Chapter 3

The Plane Piston Problem with Weak Gravitational Field in a Dusty Gas

3.1 Introduction

Dusty gas is the mixture of perfect gas and a large number of spherically small solid particles. The solid particle motion in rocket exhaust and dust flow in geophysical and astrophysical problems are the most important physical phenomena in which considered volume is a mixture of gas and dust particles. Here, we consider that solid particles are of uniform size and uniformly distributed in the gas and volume of the small solid particles is considered to be very less in comparison to the volume

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of the mixture Chadha and Jena (2014, 2015). In case of the propagation of shock wave the velocity of the mixture is very high so the dust particles present in the mixture are assumed to be a pseudo fluid. The applied gravity is dominated in stellar atmosphere that contains gas and a small amount of dust particles. The unsteady motion in a dusty gas under the effect of weak gravitational field, which is discussed in the present paper, have a great significance in the field of physical sciences. The transient process in the solar atmosphere is an important dynamical problem and is an unsteady process. The impulsive motion of the piston in backward direction causes a rarefaction wave and forward motion generates a compressive wave moving into the gas. When a shock wave induced by the motion of plane piston is propagated in a dusty gas the physical parameters change across the shock, and have a significant difference from those which arise when the shock wave induced by piston passes through an ideal gas.

The propagation of shock wave induced by piston in compressible fluid is formulated mathematically as a system of quasilinear hyperbolic system of partial differential equations. The problem of shock wave in a gaseous medium has drawn attention to a number of authors during the past decades. The most important break-through was made by Friedrichs (1984), Whitham (1956), Sedov (1959), Chisnell (1955), Chisnell *et al.* (1982) to study the shock wave for ideal isentropic gas dynamics. Pai (1977), Miura and Glass (1983), Miura (1972), Carrier (1958), Pai *et al.* (1983), Pai *et al.* (1980), Vishwakarma and Nath (2009), Jena and Sharma (1999), Vishwakarma *et al.*(2017), Singh *et al.* (2012), Anand (2014a, 2014b) have studied the shock wave in a dusty gas. Arora and Siddiqui (2013), Arora *et al.* (2012), Bira and Sekhar (2015) examined the behaviour of shock wave in non-ideal gas. Bira and Sekhar (2013) have studied the nature of shock wave in magnetogasdynamics. Sharma and Shyam (1981) discussed the growth and decay of weak discontinuity in radiating gas dynamics.

Singh and Jena (2016) evaluated the behaviour of weak shock wave in non-ideal relaxing gas. The presence of gravity in the transient process of astrophysics play very important role, so the consideration of gravity is important and relevant. Wen-rui (1985), Singh *et al.* (2011), Nath and Sahu (2016) studied the shock wave problem in the presence of gravitational field. In the present paper, the effect of dust particles on weak and strong shock wave between the region from the piston position to shock front is analysed by using perturbation method and similarity transformation technique. The effect of dust particles on the wave front is also discussed.

3.2 Basic Equations

The basic equations governing the motion of one dimensional planar flow of a transient gas with dust particles in a local region of stellar atmosphere may be written in the following form Wen-rui (1985)

$$\frac{\partial \varrho}{\partial t} + \vartheta \frac{\partial \varrho}{\partial x} + \varrho \frac{\partial \vartheta}{\partial x} = 0, \tag{3.2.1}$$

$$\frac{\partial \vartheta}{\partial t} + \vartheta \frac{\partial \vartheta}{\partial x} + \frac{1}{\varrho}\frac{\partial p}{\partial x} + \frac{GM}{x^2} = 0, \tag{3.2.2}$$

$$\frac{\partial p}{\partial t} + \vartheta \frac{\partial p}{\partial x} + a^2 \varrho \frac{\partial \vartheta}{\partial x} = 0, \tag{3.2.3}$$

