\sum_{i=0}^{n-1}\left(\sum_{j=0}^{i}\frac{\partial u_j}{\partial x}\frac{\partial u_{i-j}}{\partial x}\right)u_{n-1-i} - \frac{\partial^2}{\partial x^2}u_{n-1} \\ &\quad + \beta\frac{\partial u_{n-1}}{\partial x} + 3\sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} u_j u_{i-j}\right) \\ &\quad \frac{\partial^2}{\partial x^2}u_{n-1-i} + k\left(u_{n-1} - \sum_{i=0}^{n-1} u_i u_{n-1-i}\right) \end{aligned} \quad (15)$$

Taking $u_0(x,0) = x$, we get

$$u_1(x,t) = \frac{\hbar t^\alpha}{\Gamma(\alpha+1)}\left((6x+\beta) + k(x-x^2)\right)$$

$$\begin{aligned} u_2(x,t) = &\frac{-(1+\hbar)\hbar t^\alpha}{\Gamma(\alpha+1)}\left(6x+\beta+k(x-x^2)\right) + \frac{\hbar^2 t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\left(-2k^2x^3 + (k^2-24k)x^2 + (k^2+12k+108)x \right.\\ &\left. + (12\beta + 3k\beta + 2k)\right) \end{aligned}$$

$$\begin{aligned} u_3(x,t) = &\frac{\hbar(1+\hbar)^2 t^\alpha}{\Gamma(\alpha+1)}((6x+\beta)+k(x-x^2)) \\ &+ \frac{\hbar(1+\hbar)^2 t^{2\alpha}}{\Gamma(2\alpha+1)}(2k^2x^3 + x^2(8k^2+60k) \\ &+ x(-4k^2 - 48k - 216 + 2\beta) \\ &- (7k\beta - 24\beta)) + \frac{\hbar^3 t^{3\alpha}}{\Gamma(3\alpha+1)} \\ &(-4k^3x^4 - x^3(12k^3 + 132k^2) \\ &+ x^2(-3k^3 - (150+6\beta)k^2 - 504k) \\ &+ x(k^3 - k^2(14+4\beta) + k(180-24\beta) + 1844) \\ &+ k^2(-38+4\beta) + k(-180+42\beta) + 72\beta) \\ &+ \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2} \\ &(12((6x+\beta)+k(x-x^2))(x+k(1-2x)) \\ &+ 6x(6+k(1-2x))^2 \\ &+ 12kx((6x+\beta)+k(x-x^2)) \\ &+ k((6x+\beta)+k(x-x^2))^2) \end{aligned}$$

Proceeding in the similar manner, we can find $u(x,t)$, $n>3$. Finally, the approximate solution of $u(x,t)$ is obtained as

$$u(x,t) = \lim_{N\to\infty} \varphi_N(x,t) \quad (16)$$

where $\varphi_N(x,t) = \sum_{m=0}^{N-1} u_m(x,t)$.

Choosing the values of the auxiliary parameter $\hbar$ introduced in the zero-order deformation equation, the region of convergence of the series as well as the rate of this convergence can be influenced and thus the parameter $\hbar$ is called the convergence control parameter. To obtain the appropriate values of $\hbar$ using the $\hbar-$curve, the optimization method is applied by minimizing the squared residual of the governing equation defined by

$$E_n = \int_\omega \left(N\sum_{j=0}^{n} u_j(x,t)\right)^2 dt \quad (17)$$

To evaluate the effective region of $\hbar$, we use

$$\zeta_\hbar = \left\{\hbar : \lim_{n\to\infty} \mathrm{E_n}(\hbar) = 0\right\} \quad (18)$$

The convergence of series (16) can be obtained using those values of $\hbar$ which are distinct from the optimal value but belonging to the effective region. Thus,

