Chapter 1

Introduction

In the beginning, I would like to mention that the thesis is divided into three parts. In the first part (Chapter 2), some singular and non-singular linear and non-linear problems have been solved. In the second part (Chapter 3 and Chapter 4), we construct the approximate solutions of the hyperbolic partial differential equations. Finally in the third part (Chapter 5), the approximate solution of fractional order differential equation has been deduced.

In the introductory chapter of the thesis, firstly, basic definitions of some well known polynomials such as Legendre polynomials, Lagrange polynomials, and Euler polynomials are introduced. After that, the review of operational matrices, ODEs and PDEs is provided. Finally, we have discussed about the fractional derivative. The brief introduction to the aforementioned polynomials is as follows.

1.1 Legendre polynomials

Legendre polynomials were discovered in 1782 by Adrien-Marie Legendre, which form a system of complete and orthogonal polynomials in the domain $[-1, 1]$, with a vast number of mathematical properties, and numerous applications with the orthogonality property as

$$\int_{-1}^{1} P_n(x)P_(x) = \frac{2}{2n+1}\delta_{mn},$$

where, δ_{mn} is the Kronecker delta. We can define the Legendre polynomials using different aspects. Let us define Legendre polynomials as follows.

Definition 1.1.1. The Legendre polynomials can be defined as the coefficients in a formal expansion in powers of t of the generating function

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

The coefficient of each t^n is a polynomial of degree n.

Definition 1.1.2. The Legendre polynomial is the series solution of Legendre's differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right] + n(n+1)P_n(x) = 0.$$

Some of the Legendre polynomials are

$P_0(x) = 1,$

$P_1(x) = x,$

$P_2(x) = \frac{1}{2}(3x^2 - 1),$

$P_3(x) = \frac{1}{2}(5x^3 - 3x),$

$P_4(x) = \frac{1}{2}(35x^4 - 30x^2 + 3).$

1.2 Lagrange polynomials

Definition 1.2.1. Let $x_0, x_1, ..., x_N$ be the node points such that no two $x_i's$ are same, then the Lagrange interpolating polynomial is defined as

$$l_j(x) = \prod_{\substack{i=0 \\ i\neq j}}^{N} \frac{(x-x_i)}{(x_j-x_i)} \ ,$$

which satisfies the following Kronecker delta property $l_i(x_j) = \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

1.3 Euler polynomials

Definition 1.3.1. The Euler polynomial $E_n(x)$ is given by the Appell sequence with

$$g(t) = \frac{1}{2}(e^t + 1),$$

giving the generating function as

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{(n!)}.$$

The first few Euler polynomials are

$E_0(x) = 1,$

$E_1(x) = x - \frac{1}{2},$

$E_2(x) = x^2 - x,$

$E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4},$

$E_4(x) = x^4 - 2x^3 + x,$

$E_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2}.$

After the brief discussion about various polynomials, a brief introduction about ordinary differential equations (ODEs) and partial differential equations (PDEs) is given below.

1.4 Review of ordinary differential equations and partial differential equations

1.4.1 Ordinary differential equations

The mathematical modeling of real word problems often leads to ordinary differential equations (ODEs). ODEs play important role in the area of applied sciences. Various phenomena like mechanics of particle (physics), population dynamics (ecology), stellar structure (astronomy), birth-death model (biology), etc. are described by ODEs. Further, the ODEs are classified into two categories: initial value problems (IVPs) and boundary values problems (BVPs), depending upon the conditions specified at the points of the domain. Several methods have been developed to solve IVPs analytically. But, it is not possible to solve all IVPs with these methods, especially nonlinear and IVPs with singularities at some points of the domain. So, one needs to develop numerical methods to solve the IVPs. The simplest and oldest methods to solve IVPs are Euler's method, Improved Euler's method and Runge Kutta methods [3]. Moreover, many researchers have established schemes to solve IVPs numerically [4–6]. Recently, the operational matrix approaches have been developed by various researchers [7–10]. In chapter 2, two schemes based on operational matrices of Lagrange polynomials have been developed to solve IVPs numerically.

1.4.2 Partial differential equations

Partial differential equations (PDEs) are fundamental to the modeling of natural phenomena since they arise in every field of science and engineering [11–13]. Most of the physical phenomena like wave propagation, heat transfer, electromagnetic field, flow of traffic [14–16], etc., are described by PDEs. Besides, the more complex phenomena in science and engineering field which depends upon more than one space variable are also described by the PDEs. Therefore, mathematicians always desire to understand the solutions of these equations. Consequently, the desire to understand the solutions of these equations has always a prominent place in the efforts of mathematicians. Like algebra, topology, and rational mechanics, PDEs are the core area of mathematics.