where ϱ, ϑ and p are density, velocity and pressure of the dusty gas in the local region respectively and t is the time and x is spatial coordinate. In the present study the centre of the star is assumed as origin and x-axis is taken in the direction of stellar radius. G and M stand for the universal gravitational constant and stellar mass respectively. The equation of state for the dusty gas flow is given by Pai (1977) as

$$p = \frac{1-k_p}{1-Z}\varrho \Re T,$$

where $\Re$ is gas constant. The entity $a = (\Gamma p/ (\varrho (1-Z)))^{1/2}$ is the speed of sound in the dusty gas, where $Z = V_{sp}/V_g$, denotes the volume fraction of solid particles with V_{sp} and V_g are the volume of the dust particles and the gas respectively. The specific heat of the dusty gas at constant pressure is given by $c_{pd} = k_p c_{sp} + (1-k_p)c_p$, where c_p and c_{sp} stand for specific heat of gas and specific heat of solid particles respectively and $k_p = m_{sp}/m_g$ is the mass fraction of solid particles with m_{sp} and m_g are the masses of solid particles and gas respectively. If c_{vd} denotes the specific heat of the dusty gas at constant volume then the ratio of specific heats for the dusty gas is given by Pai (1977) $\Gamma = \frac{c_{pd}}{c_{vd}} = \frac{\gamma+\beta\delta}{1+\beta\delta}$, where $\delta = k_p/\left(1-k_p\right)$, $\beta = c_{sp}/c_p$, $\gamma = c_p/c_v$ with c_v as specific heat of gas at constant volume. The relation between the parameters Z and k_p is $k_p = Z\varrho_{sp}/\varrho$, where ϱ_{sp} stands for the density of solid particles in a dusty gas. Since mass fraction of solid particles must be constant in the equilibrium flow therefore $Z/\varrho =$ constant (say θ). The entities Z and k_p are also related by $Z = k_p/((1-k_p)\Omega + k_p)$, where $\Omega = \varrho_{sp}/\varrho_g$ with ϱ_{sp} and ϱ_g are the density of solid particles and gas respectively.

According to the theory of similarity and dimensional analysis Sedov (1959) the dimension of velocity may be written as

$$\xi = \frac{x}{t}. \tag{3.2.4}$$

Since in our model the gravitational field is also present which leads another quantity for velocity dimension given as Wen-rui (1985)

$$\vartheta_g = \sqrt{\frac{GM}{x}}. \tag{3.2.5}$$

In the presence of weak gravitational field, the gravitational velocity (3.2.5) is lesser than both the sound and plasma velocity. To discuss the basic flow properties, the following non-dimensional parameters are introduced

$$\widetilde{\vartheta} = \frac{\vartheta}{\vartheta_*}, \quad \widetilde{a} = \frac{a}{a_*}, \quad \widetilde{t} = \frac{t}{t_*}, \quad \widetilde{x} = \frac{x}{x_*}, \tag{3.2.6}$$

and

$$\varepsilon = \frac{\vartheta_g^2}{\vartheta_*^2} \ll 1, \tag{3.2.7}$$

where ϑ_* represents the typical velocity and x_*, t_* stand for the space and time. For isentropic flow, the equations (3.2.1 - 3.2.3) may be written in terms of non-dimensional parameters defined by (3.2.6 - 3.2.7) and suppressing the tilde sign as

$$\frac{\partial a}{\partial t} + \vartheta \frac{\partial a}{\partial x} + \frac{(\Gamma - 1 + 2Z)}{2(1-Z)} a \frac{\partial \vartheta}{\partial x} = 0, \tag{3.2.8}$$

$$\frac{\partial \vartheta}{\partial t} + \vartheta \frac{\partial \vartheta}{\partial x} + \frac{2(1-Z)}{(\Gamma - 1 + 2Z)} a \frac{\partial a}{\partial x} = -\frac{\varepsilon}{x^2}. \tag{3.2.9}$$

