Chapter 1

Introduction

1.1 Ancient History of Fracture Mechanics

Originally, research in the field of the fracture mechanics rarely found prior to World War II. Cracks were thought to be a very small, not significant nuisances that could never be a hazard to large structures like aero-ships, aircraft, aerospace and aeroplane wings etc. Eventually, during the World War II, many ships, aerospaces and aircrafts were failed suddenly in incomprehensible ways. Ultimately, it was obtained that the failures were because of caused by cracks and notch in their aircraft and aeroplane structures.

In 1950, the most famous case of crack related aircraft failure of the aviation industry had been occurred in de Havilland DH 106 Comet of U. K. Three fatal comet-1 were crashed one by one under a twelve months period which led to the grounding of the entire Comet fleet. The crashes were found to be caused by crack growth due to the square fuselage windows (Figure 1.1 [Ref. 103]). The square corners served as stress risers, accelerating the crack formation and growth in a fuselage stressed under pressurization duration of the highest altitude flight.

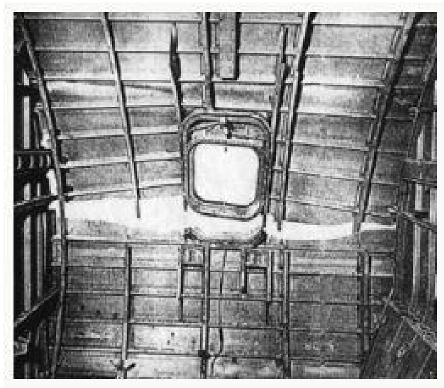

Figure 1.1: Square corners of de Havilland Comet

One of the biggest brittle failures, also known as Titanic marine disaster with the loss of 1,500 people, was happened on April 15, 1912 [Ref. 110]. According to the literature review, scientists thought that the tragedy caused by long gash torn with the ship's hull by Atlantic's iceberg. Although ship's wreckage was discovered in 1985 using undersea machines, robots, but there was no evidence found for such a long deep slash. Meanwhile, the robots were able to return a specimen of the ship's steel material whose investigation has given a hike to modify explanation.

After the collision of Titanic ship with the Atlantic's iceberg, the Titanic breaks into two pieces due to the longitudinal force (Tensile load) after sinking approximately 3/4-th part of it on the one side. This type of crack or fracture is said to be Fracture Mode-I which is opening mode (Tensile load) and shown in Figure 1.2 [Ref. 105] and Figure 1.3 [Ref. 104].

Fracture mechanics is based on the implicit form of assumption that there exists a crack or notch in a work component. The crack is created because of man-made mistakes like a notch, a slot, a hole, a re-entrant corner, *etc.*

After all, for practical purpose, the modern fracture mechanics was born in 1948, after the result of George Irwin (1948), who introduced remarkable parameters like stress intensity factor and energy release rate. Thereafter, many investigators start taking interest in fracture mechanics and it became an important topic of discussion and research. Irwin's derivation was basically for brittle or semi- ductile materials. Other parameters as crack tip opening displacement derived by Wells (1961) and J- integral derived by Rice (1968), were determined to account for the large plastic zone at the tip of the crack for ductile materials.

Nowadays fracture mechanics is applied extensively to the important fields viz., piping, spaceships, rockets, seaward structures, etc. Critical components in nuclear power plants are made from very tough materials like cast iron, high carbon steels, ceramics and some polymers; but those have failed catastrophically as well once in a while. There are several types of failure of structures, which are shown in the Table 1.1 [Ref. 109].

Figure 1.2: "Titanic" Marine disaster


The Times Dispatch

Titanic, Giant White Star Liner, Sinks After Collision With Iceberg on Her Maiden Voyage, and 1,800 Lives Are Reported Lost in World's Greatest Marine Disaster

WIRELESS CALLS SEND VESSELS RUSHING TO AID OF SEA COLOSSUS

OF ALL ON BOARD ONLY 675 KNOWN TO HAVE ESCAPED DEATH IN OCEAN

COL. JOHN JACOB ASTOR IS AMONG DROWNED

CARRIED NOTABLE PASSENGER LIST

FINANCIAL BLOW IS VERY SEVERE

OFFICIALS CONCEDE GREAT LOSS OF LIFE

PEOPLE ON SHIPS GET FIRST NEWS

Figure 1.3: Titanic disaster

Table 1.1: Causes of Failures

Failure of Structures	
Yielding Dominant	Fracture Dominant
• General plasticity • Significant Defects are those controlling resistance to plastic flow, E.G.: -interstitials -grain boundaries -precipitates -dislocation networks	• Highly localised plasticity • Significant Defects are essentially microscopic, E. G.: -weld flows -porosity -fatigue and stress -corrosion cracks

1.1.1 Mode of Fracture

Three types of linearly Independent cracking modes are used in fracture mechanics viz., Mode-I, Mode-II and Mode-III. For Mode-I (opening mode), the cracked surface is moving apart and acts normal to the plane of the crack. In Mode-II (sliding mode) cracked surfaces slide to each other and parallel to the plane of the crack and perpendicular to the crack front. Mode-III (tearing mode) acts parallel to the plane of the crack and parallel to the crack front (see Figure 1.4) [Ref. 106].

