Chapter 3

Interaction three Interfacial Cracks between an Orthotropic Half-Plane Bonded to a Dissimilar Orthotropic Layer with a Punch

3.1 Introduction

Mechanics of fracture broadly deals with the science of strength of materials relating to the study of the load-bearing capacity of a body with or without the presence of cracks and various principals governing the crack development. The development of pre-existing cracks in a body may depend on the basic parameters like material,

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shape and dimension of the body, the mode of applying an external load, time and number of cycles of load, temperature, the degree of environmental reactivity, strain rate and deformation history. The main objective of the chapter is to present the method of analysis and their solutions, and also expressions of the quantities of physical interest e.g., stress intensity factor (SIF), stress magnification factor, etc. This discussion centres around the framework of the classical linear theory of elasticity which takes account of small displacement and strains. Furthermore, the parameters of the solid are considered to be independent of temperature and its state of stress.

The study of fracture behaviour of materials is necessary to design the critical structures of the material. The punch problem is not only restricted up to the fields of Mechanical and Civil engineering but also it is an important branch of Applied Mathematics. The punch, connected with fracture mechanics, at the end of which an infinite tensile stress occurs to initiate a crack or pre-existed crack is developed due to the action of the punch. As it is hard to get an analytical solution to the problem of crack containing half-plane subjected to the punch, some numerical methods or semi-analytical methods are required to solve such types of punch problems. Study of crack problems is extremely important for the sake of safety and security of structural components. Moving punch problems in an isotropic elastic strip had been solved by Tait and Moodie (1981) using the complex variable method to obtain closed form solution, and also solved by Singh and Dhaliwal (1984) using integral transform method. Lobada and Tauchert (1985) have solved the elastic contact problem for dissimilar orthotropic infinite and semi-infinite strips by reducing the system to singular integral equations using Fourier Transform. The new complex variable approach was proposed by Viola and Piva (1986-1988) to solve elasto-dynamic crack problems in an orthotropic medium. Another significant contribution about the problem of three interfacial cracks in anisotropic media was given by Sadowski *et al.* (2016) and Craciun *et al.* (2014). They considered Mode I and Mode II loading conditions, respectively, and they showed some interesting facts as evidence regarding the effects of the interactions between cracks according to their relative distances and positions. Morini *et al.* (2013) have shown the Stroh formalism in the analyses of skew-symmetric and symmetric weight functions for interfacial cracks. Das and Debnath (2005) have found the stress intensity factors around a Griffith crack in an orthotropic punched layer. In 2010, Choi and Paulino (2010) have investigated the interfacial cracking in a graded coating/substrate system loaded by a frictional sliding flat punch. Das and Patra (1999) have studied the crack between an orthotropic half planes bonded to a dissimilar orthotropic layer with a punch. In 2013, the transient deformations and stress intensity factors at the crack tip in presence of a punch had been investigated by Periasamy and Tippur (2013) using digital gradient sensing technique. In 2016, the interaction effect of Griffith cracks was also studied by Mishra *et al.* (2016). But to the best of my knowledge study of the combination of interfacial cracks and a punch is first of its kind.

The oscillatory behaviour of the in-plane linear elastic stress field near the tip of an interfacial crack was first noticed by Williams (1959). England (1965) noted that the oscillatory singularity of the interfacial fields leads to crack surface overlapping and wrinkling, which are viewed as physically inadmissible although those are confined in a very small region close to the crack tip.

Later, England (1965) mathematical justified that the oscillatory singularity of the interfacial crack causes cracks to surface overlapping and wrinkling in a small region close to the crack tip. The attempts towards removal of oscillations had been taken by Comninou (1977) and Atkinson (1977) by considering contact between crack surfaces and transition of material properties at the interface, respectively. In terms of removing stress oscillation and surface overlapping both the models work through stress singularity remains at the crack tip.

In hybrid displacement element method proposed by Atluri and Nakagaki (1975), the assumed displacement functions are independent of the stress functions. To make the formulation more flexible, Chow *et al.* (1995) considered no crack face overlap or non-zero traction on the crack face. Pazis *et al.* (1988) explained the matter in other way, saying that to find physically acceptable solution, additional conditions $K_I = 0$ and $K_{II} = 0$ may be imposed, i.e. the singular stress field at the crack tip should vanish. The overlapping phenomena in the different geometry of the crack/cracks at the interface can be found in the works of Noda and Xu (2008), Markides *et al.* (2011), Chadegani and Batra (2011), Zhang and Deng (2006).

In the present chapter, the stress magnification factors have been determined at the vicinity of the collinear Griffith cracks situated at the interface of an orthotropic half-plane and a dissimilar orthotropic layer with a punch on another face. The interaction effects of the interfacial cracks under constant loading due to the presence of linear punch are presented graphically and discussed in section 3.4.

3.2 Problem Formulation

Consider the elasto-static problem of three collinear Griffith cracks of finite length situated at the interface of an orthotropic strip 1 of thickness h with a punch and half-plane 2. Under the assumption of plane strain in an orthotropic medium, the

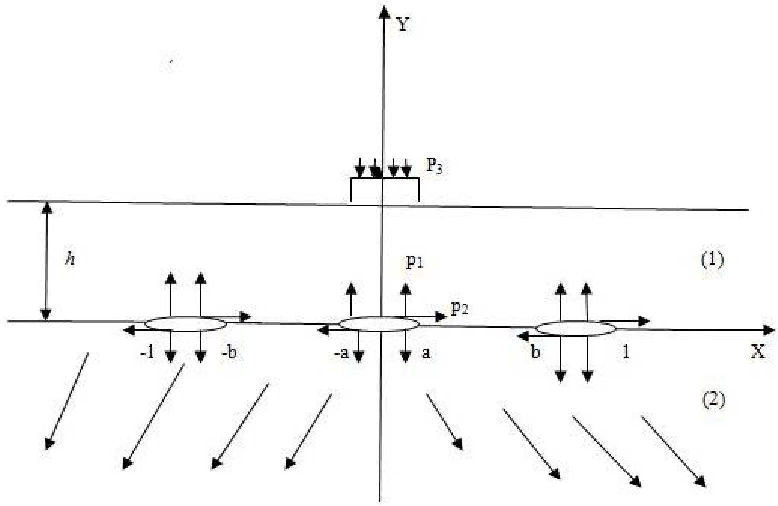

Figure 3.1: The geometry of the Problem

displacement equations of motion are given by

$$C_{11}^{(i)}\frac{\partial^2 u^{(i)}}{\partial x^2}+C_{66}^{(i)}\frac{\partial^2 u^{(i)}}{\partial y^2}+(C_{12}^{(i)}+C_{66}^{(i)})\frac{\partial^2 v^{(i)}}{\partial x\partial y}=0, \tag{3.1}$$

$$C_{22}^{(i)}\frac{\partial^2 v^{(i)}}{\partial y^2}+C_{66}^{(i)}\frac{\partial^2 v^{(i)}}{\partial x^2}+(C_{12}^{(i)}+C_{66}^{(i)})\frac{\partial^2 u^{(i)}}{\partial x\partial y}=0, \tag{3.2}$$

where $C_{jk}^{(i)}$'s are the elastic constants. Here, superscripts i=1,2 refer to the media 1 and 2, respectively. The stress-displacement relations are given by

$$\sigma_{xx}^{(i)}(x,y)=C_{11}^{(i)}\frac{\partial u^{(i)}}{\partial x}+C_{12}^{(i)}\frac{\partial v^{(i)}}{\partial y}, \tag{3.3}$$

$$\sigma_{yy}^{(i)}(x,y)=C_{12}^{(i)}\frac{\partial u^{(i)}}{\partial x}+C_{22}^{(i)}\frac{\partial v^{(i)}}{\partial y}, \tag{3.4}$$

$$\sigma_{xy}^{(i)}(x,y)=C_{66}^{(i)}\Big(\frac{\partial u^{(i)}}{\partial y}+\frac{\partial v^{(i)}}{\partial x}\Big). \tag{3.5}$$

where $i=1,2$.