To construct the solution we introduce the following expansion of flow variables in terms of small parameter ε as

$$a = a^{(0)} + \varepsilon a^{(1)} + \varepsilon^2 a^{(2)} + ..., \tag{3.2.10}$$

$$\varrho = \varrho^{(0)} + \varepsilon \varrho^{(1)} + \varepsilon^2 \varrho^{(2)} + ..., \tag{3.2.11}$$

$$\vartheta = \vartheta^{(0)} + \varepsilon \vartheta^{(1)} + \varepsilon^2 \vartheta^{(2)} + ... \tag{3.2.12}$$

Using the expansion (3.2.10 - 3.2.12) of the flow variables in equations (3.2.8) and (3.2.9) and collecting the terms of zero order, we have

$$\frac{\partial a^{(0)}}{\partial t}+\vartheta^{(0)}\frac{\partial a^{(0)}}{\partial x}+\frac{(\Gamma-1+2Z_0)}{2(1-Z_0)}a^{(0)}\frac{\partial \vartheta^{(0)}}{\partial x}=0, \tag{3.2.13}$$

$$\frac{\partial \vartheta^{(0)}}{\partial t}+\vartheta^{(0)}\frac{\partial \vartheta^{(0)}}{\partial x}+\frac{2(1-Z_0)}{(\Gamma-1+2Z_0)}a^{(0)}\frac{\partial a^{(0)}}{\partial x}=0. \tag{3.2.14}$$

The equations (3.2.13) and (3.2.14) may be transformed in terms of similarity variable ξ in the following form

$$\left(\vartheta^{(0)}-\xi\right)\frac{\partial a^{(0)}}{\partial \xi}+\frac{(\Gamma-1+2Z_0)}{2(1-Z_0)}a^{(0)}\frac{\partial \vartheta^{(0)}}{\partial \xi}=0, \tag{3.2.15}$$

$$\left(\vartheta^{(0)}-\xi\right)\frac{\partial \vartheta^{(0)}}{\partial \xi}+\frac{2(1-Z_0)}{(\Gamma-1+2Z_0)}a^{(0)}\frac{\partial a^{(0)}}{\partial \xi}=0. \tag{3.2.16}$$

The Riemann invariants for the above system are given as

$$\vartheta^{(0)}\pm\frac{2(1-Z_0)}{\Gamma-1}=const. \tag{3.2.17}$$

The solution of the problem is determined under consideration that the velocity of the piston is constant and the flow variables ahead of the shock is uniform i.e.

$$\vartheta^{(0)}=const., \qquad a^{(0)}=const. \tag{3.2.18}$$

Collecting the terms of first–order from the expansion of equation (3.2.10 - 3.2.12), we have

$$\frac{\partial a^{(1)}}{\partial t}+\vartheta^{(0)}\frac{\partial a^{(1)}}{\partial x}+\frac{(\Gamma-1+2Z_0)}{2(1-Z_0)}a^{(0)}\frac{\partial \vartheta^{(1)}}{\partial x}=0, \tag{3.2.19}$$

$$\frac{\partial \vartheta^{(1)}}{\partial t}+\vartheta^{(0)}\frac{\partial \vartheta^{(1)}}{\partial x}+\frac{2(1-Z_0)}{(\Gamma-1+2Z_0)}a^{(0)}\frac{\partial a^{(1)}}{\partial x}=-\frac{1}{x^2}, \tag{3.2.20}$$

and similarly for all higher order terms. In this method, all relations of higher order, excluding the zeroth order relation, are linear. So, the effect of the gravity on the transient process in astrophysics and space science for all higher order equations is linear. By dimensional methods, the solution of the first order relations may be taken as

$$\vartheta^{(1)}(\xi,t) = t^{-m} f(\xi), \qquad a^{(1)}(\xi,t) = t^{-n} g(\xi). \tag{3.2.21}$$