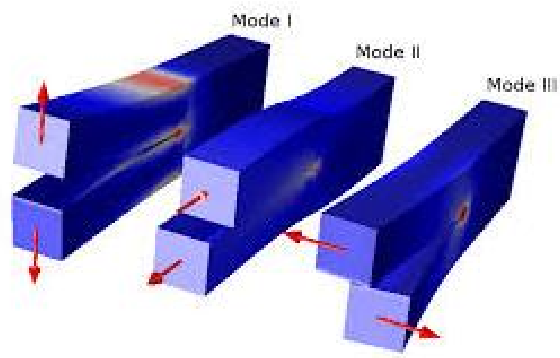

Figure 1.4: Mode of Fracture

1.1.2 Hook's law

In 1676, the English physicist Robert Hooke's presented a law of elasticity, known as Hooke's law. According to this law, for small deformations, the stress and strain are proportional to each other. Thus, stress $\propto$ strain or stress $= k \times$ strain, where k is the proportionality constant and is also known as modulus of elasticity.

1.1.3 Stress and Strain Curve

The relationship between stress and strain for any material under tensile loading can be found experimentally. A standard test had been considered of tensile properties taking a body which is stretched by an applied force. The applied force is gradually increased whereas a change in the length of the body is shown in the Figure 1.5 [Ref. 108], which is the graphical representation of the stress and the produced strain for a metal. The stress-strain curves vary from material to material. These curves aid us to understand what is the relation between stress and stain for different materials. According to the graph, we see that from region point O to point A, the curve is linear. This region follows the Hooke's law of elasticity if the applied force is removed then body regain its original state. In the region, A to B, stress and strain are not proportional to each other. But, the body still regains its original state when force is removed. The point B is a yield (also called elastic limit) and at this point, stress is called yield strength (S_y) of the material.

If increasing the small load after this point B the strain increases rapidly, the portion of B to D which is shown through. At the point D even when put stress is zero then strain never becomes zero at this point. This type of deformation is plastic deformation and the point D is ultimate tensile strength (S_u) of the material. If the ultimate tensile strength D and fracture point E are very close then the material is said to be brittle like glass, silica, diamond. If they are far apart then the material is said to be ductile (steel, metal, brass). The phenomena are shown through the Figure 1.6 [Ref. 107].

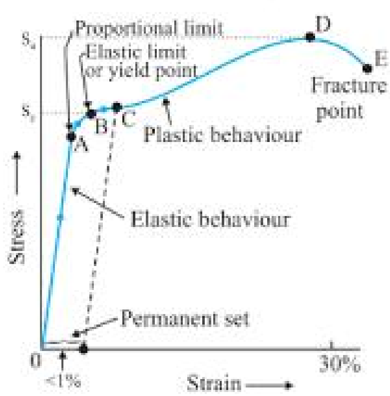

Figure 1.5: Stress-Strain curve

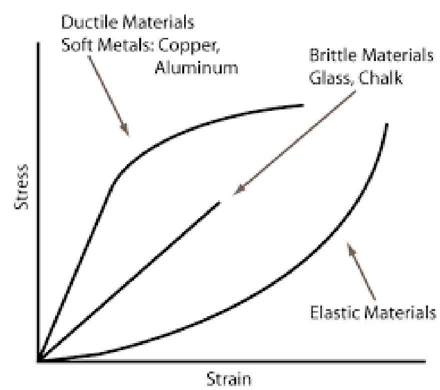

Figure 1.6: Brittle, Ductile and Elastic material curves

1.2 Energy Based Theory

1.2.1 Energy Based Approach

At beginning of fracture mechanics Alan Arnold Griffith developed the energy-based approach in the field of fracture mechanics, thereafter George Irwin modifies Griffith's energy-based approach, Nowadays researchers and scientists are using Irwin modified energy-based approach [Ref. 88]. The energy approach is energy release rate G which is defined as rate of change in potential energy with the crack area for a linear elastic material. At the moment of fracture $G = G_c$, the critical energy release rate which measures the fracture toughness. For a crack of finite crack length $2a$ in an infinite plate under the remote tensile load, the energy release rate is given by

$$G = \frac{\pi \sigma^2 a}{E}, \tag{1.1}$$

where E is Young's modulus, σ is the remotely applied stress, and a is the half crack length.

At fracture $G = G_c$, the above equation (1.1) obtains the critical combinations of stress and crack size for the failure as

$$G_c = \frac{\pi \sigma_f^2 a_c}{E}. \tag{1.2}$$

For the constant value of G_c, the failure stress σ_f varies with $1/\sqrt{a}$. The energy release rate G is the driving force for fracture, whereas G_c the material's resistance to fracture.