It is assumed that at the interface $y=0$, the central crack is defined by $\mid x\mid< a$ and the outer cracks are defined by $b<\mid x\mid<1$ are opened by internal normal and shearing tractions $p_1(x)$ and $p_2(x)$, respectively, the punch defined by $\mid x\mid\leq a$, $y=h$

is subjected to normal pressure $p_3(x)$. The boundary conditions on $y = 0$ are given by

$$\sigma_{yy}^{(1)}(x,0) = -p_1(x),\ \mid x \mid\leq a,\ b \leq\mid x \mid\leq 1, \tag{3.6}$$

$$\sigma_{xy}^{(1)}(x,0) = -p_2(x),\ \mid x \mid\leq a,\ b \leq\mid x \mid\leq 1, \tag{3.7}$$

$$u^{(1)}(x,0) = u^{(2)}(x,0),\ a <\mid x \mid< b,\ \mid x \mid> 1, \tag{3.8}$$

$$v^{(1)}(x,0) = v^{(2)}(x,0),\ a <\mid x \mid< b,\ \mid x \mid> 1, \tag{3.9}$$

$$\sigma_{yy}^{(1)}(x,0) = \sigma_{yy}^{(2)}(x,0),\ -\infty < x < \infty, \tag{3.10}$$

$$\sigma_{xy}^{(1)}(x,0) = \sigma_{xy}^{(2)}(x,0),\ -\infty < x < \infty. \tag{3.11}$$

The boundary conditions on $y = h$ are

$$v^{(1)}(x,h) = p_3(x),\ \mid x \mid\leq a, \tag{3.12}$$

$$\sigma_{xy}^{(1)}(x,h) = 0,\ \mid x \mid< \infty, \tag{3.13}$$

$$\sigma_{yy}^{(1)}(x,h) = 0,\ \mid x \mid> a. \tag{3.14}$$

The appropriate integral solution of equations (3.1) and (3.2) can be taken as

$$u^{(i)}(x,y) = \int_0^\infty A^{(i)}(s,y)\sin sx\, ds, \tag{3.15}$$

$$v^{(i)}(x,y) = \int_0^\infty B^{(i)}(s,y)\cos sx\, ds. \tag{3.16}$$

For the strip 1, the solution of the above equation is given as

$$A^{(1)}(s,y) = A_1^{(1)}(s)ch(\gamma_1^{(1)}sy) + A_2^{(1)}(s)ch(\gamma_2^{(1)}sy) + C_1^{(1)}(s)sh(\gamma_1^{(1)}sy) + C_2^{(1)}(s)sh(\gamma_2^{(1)}sy), \quad (3.17)$$

$$B^{(1)}(s,y) = B_1^{(1)}(s)sh(\gamma_1^{(1)}sy) + B_2^{(1)}(s)sh(\gamma_2^{(1)}sy) + D_1^{(1)}(s)ch(\gamma_1^{(1)}sy) + D_2^{(1)}(s)ch(\gamma_2^{(1)}sy), \quad (3.18)$$

and for the half-plane 2, the solution is given as

$$A^{(2)}(s,y) = A_1^{(2)}(s)e^{\gamma_1^{(2)}sy} + A_2^{(2)}(s)e^{\gamma_2^{(2)}sy}, \quad (3.19)$$

$$A^{(2)}(s,y) = B_1^{(2)}(s)e^{\gamma_1^{(2)}sy} + B_2^{(2)}(s)e^{\gamma_2^{(2)}sy}, \quad (3.20)$$

where $\gamma_1^{(i)}$ and $\gamma_2^{(i)}$, i=1,2 are the roots of the equation

$$C_{66}^{(i)}C_{22}^{(i)}\gamma^4 + [(C_{12}^{(i)} + C_{66}^{(i)})^2 - C_{22}^{(i)}C_{11}^{(i)} - (C_{66}^{(i)})^2]\gamma^2 + C_{11}^{(i)}C_{66}^{(i)} = 0. \quad (3.21)$$

The materials are chosen so that $\gamma_1^{(i)}$ and $\gamma_2^{(i)}(< \gamma_1^{(i)})$ depend on the elastic constants, are the positive roots of the above equation and $B_j^{(i)}(s)$, $D_j^{(i)}(s)$ are related to the arbitrary functions $A_j^{(i)}(s)$ and $C_j^{(i)}(s)$ by

$$B_j^{(i)}(s) = -\alpha_j^{(i)}A_j^{(i)}(s)/\gamma_j^{(i)}, \;\; D_j^{(i)}(s) = -\alpha_j^{(i)}C_j^{(i)}(s)/\gamma_j^{(i)} \quad (3.22)$$

with

$$\alpha_j^{(i)} = (C_{11}^{(i)} - (\gamma_j^{(i)})^2C_{66}^{(i)})/(C_{12}^{(i)} + C_{66}^{(i)}), \;\; i = 1,2;\; j = 1,2.$$

The expressions of stresses for the strip are

$$\sigma_{xy}^{(1)}(x,y) = C_{66}^{(1)} \int_0^\infty \left[\frac{\beta_1^{(1)}}{\gamma_1^{(1)}} A_1^{(1)}(s) sh(\gamma_1^{(1)} sy) + \frac{\beta_2^{(1)}}{\gamma_2^{(1)}} A_2^{(1)}(s) sh(\gamma_2^{(1)} sy) \right.$$
$$\left. + \frac{\beta_1^{(1)}}{\gamma_1^{(1)}} C_1^{(1)}(s) ch(\gamma_1^{(1)} sy) + \frac{\beta_2^{(1)}}{\gamma_2^{(1)}} C_2^{(1)}(s) ch(\gamma_2^{(1)} sy) \right] s \sin sx \, ds, \quad (3.23)$$