Using equation (3.2.21) in equations (3.2.19) and (3.2.20), we have

$$m = n = 1.$$

If we consider the typical velocity ϑ_* as the plane piston velocity ϑ_p, which is assumed to be a constant, the initial velocity becomes

$$\vartheta^{(0)} = 1,$$

and the boundary condition at the plane piston will be

$$f(1) = 0. \tag{3.2.22}$$

Let $\xi = \frac{x/t}{\vartheta_*}$, then equations (3.2.19) - (3.2.20) may be written in terms of ξ and say it is equivalent to $\pounds_1$ and $\pounds_2$ as

$$\pounds_1(f,g) \equiv (1-\xi)\frac{\partial g(\xi)}{\partial \xi} + \frac{(\Gamma - 1 + 2Z_0)}{2(1-Z_0)} a^{(0)} \frac{\partial f(\xi)}{\partial \xi} - g(\xi) = 0, \tag{3.2.23}$$

$$\pounds_2(f,,g) \equiv (1-\xi)\frac{\partial f(\xi)}{\partial \xi} + \frac{2(1-Z_0)}{(\Gamma - 1 + 2Z_0)} a^{(0)} \frac{\partial g(\xi)}{\partial \xi} - f(\xi) = 1/\xi^2. \tag{3.2.24}$$

3.3 Jump Conditions for Weak Shocks

The Rankine-Hugoniot jump conditions for the dusty gas may be written as Anand (2014a; 2014b)

$$\varrho = \left[\frac{\Gamma - 1 + 2Z_0}{\Gamma + 1} + \frac{2\,(1 - Z_0)}{\Gamma + 1}\left(\frac{a_0}{s}\right)^2\right]^{-1} \varrho_0, \tag{3.3.1}$$

$$\vartheta = \frac{2\,(1 - Z_0)}{\gamma + 1}\left[s^2 - a_0^2\right]\frac{1}{s}, \tag{3.3.2}$$

$$p = p_0 + \frac{2\,(1 - Z_0)}{\Gamma + 1}\left[s^2 - a_0^2\right]\varrho_0, \tag{3.3.3}$$

where the subscript "0" denotes the quantity evaluated in undisturbed region and s is the shock speed. Also p, ϑ, ϱ are the pressure, velocity and density in the disturbed region. Expanding the variables p and s in terms of ε similar to the equations (3.2.10 - 3.2.12), the zero order Rankine-Hugoniot jump relations may be expressed as

$$\varrho^{(0)} = \left[\frac{\Gamma - 1 + 2Z_0}{\Gamma + 1} + \frac{2\,(1 - Z_0)}{\Gamma + 1}\left(\frac{a_0}{s^{(0)}}\right)^2\right]^{-1} \varrho_0, \tag{3.3.4}$$

$$\vartheta^{(0)} = \frac{2\,(1 - Z_0)}{\Gamma + 1}\left[s^{(0)2} - a_0^2\right]\frac{1}{s^{(0)}}, \tag{3.3.5}$$

$$p^{(0)} = p_0 + \frac{2\,(1 - Z_0)}{\Gamma + 1}\left[s^{(0)2} - a_0^2\right]\varrho_0. \tag{3.3.6}$$

Also the first order Rankine–Hugoniot jump relations may be expressed as

$$\varrho_s^{(1)} = \frac{\frac{4(1-Z_0)}{\Gamma+1}\left(\frac{a_0}{s^{(0)}}\right)^2 \varrho_0}{\left[\frac{\Gamma-1+2Z_0}{\Gamma+1} + \frac{2(1-Z_0)}{\Gamma+1}\left(\frac{a_0}{s^{(0)}}\right)^2\right]^2}\frac{s^{(1)}}{s^{(0)}}, \tag{3.3.7}$$

$$\vartheta_s^{(1)} = \frac{2\,(1 - Z_0)}{\Gamma + 1}\frac{1}{s^{(0)}}\left[s^{(0)2} + a_0^2\right]\frac{s^{(1)}}{s^{(0)}}, \tag{3.3.8}$$