1.2.2 Stress Concentration by Inglis (1913)

Stress concentration effect of flow by Charles Edward Inglis, who analysed elliptical hole in flat plates. He analysed elliptic hole with $2a$ major axes and $2b$ minor axes under applied stress perpendicular to the major axes of the ellipse. He assumed elliptic hole which was not affected by plate boundary, i.e., the plate width much greater than $2a$ and plate height much greater than $2b$. The stress at the tip of the major axis at the point A as shown in Figure 1.7 [Ref. 88] is given by

$$\sigma_A = \sigma\left(1 + \frac{2a}{b}\right). \tag{1.3}$$

The ratio σ_A/σ is defined as stress concentration factor k_t. When $a = b$, then the hole is circular and in that case $k_t = 3.0$, which a well-known result.

As major axis, a increases relatively to b, then elliptical hole becomes to a sharp crack. For this case, C. E. Inglis found it in a more convenient way to express equation (1.3) in terms of the radius of curvature ρ as

$$\sigma_A = \sigma\left(1 + 2\sqrt{\frac{a}{\rho}}\right), \tag{1.4}$$

where

$$\rho = \frac{a^2}{b}. \tag{1.5}$$

When $a >> b$, the equation (1.4) becomes

$$\sigma_A = 2\sigma\sqrt{\frac{a}{\rho}}. \tag{1.6}$$

C. E. Inglis derived an approximate solution in equation (1.6) for stress concentration due to the notch except at the crack tip of the semi-infinite plate.

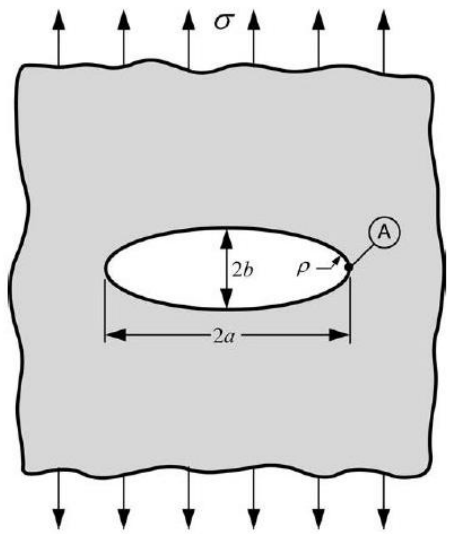

Figure 1.7: Elliptic crack in infinite plate

The above approximation proves that if $\rho \to 0$ then $\sigma_A \to \infty$. This theory is not reality-based because no material exists in this world which is bearable for infinite stress or residual load without any failure.

1.2.3 Energy Release Rate

In 1920, Alan Arnold Griffith introduced energy-based approach of cracks in the field of fracture mechanics. He was highly motivated by C. E. Inglis's linear elastic solution for stresses and residual load around an elliptical hole in which he found that the stress level tends to infinity whereas the ellipse becomes flat from the crack.

Inglis's work did not support Griffith's energy-based approach because there does not exist any material which supports an applied infinite stress or any residual load without yielding and failing. Therefore, the material or structure immediately gets failure under the smallest load if any crack is present (but practically this does not happen). Further, Griffith's energy-based approach was failed and after the Inglis's infinite-stress prediction, nevertheless, there exists the direct use of the solution of Griffith's theory of linear elastic.

Energy release per unit area increases during crack growth is said to be energy release rate. This formula can be deduced as follows [Ref. 63].

Consider an incremental increase in the area ΔA for the crack growth, an incremental external work ΔW_{ext} is done under the external load and strain energy within the body of the component increases by ΔU, the available energy is $G\Delta A$, then energy balance as given by

$$G\Delta A = G\Delta W_{ext} - \Delta U. \tag{1.7}$$

Dividing the above equation by ΔA and taking the limit $\Delta A \to 0$, we get

$$G = -\frac{\mathrm{d}(U - W_{ext})}{\mathrm{d}A}, \tag{1.8}$$

the negative sign was intentionally taken out the differential term because $(U - W_{ext})$ is commonly used as a potential function Π. Therefore, the equation is written as

$$G = -\frac{\mathrm{d}\Pi}{\mathrm{d}A}. \tag{1.9}$$

The above equation evaluates the energy release rate of a system. The energy release rate is always positive for a crack studied for its probable growth. The dimension and unit of energy release rate are $[MT^{-2}]$, energy per unit area

1.2.4 Crack Resistance

The energy requirement for a crack to grow per unit area extension is called crack resistance and denoted by R. Crack resistance is required in the sum of the energies to form two new surfaces and to cause an elastic deformation [Ref. 63].