$$\sigma_{yy}^{(1)}(x,y) = \int_0^\infty \left[(C_{12}^{(1)} - C_{22}^{(1)} \alpha_1^{(1)}) \{ A_1^{(1)}(s) ch(\gamma_1^{(1)} sy) \right.$$
$$+ C_1^{(1)}(s) sh(\gamma_1^{(1)} sy) \} + (C_{12}^{(1)} - C_{22}^{(1)} \alpha_2^{(1)}) \{ A_2^{(1)}(s) ch(\gamma_2^{(1)} sy)$$
$$\left. + C_2^{(1)}(s) sh(\gamma_2^{(1)} sy) \} \right] s \cos sx \, ds, \quad (3.24)$$

and for the half-plane are

$$\sigma_{xy}^{(2)}(x,y) = C_{66}^{(2)} \int_0^\infty \left[\frac{\beta_1^{(2)}}{\gamma_1^{(2)}} A_1^{(2)}(s) e^{\gamma_1^{(2)} sy} + \frac{\beta_2^{(2)}}{\gamma_2^{(2)}} A_2^{(2)}(s) e^{\gamma_2^{(2)} sy} \right] s \sin sx \, ds, \quad (3.25)$$

$$\sigma_{yy}^{(2)}(x,y) = \int_0^\infty \left[(C_{12}^{(2)} - C_{22}^{(2)} \alpha_1^{(2)}) A_1^{(2)}(s) e^{\gamma_1^{(2)} sy} \right.$$
$$\left. + (C_{12}^{(2)} - C_{22}^{(2)} \alpha_2^{(2)}) A_2^{(2)}(s) e^{\gamma_2^{(2)} sy} \right] s \cos sx \, ds \quad (3.26)$$

with $\beta_j^{(i)} = \alpha_j^{(i)} + (\gamma_j^{(i)})^2, \;\; i,j = 1,2.$

The boundary conditions (3.8) and (3.9) with the help of (3.10) and (3.11) give rise to

$$\int_0^\infty \left[l_1 A_1^{(1)}(s) + l_2 A_2^{(1)}(s) - l_3 C_1^{(1)}(s) - l_4 C_2^{(1)}(s) \right] \sin sx \, ds = 0,$$
$$a <| x |< b, | x |> 1, \quad (3.27)$$

$$\int_0^\infty \left[m_1 A_1^{(1)}(s) + m_2 A_2^{(1)}(s) - m_3 C_1^{(1)}(s) - m_4 C_2^{(1)}(s) \right] \cos sx \, ds = 0,$$
$$a <| x |< b, | x |> 1. \quad (3.28)$$

The expressions of l_i and m_i $(i = 1, 2, 3, 4)$ are given in the Appendix.

The boundary condition (3.14) gives

$$\int_0^\infty \Big[\eta_1^{(1)} A_1^{(1)}(s) sh(\gamma_1^{(1)} sy) + \eta_2^{(1)} A_2^{(1)}(s) sh(\gamma_2^{(1)} sy) + \eta_1^{(1)} C_1^{(1)}(s) ch(\gamma_1^{(1)} sy) + \eta_2^{(1)} C_2^{(1)}(s) ch(\gamma_2^{(1)} sy)\Big] \cos\ sy\, ds = 0,$$

$$a <| x |< b, | x |> 1. \quad (3.29)$$

Now setting,

$$l_1 A_1^{(1)}(s) + l_2 A_2^{(1)}(s) - l_3 C_1^{(1)}(s) - l_4 C_2^{(1)}(s) = \frac{1}{s}\int_0^a f_1(t)\cos(st)\, dt + \frac{1}{s}\int_b^1 f_1(t)\cos(st)\, dt, \quad (3.30)$$

$$m_1 A_1^{(1)}(s) + m_2 A_2^{(1)}(s) - m_3 C_1^{(1)}(s) - m_4 C_2^{(1)}(s) = \frac{1}{s}\int_0^a f_2(t)\sin(st)\, dt + \frac{1}{s}\int_b^1 f_2(t)\sin(st)\, dt, \quad (3.31)$$

$$\eta_1^{(1)} A_1^{(1)}(s) sh(\gamma_1^{(1)} sy) + \eta_2^{(1)} A_2^{(1)}(s) sh(\gamma_2^{(1)} sy) + \eta_1^{(1)} C_1^{(1)}(s) ch(\gamma_1^{(1)} sy) + \eta_2^{(1)} C_2^{(1)}(s) ch(\gamma_2^{(1)} sy) = \frac{1}{s}\int_0^a f_3(t)\sin(st)\, dt, \quad (3.32)$$

Equation (3.32) with the help of boundary conditions (3.13) yields

$$A_1^{(1)}(s) = -[1 + \delta_1(s)] C_1^{(1)}(s) - \frac{\delta_2(s)}{s}\int_0^a f_3(t)\sin st\, ds, \quad (3.33)$$

$$A_2^{(1)}(s) = -[1 + \delta_3(s)] C_2^{(1)}(s) - \frac{\delta_4(s)}{s}\int_0^a f_3(t)\sin st\, ds \quad (3.34)$$

with

$$\delta_1(s) = \frac{2e^{-2\gamma_1^{(1)}sh}}{1-e^{-2\gamma_1^{(1)}sh}},\ \delta_2(s) = \frac{\mu_2^{(1)}e^{-\gamma_1^{(1)}sh}}{(\mu_2^{(1)}\eta_1^{(1)} - \mu_1^{(1)}\eta_2^{(1)})(1-e^{-2\gamma_1^{(1)}sh})}, \tag{3.35}$$

$$\delta_3(s) = \frac{2e^{-2\gamma_2^{(1)}sh}}{1-e^{-2\gamma_2^{(1)}sh}},\ \delta_4(s) = \frac{\mu_1^{(1)}e^{-\gamma_2^{(1)}sh}}{(\mu_2^{(1)}\eta_1^{(1)} - \mu_1^{(1)}\eta_2^{(1)})(1-e^{-2\gamma_2^{(1)}sh})}. \tag{3.36}$$

Equations (3.27)-(3.29) are identically satisfied if

$$\int_{-a}^{a} f_i(t)\,dt = 0,\ i = 1,2,3 \text{ and } \int_{b}^{1} f_i(t)\,dt = 0,\ i = 1,2. \tag{3.37}$$

Equations (3.30) and (3.31), with the aid of (3.33) and (3.34), give

$$w_{11}(s)C_1^{(1)}(s) + w_{12}(s)C_2^{(1)}(s) = -\frac{1}{s}\Big[\int_0^a f_1(t)\cos st\,dt + \int_b^1 f_1(t)\cos st\,dt + w_{13}(s)\int_0^a f_3(t)\sin st\,dt\Big], \tag{3.38}$$

$$w_{21}(s)C_1^{(1)}(s) + w_{22}(s)C_2^{(1)}(s) = -\frac{1}{s}\Big[\int_0^a f_2(t)\sin st\,dt + \int_b^1 f_2(t)\sin st\,dt + w_{23}(s)\int_0^a f_3(t)\sin st\,dt\Big]. \tag{3.39}$$

The expressions of $w_{ij}(s)$ are given in the Appendix.