,

$$p_s^{(1)} = \frac{4\,(1 - Z_0)}{\Gamma + 1}s^{(0)}s^{(1)}\varrho_0, \tag{3.3.9}$$

where the subscript "s" stands for the value evaluated at the shock $\xi = \xi_s$. With the help of (3.2.10 - 3.2.12), the zeroth order speed of sound may be written as

$$a^{(0)} = \left(\Gamma p^{(0)} / \left(\varrho^{(0)}\left(1 - \theta\varrho^{(0)}\right)\right)\right)^{1/2}. \tag{3.3.10}$$

Also the first order sound speed may be written as

$$a^{(1)} = \frac{a^{(0)}}{2}\left[\frac{p^{(1)}}{p^{(0)}} - \delta\frac{\varrho^{(1)}}{\varrho^{(0)}}\right], \tag{3.3.11}$$

with $\delta = \left(1 - 2\theta\varrho^{(0)}\right) / \left(1 - \theta\varrho^{(0)}\right)$. From equations (3.3.8) and (3.3.9), the speed of sound at the shock is given by

$$\begin{aligned} a^{(1)} &= \frac{a^{(0)}}{2}\left[\frac{4\left(1 - Z_0\right)\varrho_0 {s^{(0)}}^2}{\left(\Gamma + 1\right)p_0 + 2\left(1 - Z_0\right)\left({s^{(0)}}^2 - a_0^2\right)\varrho_0} - \frac{4\delta\left(1 - Z_0\right)a_0^2}{\left(\Gamma - 1 + 2Z_0\right){s^{(0)}}^2 + 2\left(1 - Z_0\right)a_0^2}\right] \\ &\times \frac{\left({s^{(0)}}^2 - a_0^2\right)}{\left({s^{(0)}}^2 + a_0^2\right)}\frac{\vartheta_s^{(1)}}{\vartheta^{(0)}}. \end{aligned} \tag{3.3.12}$$

From equations (3.3.10) and (3.3.12), we have a relation between $f\left(\xi_s\right)$ and $g\left(\xi_s\right)$ as

$$f\left(\xi_s\right) = \alpha g\left(\xi_s\right), \tag{3.3.13}$$

where the constant α is given by the following relation

$$\begin{aligned} \frac{1}{\alpha} &= \frac{2a^{(0)}}{\vartheta^{(0)}}\left[\frac{\left(1 - Z_0\right)\varrho_0 {s^{(0)}}^2}{\left(\Gamma + 1\right)p_0 + 2\varrho_0\left(1 - Z_0\right)\left({s^{(0)}}^2 - a_0^2\right)} - \delta\frac{\left(1 - Z_0\right)a_0^2}{\left(\Gamma - 1 + 2Z_0\right){s^{(0)}}^2 + 2\left(1 - Z_0\right)a_0^2}\right] \\ &\times \left(\frac{{s^{(0)}}^2 - a_0^2}{{s^{(0)}}^2 + a_0^2}\right). \end{aligned} \tag{3.3.14}$$

With the help of equations(3.2.22) and (3.3.12), perturbation state equations (3.2.23) - (3.2.24) may be solved for the region $1 \leq \xi \leq s^{(0)}/\vartheta_p$.