Parameters viz., energy release rate (G) and crack resistance (R) are very important to study the possibility of a crack becoming critical. It is obvious that if the crack has to have a chance to grow due to load, there must be energy release rate is greater than the crack resistance. If the energy release rate exceeds the crack resistance, then crack acquires kinetic energy and it may grow at a faster rate than the speed of a supersonic aeroplane.
Thus, for a crack to grow and becomes critical, there are two conditions necessary given as follows:

$$G \geq R, \tag{1.10}$$

$$\frac{\mathrm{d}G}{\mathrm{d}a} \geq \frac{\mathrm{d}R}{\mathrm{d}a}, \tag{1.11}$$

where a is the half crack length.

1.3 Linear Elastic Fracture Mechanics

1.3.1 Stress Intensity Factor

In 1957, George Irwin, the man nowadays noticed to be the father of fracture mechanics had introduced Stress intensity factor. The abbreviation of stress intensity factor is SIF and symbolized by K. It is the most useful and considerable fundamental parameter of all the fields of fracture mechanics. Basically, stress intensity factor is concerned with the stress state at the crack tip which is considered as the rate of crack growth and it is used to calculate the failure criteria due to the fracture. Irwin derived the definition of K at the vicinity of crack tip under the remote load or residual stresses.

The stress intensity factor depends on the geometry of the problem, size and shape of the crack, magnitude, distribution and direction of the load. Stress intensity factor is the single parameter characterization of the crack tip of any material under the stress field. Stress intensity factor can be determined by an integral transform method, Finite element method and Weight function method.

Embedded crack produces $r^{-1/2}$ singularity at the crack tip whereas interfacial crack produces $r^{-1/2+i\omega}$ singularity at the crack tip, where ω is real.

In all the equations of stress and displacement, σ and a coexist as $\sigma\sqrt{a}$. Now with several decades of research work, it is found that it is advantageous to do so. This credit goes to George Irwin, who defined the new parameter stress intensity factor, and used the symbol K after the name of his collaborator Kies. He defined K as

$$K_I = \sigma(\pi a)^{1/2}. \tag{1.12}$$

There is no reason to have π in the above definition. It was included in the expression because of some historical reasons. However, the stress intensity factor K_I is formally defined as

$$K_I = (2\pi r)^{1/2}\sigma_{22}(r, \theta = 0) \text{ as } r \to 0. \tag{1.13}$$

Stress can be measured at every point of the body except at the tip of the crack of the body.

The stress intensity factors for Mode-I, Mode-II and Mode-III are formally defined as [Ref. 111]

$$\begin{aligned} K_I =& Lim_{r\to 0}\sqrt{2\pi r}\sigma_{yy}(r,0), \\ K_{II} =& Lim_{r\to 0}\sqrt{2\pi r}\sigma_{xy}(r,0), \\ K_{III} =& Lim_{r\to 0}\sqrt{2\pi r}\sigma_{yz}(r,0). \end{aligned} \tag{1.14}$$

1.3.2 Relation between SIF and Energy Release Rate

The strain energy release rate (G) for a crack under Mode-I loading is related to the stress intensity factor as [Ref. 111]

$$G = K_I^2\left(\frac{1-\nu}{E}\right), \tag{1.15}$$

where E is the Young's modulus and ν is the Poission's ratio of the material. the material supposed to be an isotropic, homogeneous, and linear elastic.
For Mode-II loading, we have

$$G = K_{II}^2\left(\frac{1-\nu}{E}\right) \quad \text{or} \quad G = K_{II}^2\left(\frac{1}{E}\right),$$

and for pure fracture Mode-III loading, we have

$$G = K_{III}^2\left(\frac{1}{2\mu}\right),$$

where μ is the shearing modulus.
In plane strain, the relationship between the strain energy and the stress intensity

factors for the three modes is given as

$$G = K_I^2\left(\frac{1-\nu}{E}\right) + K_{II}^2\left(\frac{1-\nu}{E}\right) + K_{III}^2\left(\frac{1}{2\mu}\right). \tag{1.16}$$

1.3.3 Punch

Punch is a tool or machine for impressing a design or stamping a die on a material. In other words, A device or machine for making holes in materials such as paper, leather or metal.

1.4 Mathematical Methods, Terms and Definitions

1.4.1 Integral Transform and its Properties

A general integral transform is defined as fallows

$$g(\alpha) = \int_a^b f(t)k(\alpha,t)dt, \tag{1.17}$$

where $k(\alpha, t)$ is called the integral kernel of the transform.

1.4.1.1 Laplace Transform

If the kernel $k(\alpha, t)$ in equation (1.17) is defined as

$$k(p,t) = \begin{cases} 0, & for\ t < 0, \\ e^{-pt}, & for\ t \geq 0, \end{cases} \tag{1.18}$$

then $F(p) = \int_0^\infty e^{-pt} f(t)dt$. The function $F(p)$ defined by the integral (1.18) is called the Laplace transform of the function $f(t)$ and is also denoted by $L\{F(t)\}$ or $F(p)$. thus, Laplace transform is a function of a new variable (or parameter) p given by (1.18).