Equations (3.38) and (3.39) give rise to

$$C_1^{(1)}(s) = -\frac{\omega_{22}(s)}{s}\int_0^a f_1(t)\cos\, st\,dt - \frac{\omega_{22}(s)}{s}\int_b^1 f_1(t)\cos\, st\,dt + \frac{\omega_{12}(s)}{s}\int_0^a f_2(t)\sin\, st\,dt + \frac{\omega_{12}(s)}{s}\int_b^1 f_2(t)\sin\, st\,dt + \frac{\omega_{13}(s)}{s}\int_0^a f_3(t)\sin\, st\,dt, \tag{3.40}$$

$$C_2^{(1)}(s) = \frac{\omega_{21}(s)}{s}\int_0^a f_1(t)\cos\, st\,dt + \frac{\omega_{21}(s)}{s}\int_b^1 f_1(t)\cos\, st\,dt$$
$$-\frac{\omega_{11}(s)}{s}\int_0^a f_2(t)\sin\, st\,dt - \frac{\omega_{11}(s)}{s}\int_b^1 f_2(t)\sin\, st\,dt$$
$$-\frac{\omega_{23}(s)}{s}\int_0^a f_3(t)\sin\, st\,dt, \quad (3.41)$$

where $\omega_{ij}(s) = \frac{w_{ij}(s)}{(w_{11}(s)w_{22}(s) - w_{12}(s)w_{21}(s))}$, $i = 1,2,\ j = 1,2,3;$

and

$$\omega_{ij}(\infty) = \lim_{s\to\infty}\omega_{ij}(s) \quad \text{and} \quad \omega_{ij}^{(1)} = \omega_{ij}(s) - \omega_{ij}(\infty),\ i = 1,2,\ j = 1,2, \qquad (3.42)$$

with

$$\omega_{1i}(\infty) = 1 + \frac{\left[\frac{\alpha_i^{(1)}}{\gamma_i^{(1)}}(\mu_1^{(2)} - \mu_2^{(2)}) + \mu_i^{(1)}\left(\frac{\alpha_1^{(2)}}{\gamma_1^{(2)}} - \frac{\alpha_2^{(2)}}{\gamma_2^{(2)}}\right)\right]}{\left[\mu_2^{(2)}\frac{\alpha_1^{(2)}}{\gamma_1^{(2)}} - \mu_1^{(2)}\frac{\alpha_2^{(2)}}{\gamma_2^{(2)}}\right]},$$

$$\omega_{2i}(\infty) = \eta_i^{(1)} + \frac{\left[\frac{\alpha_i^{(1)}}{\gamma_i^{(1)}}(\eta_1^{(2)}\mu_2^{(2)} - \eta_2^{(2)}\mu_1^{(2)}) + \mu_i^{(1)}(\eta_1^{(2)}\mu_2^{(2)} - \eta_2^{(2)}\mu_1^{(2)})\right]}{\left[\mu_2^{(2)}\frac{\alpha_1^{(2)}}{\gamma_1^{(2)}} - \mu_1^{(2)}\frac{\alpha_2^{(2)}}{\gamma_2^{(2)}}\right]}. \qquad (3.43)$$

The boundary conditions (3.6), (3.7) and (3.12) with the aid of above equations give rise to the following singular integral equations for the determinations of $f_i(t)$, i=1,2,3 as

$$a_1 f_1(x) + \frac{1}{\pi b_1}\int_{-a}^{a}\frac{f_2(t)dt}{t-x} + \frac{2}{\pi b_1}\int_b^1\frac{f_2(t)dt}{t-x} + \frac{1}{\pi}\int_{-a}^{a}[K_{11}(x,t)f_1(t)$$
$$+ K_{12}(x,t)f_2(t) + K_{13}(x,t)f_3(t)]dt + \frac{2}{\pi}\int_b^1[K_{11}(x,t)f_1(t)$$
$$+ K_{12}(x,t)f_2(t) + K_{13}(x,t)f_3(t)]dt = \frac{2p_1(x)}{\pi}, 0 < x < a,\ b < x < 1, \quad (3.44)$$

$$c_1 f_2(x) - \frac{1}{\pi d_1}\int_{-a}^{a}\frac{f_1(t)dt}{t-x} - \frac{2}{\pi d_1}\int_{b}^{1}\frac{f_1(t)dt}{t-x} + \frac{1}{\pi}\int_{-a}^{a}[K_{21}(x,t)f_1(t)$$
$$+ K_{22}(x,t)f_2(t) + K_{23}(x,t)f_3(t)]dt + \frac{2}{\pi}\int_{b}^{1}[K_{21}(x,t)f_1(t)$$
$$+ K_{22}(x,t)f_2(t) + K_{23}(x,t)f_3(t)]dt = \frac{2p_2(x)}{\pi}, 0 < x < a,\ b < x < 1, \quad (3.45)$$

$$\int_{-a}^{a}\frac{f_3(t)dt}{t-x} + \int_{-a}^{a}[K_{31}(x,t)f_1(t) + K_{32}(x,t)f_2(t) + K_{33}(x,t)f_3(t)]dt$$
$$+ \int_{b}^{1} 2[K_{31}(x,t)f_1(t) + K_{32}(x,t)f_2(t)]dt = 2p_3(x),\ 0 < x < a, \quad (3.46)$$

where

$$a_1 = -\eta_1^{(1)}\omega_{22}(\infty) + \eta_2^{(1)}\omega_{21}(\infty),\ \frac{1}{b_1} = \eta_1^{(1)}\omega_{12}(\infty) - \eta_2^{(1)}\omega_{11}(\infty), \quad (3.47)$$
$$c_1 = \mu_1^{(1)}\omega_{22}(\infty) - \mu_2^{(1)}\omega_{11}(\infty),\ \frac{1}{d_1} = \mu_1^{(1)}\omega_{22}(\infty) - \mu_2^{(1)}\omega_{21}(\infty), \quad (3.48)$$

where a_1, b_1, c_1 and d_1 are the material-dependent constants and

$$K_{ii}(x,t) = \int_0^\infty d_{ii}(s)\cos s(t-x)ds,\ i = 1, 2,$$
$$K_{ij}(x,t) = \int_0^\infty d_{ij}(s)\sin s(t-x)ds,\ i \neq j,\ i,j = 1, 2, \quad (3.49)$$
$$K_{ij}(x,t) = \int_0^\infty d_{ij}(s)\sin s(t-x)ds,\ i = 1,\ j = 3 \text{ and } i = 3, j = 2, 3,$$
$$K_{ij}(x,t) = \int_0^\infty d_{ij}(s)\cos s(t-x)ds,\ i = 1, j = 3 \text{ and } i = 3, j = 1. \quad (3.50)$$

The expressions of $d_{ij}(s)$ are given in Appendix.