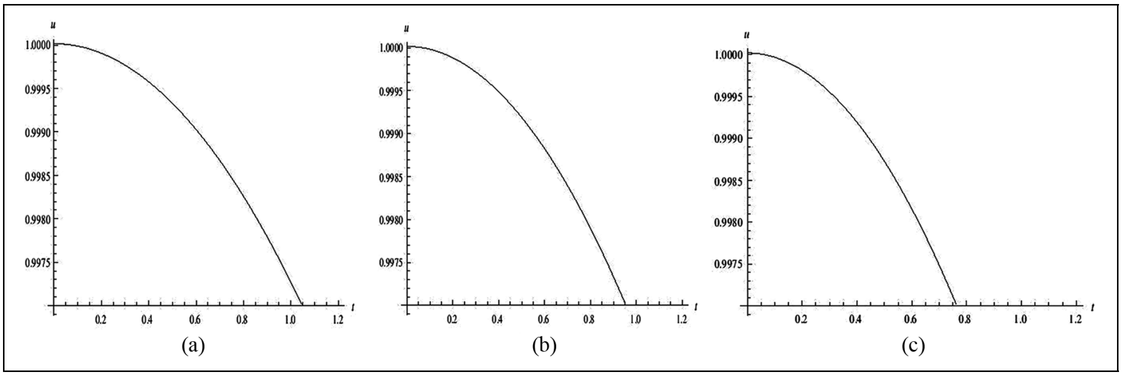

Figure 1. Plots of $u(x, t)$ versus t at $x = 1$ for $\beta = -1$, $\alpha = 0.8$: (*a*) $k = -1$, (*b*) $k = 0$ and (*c*) $k = 1$.

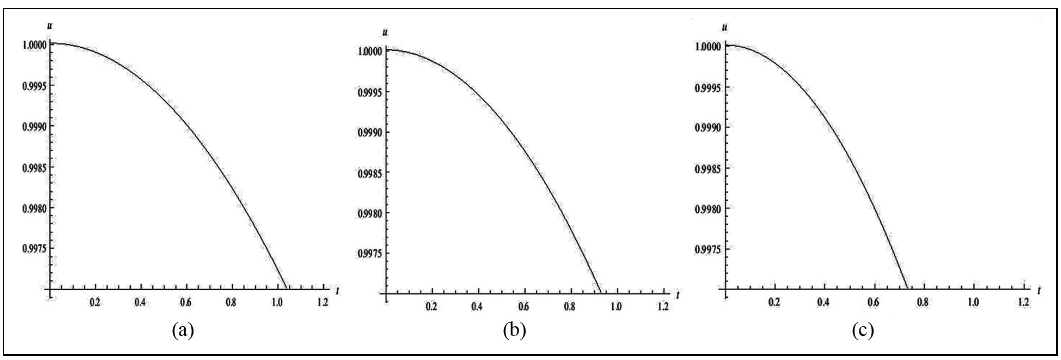

Figure 2. Plots of $u(x, t)$ versus t at $x = 1$ for $\beta = -1$, $\alpha = 0.9$: (*a*) $k = -1$, (*b*) $k = 0$ and (*c*) $k = 1$.

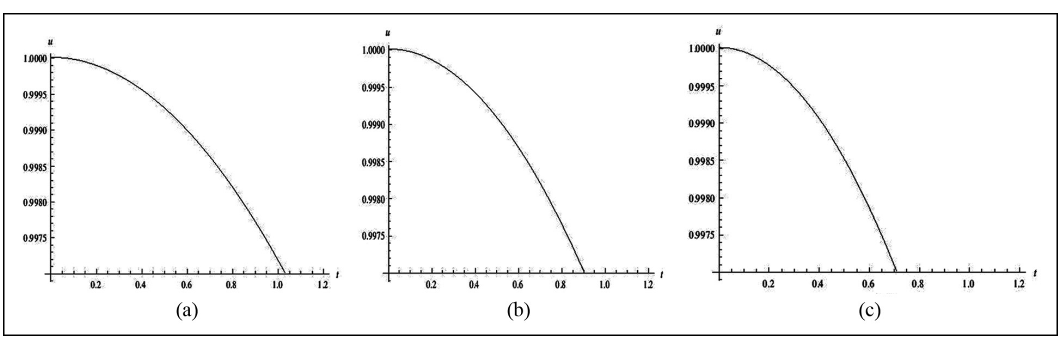

Figure 3. Plots of $u(x, t)$ versus t at $x = 1$ for $\beta = -1$, $\alpha = 1$: (*a*) $k = -1$, (*b*) $k = 0$ and (*c*) $k = 1$.

$\hbar-$curve method enables to determine the effective region of the convergence control parameter. It is to be noted that the speed of the convergence of the series is slow and never gives any guarantee to obtain a $\hbar$ which ensures the fastest convergence.