The Inverse Laplace transform is given by

$$f(t) = L^{-1}\{F\}(t) = \frac{1}{2\pi i} \lim_{h\to\infty} \int_{c-ih}^{c+ih} e^{pt} F(p)dp \tag{1.19}$$

where c is greater than the real part of all singularities of $F(p)$.

1.4.2 Hilbert Transform

Let, $x(t)\epsilon L^p(R)$ be a function for $1 \leq p < \infty$, then $H(x(t))$ is the Hilbert transform of $x(t)$ given by

$$H(x(t)) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{x(s)}{(t-s)} ds,$$

where "PV" is the Cauchy principal value of the integral.

1.4.3 Fourier Transform and Inverse Fourier Transform

The Fourier transform of a function $f(x)$ is

$$\phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x)dx,$$

the Inverse transform is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \phi(\xi)d\xi.$$

1.4.4 Dirac Delta Function

In many crack problems the boundary conditions are concerned with point loading conditions with Dirac Delta function, which are usually defined through the functional relation

$$\int_{-\infty}^{\infty} f(\xi)\delta(\xi - x)d\xi = f(x). \tag{1.20}$$

Laplace transformation of $\delta(x)$ gives

$$\int_{0}^{\infty} \delta(x)e^{-tx}dx = H(t), \tag{1.21}$$

where $H(t)$ is the Heaviside unit step function and it is used to solve the problems related to the impact loading.

1.4.5 Singular Integral Equations

When one or both limits of integration become infinite or when the kernel becomes infinite at one or more points within the range of integration, the integral equation is known as singular integral equation. There are two types of singular integral equations:

- Fredholm singular Integral Equations
- Volterra singular Integral Equations

1.4.5.1 Fredholm Singular Integral Equations

A linear singular integral equation of the form

$$a(x)\phi(x) + \frac{b(x)}{\pi i}\int_L \frac{\phi(t)}{t-x}dt + \int_L k(x,t)\phi(t)dt = f(x), \tag{1.22}$$

where $f(x), a(x), b(x)$ and $k(x,t)$ are known functions whereas $\phi(x)$ is unknown function, is called Fredholm singular integral equation and L is contour. The function $K(x,t)$ is known as the kernel of the singular integral equation.

- Fredholm singular integral equation of the first kind:

  A linear singular integral equation of the form (if $a(x) = 0$ in equation (1.22))

$$\frac{b(x)}{\pi i}\int_L \frac{\phi(t)}{t-x}dt + \int_L k(x,t)\phi(t)dt = f(x), \tag{1.23}$$

  is known as Fredholm singular integral equation of the first kind.

- Fredholm singular integral equation of the second kind:

  A linear singular integral equation of the form (if $a(x) = 1$ in equation (1.22))

$$\phi(x) + \frac{b(x)}{\pi i}\int_L \frac{\phi(t)}{t-x}dt + \int_L k(x,t)\phi(t)dt = f(x), \tag{1.24}$$

  is known as Fredholm Singular integral equation of the second kind.

- Homogeneous Singular Fredholm integral equation of the second kind:

  A linear integral equation of the form (if $f(x) = 0$ in equation (1.24))

$$\phi(x) + \frac{b(x)}{\pi i}\int_L \frac{\phi(t)}{t-x}dt + \int_L k(x,t)\phi(t)dt = 0, \tag{1.25}$$

  is known as Homogeneous singular Fredholm integral equation.

1.5 Orthogonal Polynomials

The origin of orthogonal polynomial is considered from the study of continued fractions by P. L. Chebyshev in 19-th century, which was later pursued by A. A. Markov and T. J. Stieltjes. Some of the mathematicians who have worked on orthogonal polynomials include Gábor Szegő, Sergei Bernstein, Naum Akhiezer, Arthur Erdélyi, Yakov Geronimus, Dave Gwyn, Wolfgang Hahn, Theodore Seio Chihara, Mourad Ismail, Waleed Al-Salam, and Richard Askey [Ref. 112-113].

An orthogonal polynomial sequence is a family of polynomials such that any two different polynomials in the sequence are orthogonal to each other under some inner product.

In Mathematical form, a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with degree $[p_n(x)] = n$ for each n is called orthogonal with respect to the weight function $w(x)$ on the interval (a, b) with $a < b$ if

$$\int_a^b w(x)P_m(x)P_n(x)\,dx = \delta_{mn}c_n \text{ with } \delta_{mn} := \begin{cases} 0, & m \neq n \\ 1, & m = n. \end{cases} \tag{1.26}$$

The weight function should be continuous and positive on (a, b) such that

$$c_n := \int_a^b w(x)x^n dx,\ n = 0, 1, 2, ...$$

exists. Then the integral $< f, g >:= \int_a^b w(x)f(x)g(x)dx$ denotes an inner product of the polynomials f and g. The interval (a, b) is called the interval of orthogonality. If $c_n = 1$ for each $n \epsilon \{0, 1, 2, ...\}$, the sequence of polynomials is called orthogonal, and if

$$P_n(x) = k_n x^n + \text{ lower order terms with } k_n = 1,$$

for each $n\epsilon\{0, 1, 2, ...\}$, the polynomials are called monic.