The pair of singular integral equations (3.44) and (3.45) can be reduced in the form

$$\phi_k(x) + \frac{1}{\pi i a r_k}\int_{-a}^{a}\frac{\phi_k(t)dt}{t-x} + \frac{2}{\pi i a r_k}\int_{b}^{1}\frac{\phi_k(t)dt}{t-x} + \int_{-a}^{a}[K_{k1}^*(x,t)\phi_1(t) + K_{k2}^*(x,t)\phi_2(t) + K_{k3}^*(x,t)f_3(t)]dt + 2\int_{b}^{1}[K_{k1}^*(x,t)\phi_1(t) + K_{k2}^*(x,t)\phi_2(t) + K_{k3}^*(x,t)f_3(t)]dt = g_k(x),\ 0<x<a,\ b<x<1, \quad (3.51)$$

which can be rewritten as

$$\phi_k(X) + \frac{1}{\pi i a r_k}\int_{-1}^{1}\frac{\phi_k(T)dt}{T-X} + \frac{2}{\pi i a r_k}\int_{b/a}^{1/a}\frac{\phi_k(T)dT}{T-X} + \int_{-1}^{1}[K_{k1}^*(X,T)\phi_1(T) + K_{k2}^*(X,T)\phi_2(T) + K_{k3}^*(X,T)f_3(T)]dT + 2\int_{b/a}^{1/a}[K_{k1}^*(X,T)\phi_1(T) + K_{k2}^*(X,T)\phi_2(T) + K_{k3}^*(X,T)f_3(T)]dT = g_k(X),$$

$$0<X<1,\ b/a<X<1/a, \quad (3.52)$$

where

$$\phi_k(X) = \sqrt{a_1b_1}f_1(aX) + ir_k\sqrt{c_1d_1}f_2(aX), a = \sqrt{a_1b_1c_1d_1} \quad \text{and } r_1 = 1, r_2 = -1,$$

$$2\pi K_{lm}^* = \left(\frac{1}{a_1}K_{11} + r_lr_m\frac{1}{c_1}K_{22}\right) - a^{-1}(r_mK_{21}d_1 + r_lb_2K_{12}), l, m = 1,2,$$

$$K_{k3}^* = \frac{1}{\pi}\left[\sqrt{\frac{b_2}{a_1}}K_{13} - r_k\sqrt{\frac{d_1}{c_1}}K_{23}\right], \quad (3.53)$$

$$g_k(X) = \frac{2}{\pi}\left[\sqrt{\frac{b_2}{a_1}}p_1(aX) - r_k\sqrt{\frac{d_1}{c_1}}p_2(aX)\right], k = 1,2. \quad (3.54)$$

3.3 Solution of the Problem

The solutions of integral equations (3.52) and (3.46) may be assumed as

$$\phi_k(X) = \omega_k(X) \sum_{n=0}^{\infty} C_{kn} P_n^{(\alpha_k \beta_k)}(X), \;\; k = 1, 2, \tag{3.55}$$

$$f_3(X) = R(X) \sum_{n=0}^{\infty} C_{3n} T_{2n+1}(X), \tag{3.56}$$

where $\omega(X) = (1-X)^{\alpha_k}(1+X)^{\beta_k}, \;\; \alpha_k = -\frac{1}{2} + i\omega_k, \;\; \beta_k = -\frac{1}{2} - i\omega_k,$
$\omega_k = r_k\omega, \;\; k = 1, 2$ with $w = \frac{1}{2\pi} ln\left|\frac{1+a}{1-a}\right|, \;\; R(x) = (1-X^2)^{-\frac{1}{2}}.$
Equation (3.37) implies that

$$\int_{-1}^{1} \phi_k(T)dT = 0, \; \int_{b/a}^{1/a} \phi_k(T)dT = 0 \quad \text{and} \quad C_{k0} = 0, \;\; k = 1, 2. \tag{3.57}$$

Using the orthogonality relations of Jacobi's and Chebyshev's polynomials given by Sadowski *et al.* (2016), the three singular integral equations are reduced to

$$\frac{\sqrt{1-a^2}}{2iar_k} \sum_{j=1}^{\infty} C_{kn} P_{j-1}^{(-\alpha_k, -\beta_k)}(X) + \frac{2}{\pi iar_k} \sum_{n=0}^{\infty} C_{kn}\psi_{kj}^* + \sum_{n=1}^{\infty}\sum_{m=1}^{2} C_{mn} L_{kmnj}^* + \sum_{n=0}^{\infty} C_{3n} b_{knj} = F_{kj}, j = 0, 1, 2. \tag{3.58}$$

$$\frac{\pi}{2} C_{3j} + \sum_{n=1}^{\infty}\sum_{m=1}^{2} C_{mn} L_{3mnj}^* + \sum_{n=0}^{\infty} C_{3n} a_{nj} = P_{3j}, \tag{3.59}$$

where

$$L_{kmnj}^* = \int_{-1}^{1} L_{kmn}(X)\omega_k^{-1}(X) P_j^{(-\alpha_k, -\beta_k)}(X)dX,$$
$$\psi_{kj}^* = \int_{-1}^{1} \psi_{kX}\omega_k^{-1}(X) P_j^{(-\alpha_k, -\beta_k)}(X)dX,$$

with

$$\psi_{kX} = \int_{b/a}^{1/a} \frac{\omega_k(T) P_n^{(\alpha_k,\beta_k)}(T) dT}{T - X},$$

$$L_{kmn}(X) = \int_{-1}^{1} K_{km}^{*}(X,T)\omega_m(T) P_n^{(\alpha_k,\beta_k)}(T) dT$$
$$+ \int_{b/a}^{1/a} K_{km}^{*}(X,T)\omega_m(T) P_n^{(\alpha_k,\beta_k)}(T) dT,$$

$$H_{k2n+1}(X) = \int_{-1}^{1} K_{k3}^{*}(X,T) T_{2n+1}(t)(1-T^2)^{-1/2} dT$$
$$+ \int_{b/a}^{1/a} K_{k3}^{*}(X,T) T_{2n+1}(t)(1-T^2)^{-1/2} dT,$$

$$b_{knj} = \int_{-1}^{1} H_{k2n+1}(X)\omega_k^{-1}(X) P_j^{(-\alpha_k - \beta_k)}(X) dX,$$

$$F_{kj} = \int_{-1}^{1} g_k(X)\omega_k^{-1}(X) P_j^{(-\alpha_k,-\beta_k)}(X) dX,$$

$$\theta_j^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{j!\,(2j+\alpha+\beta+1)\Gamma(j+\alpha+\beta+1)}.$$

$$L_{3mn}(X) = \int_{-1}^{1} K_{3m}(X,T)\omega_m(T) P_n^{(\alpha_k,\beta_k)}(T) dT$$
$$+ \int_{b/a}^{1/a} K_{3m}(X,T)\omega_m(T) P_n^{(\alpha_k,\beta_k)}(T) dT, (m = 1,2, n = 0,1,2,3...)$$

$$L_{3mn}(X) = \int_{-1}^{1} K_{3m}(X,T)\omega_m(T) P_n^{(\alpha_k,\beta_k)}(T) dT, (m = 3, n = 0,1,2,3...)$$

$$a_{nj} = \int_{-1}^{1} H_{2n+1}(X) U_{2j}(X)\sqrt{1-X^2} dX,$$

$$P_{3j} = \int_{-1}^{1} p_3(X) U_{2j}(X)\sqrt{1-X^2} dX.$$

The expressions for the stress intensity factors at $x = a$ are given by

$$\sqrt{\frac{b_1}{a_1}}K_I^a + ir_k\sqrt{\frac{d_1}{c_1}}K_{II}^a = \frac{i\pi\sqrt{1-a^2}}{2ar_k}\sum_{n=1}^{\infty} C_{kn}P_n^{(\alpha_k,\beta_k)}(1). \tag{3.60}$$