Results and discussion

In this section, numerical values of the field variable $u(x, t)$ for different values of the fractional and also standard motions ($\alpha = 0.8,\ 0.9,\ 1.0$) are calculated in the presence or absence of the reaction term ($k \neq 0$ or $k = 0$) at $x = 1$ for different particular cases which are depicted through Figures 1–3. It is observed from the figures that it takes more time to stabilize the field variable $u(x, t)$ due to the presence of the sink term ($k = -1$) as compared to the absence of the reaction term ($k = 0$). It is noticed that due to presence of source term ($k = 1$), it takes comparatively less time as compared to previous cases. It should also be mentioned that the presence of the advection term has a big role in the system. The presence of the sink term

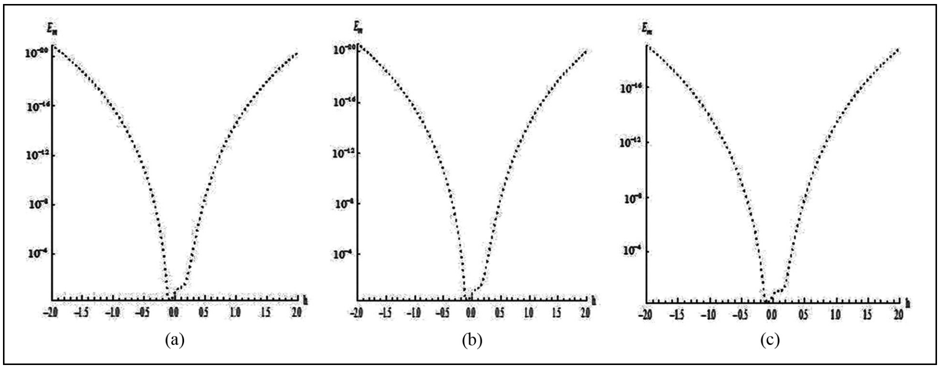

Figure 4. Plots of E_m versus $\hbar$ for $\alpha = 0.9$ in the presence of reaction term k : (a) $k = -1$, (b) $k = 0$ and (c) $k = 1$.

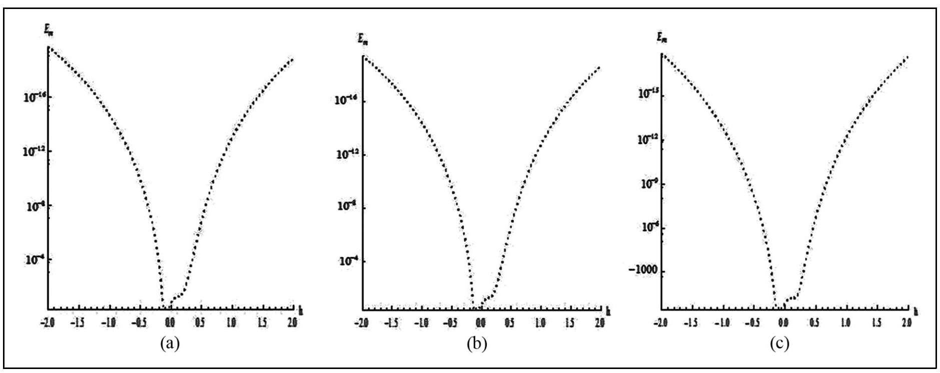

Figure 5. Plots of E_m versus $\hbar$ for $\alpha = 1$ in the presence of reaction term k : (a) $k = -1$, (b) $k = 0$ and (c) $k = 1$.

in the system is required to enhance the stability margin. For the validity of the HAM used to solve the concerned model, the numerical calculations of residual error (E_m) for different values of the convergence control parameter $\hbar$ are presented through Figures 4 and 5 for $\alpha = 0.9$ and $\alpha = 1.0$. It is seen that for $\alpha = 0.9$ and $\alpha = 1$, the magnitudes of the errors are minimum at $\hbar = -0.0715694$, -0.0810788, -0.10886799 and at $\hbar = -0.0777697$, -0.0874554, -0.1182229, respectively, for $k = -1, 0, 1$ when four terms of the series solution are taken. It is seen from the figures that the sink term minimizes the magnitude of the error more in the case of the fractional order cases compared to the standard order case.

Conclusion

Three goals have been achieved through this study. The first one is the demonstration of damping of the field variable $u(x, t)$ through the use of the reaction term in the presence of the advection term in the C-H equation for fractional order as well as standard order using the powerful and convenient HAM. The second one is the graphical presentation of acceleration of convergence control parameter through error analysis, which clearly depicts the potential of the concerned method even in the presence of the reaction term for both the cases. The third goal is the graphical exhibition that less time is required to stabilize the probability density function when the system approaches from standard order to fractional order in the presence of sink term.

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