The most widely used orthogonal polynomials are the classical orthogonal polynomials consisting of the Hermite polynomials, the Laguerre polynomials, the Jacobi polynomials together with their special cases the Gegenbauer polynomials, the Chebyshev polynomials, and the Legendre polynomials.

These are formulated in the Table 1.2.

1.5.1 Jacobi Polynomial and its Conditions

In mathematics, Jacobi polynomials (also called the hypergeometric polynomials) $P_n^{(\alpha,\beta)}(x)$ are a class of classical orthogonal polynomials. They are orthogonal with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ on the interval $[-1, 1]$. The Gegenbauer polynomials, and thus also the Legendre, Zernike and Chebyshev polynomials are special cases of the Jacobi polynomials [Ref. 114]. The Jacobi polynomials were introduced by Carl Gustav Jacob Jacobi . Jacobi Polynomial is defined as

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1 - x)^{-\alpha}(1 + x)^{-\beta}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(1 - x)^{n+\alpha}(1 + x)^{n+\beta}], \quad (1.27)$$

where $\alpha, \beta > -1$ and $-1 < x < 1$.
There are some cases as follows

- when $\alpha = \beta = 0$, this implies the Lagendre Polynomial.
- when $\alpha = \beta = \frac{1}{2}$, this implies the Chebyshev Polynomial of the first kind.
- when $\alpha = \beta = -\frac{1}{2}$, this implies the Chebyshev Polynomial of the second kind.
- when $\alpha = \beta$, we have the Gegenbauer Polynomial.

Table 1.2: Orthogonal Polynomials

Name	$f_i(x)$	**Interval**	$W(x)$	c_n
Jacobi	$P_n^{(\alpha,\beta)}(x)$	$[-1,1]$	$(1-x)^{\alpha}(1+x)^{\beta}$, $(\alpha,\beta>-1)$	$\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}$
Chebyshev (First kind)	$T_n(x)$	$[-1,1]$	$(1-x^2)^{-1/2}$	$\begin{cases}\pi, & for\ \text{n}=0\\ 1/2\pi, & otherwise\end{cases}$
Chebyshev (Second kind)	$U_n(x)$	$[-1,1]$	$(1-x^2)^{1/2}$	$\pi/2$
Legendre	$P_n(x)$	$[-1,1]$	1	$2/(2n+1)$
Laguerre	$L_n(x)$	$[0,\infty)$	e^{-x}	1
Hermite	$H_n(x)$	$(-\infty,\infty)$	e^{-x^2}	$\sqrt{\pi}2^n n!$
Gegenbauer Polynomial	$C_n^{(\lambda)}(x)$	$[-1,1]$	$(1-x^2)^{\alpha-\frac{1}{2}}$	$\begin{cases}\frac{2^{1-2\alpha}\pi\Gamma(n+2\alpha)}{n!(n+\alpha)(\Gamma(\alpha))^2}, & for\ \alpha\neq 0\\ 2\pi/n^2, & for\ \alpha=0\end{cases}$
Generalized Laguerre Polynomial	$L_n^{(k)}(x)$	$[0,\infty)$	$x^k e^{-x}$	$\frac{(n+k)!}{n!}$

The Jacobi polynomial satisfies the orthogonality conditions

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}$$
$$\frac{\Gamma(n+\alpha+1)(n+\beta+1)}{(n+\alpha+\beta+1)n!}\delta_{mn}, \alpha\beta > -1.$$

Symmetry Relation:

$P_n^{(\alpha,\beta)}(-z) = (-1)^n P_n^{(\beta,\alpha)}(z)$,

$P_n^{(\alpha,\beta)}(-1) = (-1)^n \binom{n+\beta}{n}$.

Differential equation:

The Jacobi polynomial $P_n^{(\alpha,\beta)}$ is a solution of the second order linear homogeneous differential equation[1]

$$(1-x^2)y'' + (\beta-\alpha-(\alpha+\beta+2)x)y' + n(n+\alpha+\beta+1)y = 0. \tag{1.28}$$

Recurrence relation: the recurrence relation for the Jacobi polynomial of fixed α, β is

$$2n(n+\alpha+\beta)(2n+\alpha+\beta-2)P_n^{\alpha,\beta}(z) = (2n+\alpha+\beta-1)$$
$$\{(2n+\alpha+\beta)(2n+\alpha+\beta-2)z+\alpha^2-\beta^2\}P_{n-1}^{\alpha,\beta}(z) - 2(n+\alpha-1)$$
$$(n+\beta-1)(2n+\alpha+\beta)P_{n-2}^{\alpha,\beta}(z), \tag{1.29}$$

for $n = 2, 3, \ldots$

Generating function: The generating function of the Jacobi polynomial is given by

$$\sum_{n=0}^{\infty} P_n^{\alpha,\beta}(z)t^n = 2^{\alpha+\beta}R^{-1}(1-t+R)^{-\alpha}(1+t+R)^{-\beta},$$

where $R = R(z,t) = (1-2zt+t^2)^{1/2}$.