The stress intensity factors at $x = b$ and $x = 1$ are

$$\sqrt{\frac{b_1}{a_1}}K_I^b + ir_k\sqrt{\frac{d_1}{c_1}}K_{II}^b = \frac{i\pi\sqrt{1-b^2}}{2br_k}\sum_{n=1}^{\infty} C_{kn}P_n^{(\alpha_k,\beta_k)}(1), \tag{3.61}$$

$$\sqrt{\frac{b_1}{a_1}}K_I^1 + ir_k\sqrt{\frac{d_1}{c_1}}K_{II}^1 = \frac{i\pi}{2r_k}\sum_{n=1}^{\infty} C_{kn}P_n^{(\alpha_k,\beta_k)}(1). \tag{3.62}$$

Now stress magnification factors (SMF) are defined by $M_I^a = \frac{K_I^a}{K_I^{a*}}$, $M_I^b = \frac{K_I^b}{K_I^{b*}}$, $M_I^1 = \frac{K_I^1}{K_I^{1*}}$ and $M_{II}^a = \frac{K_{II}^a}{K_{II}^{a*}}$, $M_{II}^b = \frac{K_{II}^b}{K_{II}^{b*}}$, $M_{II}^1 = \frac{K_{II}^1}{K_{II}^{1*}}$, where K_I^{a*} and K_{II}^{a*} are the Mode-I and Mode-II stress intensity factors at $x = a$ due to the presence of a central crack only situated at the interface of two half-planes. K_I^{b*}, K_{II}^{b*} and K_I^{1*}, K_{II}^{1*} are the stress intensity factors at $x = b$ and $x = 1$, respectively, due to the presence of outer cracks, only situated at the interface of two half-planes. The expressions of SMF are found in the research articles of Mukherjee and Das (2007), Kobayashi and Moss (1969), Rose (1986), Senddon and Lowengrub (1969), Nisitani and Murakami (1974).

During the measurement of the distance between two surfaces of the central crack in the presences of an outer crack, we have

$$\Delta v(x,0) = v^{(1)}(x,0) - v^{(2)}(x,0) = \frac{\pi}{2}\int_x^a f_2(t)dt + \frac{\pi}{2}\int_b^1 f_2(t)dt, \ 0 < x < a \tag{3.63}$$

As the concern is to find the overlap of the central crack, therefore it is assumed that $(1 - b) \to \infty$.

Then,

$$\Delta v(x,0) = \frac{\pi a}{2\sqrt{c_1 d_1}} lm\, C_{11} \left(\frac{a-x}{2a}\right)^{\alpha_1+1} \left[\left(i\omega \sum_{n=0}^{\infty} \frac{(-\beta_1)_n}{n(\alpha_1+1+n)} \right) \left(\frac{a-x}{2a}\right)^n \right.$$
$$\left. + \left(\frac{a-x}{2a}\right) \sum_{n=0}^{\infty} \frac{(-\beta_1-1)_n}{n(\alpha_1+1+n)} \left(\frac{a-x}{2a}\right)^n - \sum_{n=0}^{\infty} \frac{(-\beta_1)_n}{n(\alpha_1+2+n)} \left(\frac{a-x}{2a}\right)^n \right],$$
$$0 < x < a, \quad (3.64)$$

where $(\lambda)_n$ is Pocchammer symbol defined by

$$(\lambda)_n = \lambda(\lambda+1)...(\lambda+n-1)$$

and $(-\lambda)_n = (-1)^n(\lambda - n + 1)_n$.

It is observed that near the end of the crack, the sign changes infinitely conform that the upper and lower surfaces will overlap with each other. Taking into account that the contact first takes place at a distance $a - x = \delta$, where $\delta > 0$ is a small quantity, we get

$$\cos\left(\omega\, ln\frac{\delta}{2a}\right) = 0, \quad (3.65)$$

which gives the length of the overlap of two lips of the central crack as

$$\delta = 2ae^{-\frac{\pi}{2\omega}}. \quad (3.66)$$

It should be noted that the quantity ω depends on the materials' constants and the loading conditions. This shows that the region of oscillation is very small; therefore, the irregularities of this solution are confined to a very small region near the ends of the crack. Proceeding in the similar way, we can find the expressions of the size of the overlap of two surfaces can be found at both the tips of outer cracks.

3.4 Results and Discussion

In this section, numerical results are used to investigate the interactions among the interfacial collinear cracks due to the presence of rigid punch for a pair of materials. The orthotropic materials strip 1 and a half-plane 2 are considered as α-Uranium and Beechwood, respectively, whose elastic constants are given in Table 3.1 (Mishra *et al*. (2016)).

Table 3.1: Elastic Constants

Materials	$C_{11}^{(i)}$ 10^{10} Pa	$C_{12}^{(i)}$ 10^{10} Pa	$C_{22}^{(i)}$ 10^{10} Pa	$C_{66}^{(i)}$ 10^{10} Pa
α-Uranium ($i = 1$)	21.47	19.86	4.65	7.43
Beech wood ($i = 2$)	0.17	1.58	0.15	0.103

The author has used the material composites while dealing with problems of fracture in composite structure. α-uranium is used for its serious engineering applications in aero-spaces and military industry. Beechwood is considered for composite material due to its strong moisture-dependent and shrinking and high abrasion resistance characteristic.

During computations, the loadings are considered as $p_1(x) = p$, $p_2(x) = 0$ and the punch is considered as $p_3(x) = x$. The effect of the presence of interfacial cracks on the contact stress field due to the presence of a linear punch is examined through stress magnification factor. The dimensionless stress magnification factors versus the thickness of the strip (h) for both Mode-I and Mode-II types at the tip of the central crack $x = a$ are described through Figures 3.2 and 3.3, respectively, for various values of dimensionless quantity b/a, once by keeping $a = 0.5$ and varying $b = 0.6(0.1)0.9$. Then, by keeping the outer crack length fixed at $b = 0.6$ and varying $a = 0.2(0.1)0.5$, the stress magnification factors versus h at the outer crack tips $x = b$ and $x = 1$ are depicted through Figures 3.4-3.5 and 3.6-3.7, respectively, for various values of b/a.