Due to the wide applications of PDEs in the area of science and engineering, a large number of mathematicians have been involved to find the solutions of the PDEs. Some well-known methods for solving PDEs analytically are characteristic method, separation of variable, Laplace transformation, homotopy perturbation method (HPM), variational iteration method (VIM), Laplace transform method (LTM) and Adomian decomposition method (ADM) [17–20]. But, there is wide range of PDEs, on which these analytical methods do not work. Keeping this into the mind, mathematicians gave attention to solve PDEs numerically. In this progress, one needs to develop some numerical techniques to solve PDEs. Several methods such as the finite difference methods, finite element methods [21–25] and spectral methods [26–28] have been applied by the various mathematicians to find the numerical solutions. Moreover, collocation methods and finite volume methods [29–32] are also applied to solve PDEs numerically.

1.5 Operational matrices

Orthogonal functions and polynomials have received considerable attention in dealing with various problems in mathematics such as Fourier series, wavelet series, signal processing, spectrofluorometric analysis etc. Orthogonal functions are frequently used towards a considerable simplification of the statistical analysis, for the linear regression model. They have played an important role in problem solving in various areas of mathematics.

One of the important roles of orthogonal functions is in operational matrix theory. Operational matrices are those which are produced by approximating a derivative or integration of a function in terms of orthogonal functions. The main advantage of this technique is that it reduces original problems into a system of algebraic equations, which can easily be solved. In the numerical analysis, operational matrices technique is a powerful technique for approximating solutions of integral and fractional differential equations (see [33–37]). There are many equations including PDEs and PIDEs which contain singularity and can not be solved by classical methods. The operational matrix technique performs nicely for solving such equations and converts the main problem into system of algebraic equations. It not only simplifies the proposed problem but also speeds up the computation (see [36, 38–41]).

The theory of the operational matrices mainly depends on two operators, differentiation and integration and the corresponding operational matrices can be evaluated in the following manner

$$\int_a^t \Phi(x)dx \approx P\Phi(t),$$

$$\frac{d\Phi(t)}{dt} \approx M\Phi(t),$$

where, P and M are the matrices of integration and differentiation, respectively of dimension $N+1$ and $\Phi(t) = [\phi_0(t), \phi_1(t), \phi_2(t), ..., \phi_N(t)]$ is the orthonormal basis which is orthonormal in the certain interval. More generally, one can write,

$$\int_a^t ... \int_a^t \Phi(x)\,(dx)^k \approx P^k\Phi(t),$$

$$\frac{d^k\Phi(t)}{dt^k} \approx M^k\Phi(t).$$

The popularity of the operational matrix approach is due to its easy implementation, high order convergence, and easy to extend in higher dimensions. Some of the advantages are discussed below,

- It can be easily extended into higher dimensions using Kronecker product.
- It reduces the given equation (PDEs, IPDEs, FPDEs etc.) into a system of algebraic equations which can be solved by well-known methods.
- Solution is convergent even though the size of increment is large.
- Removes the singularities in the equation.

Due to the above advantages, the operational matrices of differentiation and integration have been used by many researchers.

Several types of orthogonal basis functions have been used for operational matrix of integration such as block-pulse function [42], Chebyshev polynomials [43, 44], the Walsh function [45], Laguerre series [46–48], Legendre polynomials [49], Fourier series [50, 51], Bessel series [52], Haar wavelets [53], Legendre wavelets [54] and Berstein polynomials [55].

Meanwhile, the operational matrix of differentiation has also been determined for several types of orthogonal basis functions, such as Legendre polynomial [56], Jacobi operational matrix [57], Legendre wavelets [58], Chebyshew wavelets [59] and Bernoulli wavelets [60]. In chapters 2 and 4, the operational matrices have been used based on Lagrange polynomials and in chapter 3, Euler's operational matrices are used.

1.6 Survey of fractional derivatives

The concept of non-integer order differentiation is by no means new. It is evident from the letter written by Leibniz (1859) to L'Hospital in 1695 that the idea of fractional calculus is known as soon as the ideas of the classical calculus were known. Although Euler (1730), Lagrange (1772), and others made contributions even earlier but, the study in a systematic way was begun in the beginning and middle of the 19^{th} century by Liouville (1832), Riemann (1853), and Holmgren (1864).

Several definitions of the fractional derivatives like Grunwald Letnikov, Caputo, Riemann-Livioulle have been made till the date. However, one can obtain Caputo and Riemann Liouville from Grunwald Letnikov by using some suitable transformations. Out of these definitions, Caputo and Riemann-Livioulle's definitions attract the researchers because those are defined starting from $t = 0$. Other derivatives like Feller, Riesz need information starting from $t = -\infty$ which is almost a difficult case for time dependent problems.

In this thesis, the Caputo derivative of order α is used in chapter 5 due to its wide applications in the applied mathematics [30, 61, 62], which is defined as

$$ {}_0D_t^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_0^t (t-s)^{n-(\alpha+1)} \frac{d^n u(s)}{ds^n} ds, \quad n-1 < \alpha < n. $$

Another reason for using Caputo definition is that Caputo derivative of constant is zero whereas the Riemann-Liouville derivative of constant is non-zero. But, one needs derivative zero from the physical point of view and the Riemann-Liouville derivative of a constant is zero only when the lower terminal is tending to $-\infty$ i.e., the starting point is $t = -\infty$.

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