1.5.2 Chebyshev Polynomial and its Conditions

These polynomials are also used in the theory of approximation of functions. Chebyshev polynomials $T_n(x)$ of the first kind of degree n over the interval $[-1, 1]$ is defined by the relation [Ref. 115]

$$T_n = \cos\big(n\cos^{-1}x\big) = \cos n\theta,\ \text{where } x = \cos\theta.$$

From this, we have

$$T_0(x) = \cos 0^\circ = 1 \text{ and } T_1(x) = \cos\theta = x$$

Recurrence relation: $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ is satisfied by Chebyshev polynomials.

Properties of Chebyshev Polynomial $T_n(x)$:

- $T_n(-x) = (-1)^n(x)$, which is shows that $T_n(x)$ is an odd function of x if n is odd and even unction of x if n is even.
- $T_n(x)$ has n simple zeros.
  $x_\lambda = \cos[\{(2\lambda - 1)/2n\}\pi], \lambda = 1, 2, 3, ..., n$ on the interval $[-1, 1]$.
- $|T_n(x)|\leq 1,\ x\epsilon[-1, 1]$.
- $T_n(x)$ assumes extreme values at $(n + 1)$ points

  $$x_m = \cos(n\pi/n), m = 0, 1, 2, .., n;$$

  and the extreme value at x_m is $(-1)^m$.

- $\int_{-1}^{1} T_m(x)T_n(x)dx = \begin{cases} 0, & if\ m \neq n \\ \pi/2, & if\ m = n \neq 0 \\ \pi, & if\ m = n = 0, \end{cases}$

  which can be proved easily by putting $x = \cos\theta$.

  Also $T_n(x)$ is orthogonal on the interval $[-1, 1]$ with respect to the weight function $w(x) = (1 - x^2)^{-1/2}$.

- Minimax property of $T_n(x)$: Of all monic polynomials $P_n(x)$ of degree n, the polynomial $2^{1-n}T_n(x)$ has the smallest least upper bound for the absolute value in the range $-1 \leq x \leq 1$. Since $T_n(x) \leq 1$, the upper bound refered to above is 2^{1-n}.

1.6 Literature Review

Noda and Jin (1993) have considered a crack problem in an infinite non-homogeneous elastic solid material under the steady state heat flux for the crack surface in which the authors found expressions of stress intensity factors for Mode-I and they found the effect of nonhomogeneities of materials on the Stress intensity factors (SIFs) by using Fourier transform method. Jin and Paulino (2001) have studied an edge crack under the transient thermal loading in the functionally graded material, in which authors used Young's modulus, Poisson's ratio, considering the thermal properties of the material vary along the direction of the thickness of strip and found thermal stress intensity factors for TiC/SiC FGM with different volume fraction profiles of the constituent materials solved with Laplace transform and an asymptotic analysis. Hongmin *et al.* (2008) have obtained dynamic stress intensity factors for Mode-I and Mode-II under the uniform impact loading at the tip of the semi- infinite cracks in an infinite functionally graded orthotropic material using Laplace and Fourier transforms with the aid of Winner–Hopf technique. Sladek and Sladek (1997) have found T-stresses and stress intensity factors in two-dimensional stationary thermo elastic medium using conservation integral method and stress intensity factors are calculated using the path independent J-integral technique. Movchan and Jones (2006) have studied analytical expressions and numerical computations of a model problem for the thermo-elastic half space, which contains a surface breaking crack (crack surface is free from traction) under the oscillatory thermal loading in which the authors have found the amplitude as a function of the stress intensity factor at the crack vertex.

Gaikwad and Ghadle (2011) have considered a thick rectangular plate under an internal heat and found displacement and thermal stresses, in which the boundary surfaces are kept at zero temperature. The governing heat conduction equation have been solved by using integral transform method. Sih *et al.* (1962) have studied stress intensity factors for plane extension and plate bending and calculated the strengthens of stress singularities using the complex variable method. The problem considered with fractal geometry having many finite elements are created around the crack surface which are solved with the help of finite element method. The authors have found SIFs for Mode-I and Mode-II for thin plate under the loads like bending, twisting and shear loads (see Su and Sun (2002)). Xia and Hutchinson (2000) have studied a two-dimensional problem in thin film bonded to the elastic medium in which the authors have found the crack propagation path in this thin film. Ding and Liu (2018) have solved the multi-layered problem with Griffith crack under the energy flux loading in which they have found heat conduction analysis in the strip of thermo-electric material and found the effects of electric flux intensity factor and the thermal flux intensity factor on the width strip.