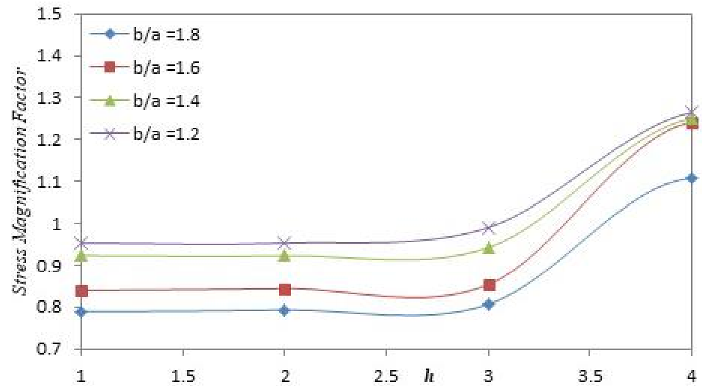

Figure 3.2: Plot of M_I^a versus h at $a = 0.5$

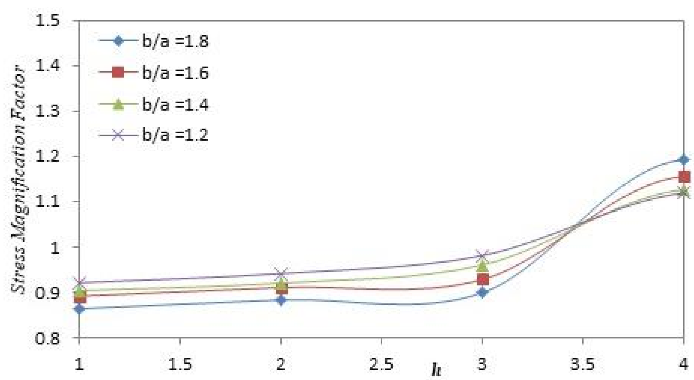

Figure 3.3: Plot of M_{II}^a versus h at $a = 0.5$

It is seen from Figures 3.2 and 3.3 that keeping the central crack fixed, if the outer crack length is decreased, then the stress magnification factors at $x = a$ are decreased. Due to the shielding ($M_I^a, M_{II}^a < 1$), there is the possibility of crack arrest. Moreover, this effect is increased with the additional effect of punch on the strip. Eventually, as the depth of the strip is increased, the effect of shielding is immediately changed to amplification ($M_I^a, M_{II}^a > 1$), which clearly exhibits the fact

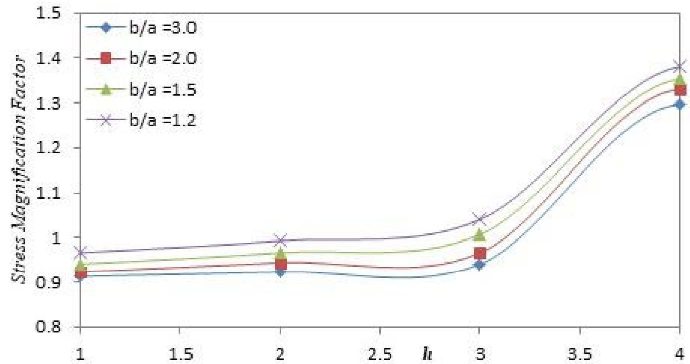

Figure 3.4: Plot of M_I^b versus h at $b = 0.6$

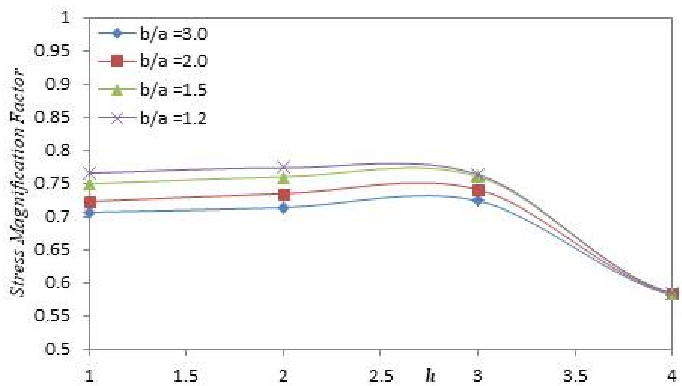

Figure 3.5: Plot of M_{II}^b versus h at $b = 0.6$

that the effect of punch through a certain depth of strip may lead to the possibilities of crack arrest.

Figures 3.4 and 3.5 show the variations of stress magnifications factors at $x = b$ with that of the central crack size when the outer crack length is fixed at $b = 0.6$. It is seen that as the size of the central crack is increased, both M_I^a and M_{II}^a are increased but as the depth of the strip increases, M_I^b increases whereas M_{II}^b decreases.

Variations of stress magnification factors at $x = 1$ for fixed outer crack length at $b = 0.6$ and different central crack lengths are shown through Figures 3.6 and 3.7. It is observed from the Figures 3.4-3.7 that nature of M_I^1 and M_{II}^1 are

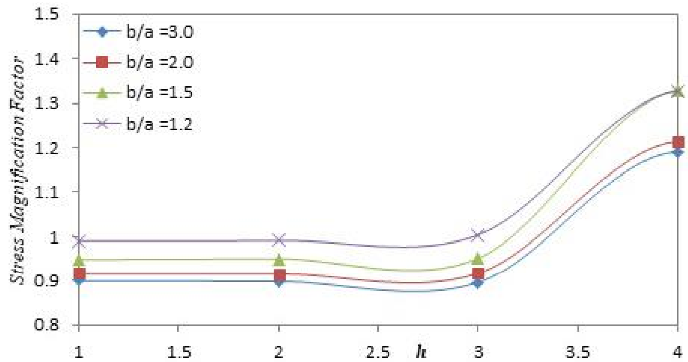

Figure 3.6: Plot of M_I^1 versus h at $b = 0.6$

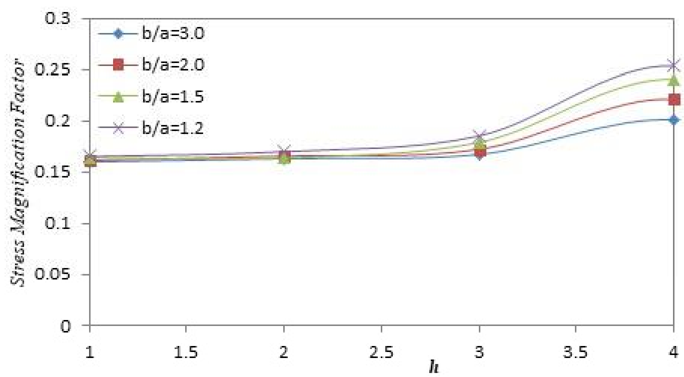

Figure 3.7: Plot of M_{II}^1 versus h at $b = 0.6$

similar to M_I^b and M_{II}^b as and when the central crack lengths keep changing. As the depth of the strip increases, the behaviour of M_I^1 is found to be similar to M_I^b whereas M_{II}^1 behaves opposite to M_{II}^b.

Figures 3.5 and 3.6 reveal that the outer crack experiences shielding effect due to the presence of a central crack. This effect is maximum when the central crack size is minimum and crack separation distance between the outer crack and the central crack is maximum. It is seen from Figure 3.5 that when $a = 0.5, b = 0.4$ i.e., when the outer crack size is two-fifth and crack separation distance is one-tenth of the central crack size, then shielding is approximately 24% when $h = 1$. When the sizes of the outer cracks are equal to the central crack length and crack separation is also equal to the central crack ($a = 0.2, b = 0.4$), then shielding is approximately

30% when $h = 1$. This effect decreases with the increase of the value of h. Figure 3.7 depicts that when the outer crack size is one-fourth and crack separation distance is one-tenth to the central crack size, then shielding is approximately 84% at $h = 1$. When the size of the outer crack is equal to the central crack and crack separation distance, then shielding is approximately 84.5% for different values of h. The effect increases with the increase of h.