Jin (2011) has studied transient of heat conduction at edge crack in a functionally graded material plate under the gradual heating and cooling on its boundaries in which author found thermal stress intensity factor at the edge crack in the functionally graded plate with the asymptotic temperature. Nabavi and Shahani (2009) have obtained thermal stress intensity factors at the depth and surface points of semi-elliptical crack in a solid cylinder under thermal shock loading with the aid of weight function. Fdelinski *et al.* (1994) have solved stationary cracks in a linear elastic material with the aid of dual boundary element method and dual reciprocity approach. The problem considered for a Griffith crack in thermal electric material strip (TEM) under energy flux load had been solved by Ding and Liu (2018). Cai and Chen (2007) have determined the dynamic stress intensity factors of the Griffith crack at the interface of viscoelastic layer bonded to an infinite elastic body under the shearing load and the problem was solved with the aid of integral transform method. Song and Paulino (2006) have solved the dynamic stress intensity factor of a cracked solid body for homogeneous and heterogeneous materials using the integral method and the finite element method. Smith *et al.* (1967) have obtained an expression of a penny-shaped crack in an infinite elastic solid under the non-axisymmetric normal loading in which the authors have done numerical computation of a crack in a penny-shaped in an infinite or finite solid plane material under the symmetric loading. Ognjanovic *et al.* (2013) have considered a circular crack with respect to radial crack in a thin plate under the thermal loading in which SIFs have been found with the aid of finite element method and other approaches like J- integral approach and the displacement method. Meyer and Schmauder (1992) have determined thermal stress intensity factors for a shearing mode of the interface cracks under the applied loads in composite materials. Itou (2014) has considered two collinear cracks parallel to one central crack situated in an infinite in a orthotropic composite media under the uniform heat flux in which stress intensity factors at the tips of the cracks have been found by using the Fourier transform technique.

Liu and Kardomateas (2005) have studied a line crack problem in anisotropic thermo-elastic half plane under the uniform heat flux in which stress intensity factors for Mode-I, Mode-II and Mode-III are found. The problem concerned with the methodology for finding the interface crack propagation isotropic and anisotropic materials had been studied by Banks-Sills (2015) in which the author has found the interfacial energy release rate at the different fractured confidence interval. Basically, this article is regarding the literature survey of other authors' articles. Agarawal and Kartson (2007) have investigated properties for interfacial crack within the framework of linear elastic fracture mechanics including interfacial fracture toughness, mode mixity, and the associated reference length, in which the expression of mode mixity with respect to crack tip in bimaterial system has been deduced. The problem associated with a two-dimensional analysis of the stress field around a crack on the plane interface between two bonded dissimilar anisotropic elastic half-spaces had been studied by Willis (1971). Shuicheng *et al.* (2008) have studied the stress intensity factors of specimen per unit load with different combinations of K_1 and K_2 with the help of the mixed hybrid finite element method, in which the authors have considered a single edge crack in the different fracture Modes.

Cui *et al.* (2017) have considered one horizontal straight and slanted cracks under the uniaxial tension in which stress intensity factor has been calculated with the aid of contour integral method at infinity. Singh *et al.* (2018) have found thermal stress intensity factors at the edge crack of an orthotropic composite media under the thermo-mechanical loading in which authors obtained the analytical expressions of SIF for concentrated point loading. The problem concerned as 3D planar cracks and Semi-elliptical cracks in functionally graded and homogeneous materials under several thermal- mechanical loadings had been solved by Memari *et al.* (2019) in which stress intensity factors have been found using co-ordinate transform method with the aid of linear weight function. Huanga *et al.* (2018) have found the stress intensity factors and T- stress at the crack tip along the circumference of the circular core under the static and transient dynamic loads, which have been solved by using the Laplace transformation method and the Durbin's inversion method.

Lenwari and Ma (2019) have found stress intensity factor for a two-tips web crack in wide-flange steel member under the linear distributive loading. The problem concerned with the V- notch have been solved by Yao *et al.* (2018) with the help of finite element method using the asymptotic expansion technique. Yue *et al.* (2017) have found the static stress intensity factors for a cracked orthotropic strip bonded to functionally graded material under the static and dynamic loads with the help of the finite element method. The problem concerned with crack propagation in the human's bone have been solved by Craciun *et al.* (2018). Sadowski *et al.* (2017) have used the multi-scale approach during the solution of a non-linear and complex two-phase ceramics under the uniaxial compression deformation. Wu *et al.* (2018) have obtained stress intensity factor and crack opening displacement solutions under the different loads.

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