It is also clear from Figures 3.5 and 3.7 that the overall values of stress magnification factors of Mode-II type are found to be very small caused due to the effect of the punch at the cracks' faces under the compressive stress field.

3.5 Conclusion

In this chapter, four important goals have been achieved. The first one is the investigation of interaction among the three collinear cracks situated at the interface of orthotropic strip bonded to the half-plane with the punch on its other face of the strip. The second one is the finding the length of the overlap of the two lips of crack faces near the crack tip. The third one is finding the analytical form of the stress intensity factors in the vicinity of the crack tips. The last one is the graphical presentations of amplification and shielding effect through the stress magnification factors, which help to find the possibilities of the arrest of the central crack due to the presence of outer cracks and vice-versa. The author is the optimist that the proposed mathematical model will be beneficial for the engineers working in the field of composite media.

3.6 Appendix

$$d_{11}(s) = -\eta_1^{(1)} w_{22}^{(1)}(s) - \eta_1^{(1)} \delta_1(s) w_{22}(s) + \eta_2^{(1)} w_{21}^{(1)}(s) + \eta_2^{(1)} \delta_1(s) w_{21}(s),$$

$$d_{12}(s) = -\eta_1^{(1)} w_{12}^{(1)}(s) - \eta_1^{(1)} \delta_1(s) w_{12}(s) + \eta_2^{(1)} w_{21}^{(1)}(s) + \eta_2^{(1)} \delta_3(s) w_{11}(s),$$

$$d_{13}(s) = -\eta_1^{(1)} \delta_2(s) + \eta_2^{(1)} \delta_4(s) + \eta_1^{(1)} [1 + \delta_1(s)] w_{13}(s) - \eta_2^{(1)} [1 + \delta_3(s)] w_{23}(s),$$

$$d_{21}(s) = -\eta_1^{(1)} w_{22}^{(1)}(s) - \eta_1^{(1)} \delta_1(s) w_{22}(s),$$

$$d_{22}(s) = \eta_2^{(1)} w_{21}^{(1)}(s) + \eta_2^{(1)} \delta_1(s) w_{21}(s),$$

$$d_{23}(s) = -\eta_1^{(1)} w_{13}(s) + \eta_2^{(1)} w_{23}(s),$$

$$d_{31}(s) = \left[-\delta_5(s) w_{22}(s) \frac{\alpha_1^1}{\gamma_1^1} + \delta_6(s) w_{21}(s) \frac{\alpha_2^1}{\gamma_2^1} \right],$$

$$d_{32}(s) = \left[-\delta_5(s) w_{12}(s) \frac{\alpha_1^1}{\gamma_1^1} + \delta_6(s) w_{11}(s) \frac{\alpha_2^1}{\gamma_2^1} \right],$$

$$d_{33}(s) = \left[-\delta_5(s) w_{13}(s) \frac{\alpha_1^1}{\gamma_1^1} + \delta_6(s) w_{23}(s) \frac{\alpha_2^1}{\gamma_2^1} \right] - \left[\delta_2(s) \frac{\alpha_1^1}{\gamma_1^1} sh(\gamma_1^{(1)} sh) + \delta_4(s) \frac{\alpha_2^1}{\gamma_2^1} sh(\gamma_2^{(1)} sh) \right],$$

with

$$w_{11}(s) = l_1[1 + \delta_1(s)] + l_3, w_{12}(s) = l_2[1 + \delta_3(s)] + l_4,$$

$$w_{13}(s) = l_1 \delta_2(s) - l_2 \delta_4(s), w_{21}(s) = m_1[1 + \delta_1(s)] + m_3,$$

$$w_{22}(s) = m_2[1 + \delta_1(s)] + m_4, w_{23}(s) = m_1 \delta_2(s) - m_2 \delta_4(s),$$

where

$$\delta_1(s) = \frac{2e^{-2\gamma_1^{(1)}}}{1 - e^{-2\gamma_1^{(1)}sh}}, \;\; \delta_2(s) = \frac{\mu_2^{(1)} e^{-\gamma_1^{(1)}sh}}{(\mu_2^{(1)} - \eta_1^{(1)} - \mu_1^{(1)}\eta_2^{(1)})(1 - e^{-2\gamma_1^{(1)}sh})},$$

$$\delta_3(s) = \frac{2e^{-2\gamma_2^{(1)}}}{1 - e^{-2\gamma_2^{(1)}sh}}, \;\; \delta_4(s) = \frac{\mu_2^{(1)} e^{-\gamma_2^{(1)}sh}}{(\mu_2^{(1)} - \eta_1^{(1)} - \mu_1^{(1)}\eta_2^{(1)})(1 - e^{-2\gamma_2^{(1)}sh})},$$

$$\delta_5(s) = -(1 + \delta_1(s))sh(\gamma_1^{(1)}sh) + ch(\gamma_1^{(1)}sh),$$

$$\delta_6(s) = -(1 + \delta_3(s))sh(\gamma_2^{(1)}sh) + ch(\gamma_2^{(1)}sh)$$

and

$$l_i = 1 + \frac{\eta_i^{(1)}(\mu_1^{(2)} - \mu_2^{(2)})}{(\mu_2^{(2)}\eta_1^{(2)} - \mu_2^{(2)}\eta_2^{(2)})}, l_{i+2} = \mu_i^{(1)} \frac{(\eta_1^{(2)} - \eta_2^{(2)})}{(\mu_2^{(2)}\eta_1^{(2)} - \mu_2^{(2)}\eta_2^{(2)})},$$

$$m_i = \eta_i^{(1)} \frac{(\alpha_1^{(2)}\mu_2^{(2)}/\gamma_1^{(2)} - \alpha_2^{(2)}\mu_1^{(2)}/\gamma_2^{(2)})}{(\mu_2^{(2)}\eta_1^{(2)} - \mu_2^{(2)}\eta_2^{(2)})},$$

$$m_{i+2} = \frac{\alpha_i^{(1)}}{\gamma_i^{(1)}} + \mu_i^{(1)} \frac{(\alpha_1^{(2)}\mu_2^{(2)}/\gamma_1^{(2)} - \alpha_2^{(2)}\mu_1^{(2)}/\gamma_2^{(2)})}{(\mu_2^{(2)}\eta_1^{(2)} - \mu_2^{(2)}\eta_2^{(2)})},$$

with $\eta_j^{(i)} = C_{12}^{(i)} - C_{22}^{(i)}\alpha_j^{(i)}$ and $\mu_j = C_{66}^{(i)} \frac{\beta_j^{(i)}}{\gamma_j^{(i)}}, i, j = 1, 2.$

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