TABLE 3.1: Numerical value of α with varying k_p at a / s = 0.50

k_p	β	γ	Ω	δ	$\frac{s}{\vartheta_p}$	$\frac{a^0}{\vartheta_p}$	$\frac{a_0}{\vartheta_p}$	α
0.0	0.5	1.66666	1000	0.00000	1.77778	1.2814	0.888889	2.0842
0.2	0.5	1.66666	1000	0.25000	1.70157	1.17808	0.850787	2.47471
0.4	0.5	1.66666	1000	0.66666	1.62013	1.06718	0.810063	3.10873

TABLE 3.2: Numerical value of α with varying k_p at a / s = 0.75

k_p	β	γ	Ω	δ	$\frac{s}{\vartheta_p}$	$\frac{a^{(0)}}{\vartheta_p}$	$\frac{a_0}{\vartheta_p}$	α
0.0	0.5	1.66666	1000	0.00000	3.04762	2.63114	2.28571	2.72098
0.2	0.5	1.66666	1000	0.25000	2.91698	2.47418	2.18774	3.27565
0.4	0.5	1.66666	1000	0.66666	2.77736	2.3064	2.08302	4.19054

TABLE 3.3: Numerical value of α with varying β at a / s =0.50

k_p	β	γ	Ω	δ	$\frac{s}{\vartheta_p}$	$\frac{a^{(0)}}{\vartheta_p}$	$\frac{a_0}{\vartheta_p}$	α
0.1	0.0	1.66666	1000	0.111111	1.77797	1.28172	0.888987	2.08294
0.1	0.5	1.66666	1000	0.111111	1.74031	1.23064	0.870153	2.25878
0.1	1.0	1.66666	1000	0.111111	1.70852	1.18745	0.854261	2.43343

TABLE 3.4: Numerical value of α with varying β at a / s =0.75

k_p	β	γ	Ω	δ	$\frac{s}{\vartheta_p}$	$\frac{a^{(0)}}{\vartheta_p}$	$\frac{a_0}{\vartheta_p}$	α
0.1	0.0	1.66666	1000	0.111111	3.04796	2.63158	2.28597	2.7194
0.1	0.5	1.66666	1000	0.111111	2.98338	2.55396	2.23754	2.96798
0.1	1.0	1.66666	1000	0.111111	2.9289	2.48846	2.19667	3.21638

TABLE 3.5: Numerical value of α with varying Ω at a / s = 0.50

k_p	β	γ	Ω	δ	$\frac{s}{\vartheta_p}$	$\frac{a^{(0)}}{\vartheta_p}$	$\frac{a_0}{\vartheta_p}$	α
0.1	0.5	1.66666	10	0.111111	1.75945	1.26246	0.879724	2.12154
0.1	0.5	1.66666	100	0.111111	1.74205	1.23353	0.871023	2.24557
0.1	0.5	1.66666	1000	0.111111	1.74031	1.23064	0.870153	2.25878

TABLE 3.6: Numerical value of α with varying Ω at a / s = 0.75

k_p	β	γ	Ω	δ	$\frac{s}{\vartheta_p}$	$\frac{a^{(0)}}{\vartheta_p}$	$\frac{a_0}{\vartheta_p}$	α
0.1	0.5	1.66666	10	0.111111	3.30162	2.59621	2.26215	2.79453
0.1	0.5	1.66666	100	0.111111	2.98636	2.5578	2.23977	2.95133
0.1	0.5	1.66666	1000	0.111111	2.98338	2.55396	2.23754	2.96798

The solution of the piston problem in the mixture of gas and dust particles may be reduced in two elementary solutions as Wen-rui (1985)

$$\pounds_1\left(f_1,,g_1\right)=0,\qquad \pounds_2\left(f_1,,g_1\right)=-\frac{1}{\xi_2},\qquad f_1(1)=0,\qquad g_1(1)=0,$$

and

$$\pounds_1\left(f_2,,g_2\right)=0,\qquad \pounds_2\left(f_2,,g_2\right)=0,\qquad f_2(1)=0,\qquad g_2(1)=1.$$

Since the equations (3.2.23) and (3.2.24) and relation (3.3.13) are linear, so the solution of plane piston problem in a dusty gas may be given as

$$f(\xi)=f_1(\xi)+Af_2(\xi),\qquad\qquad g(\xi)=g_1(\xi)+Ag_2(\xi),$$

where A is arbitrary constant, which is determined with the help of equation (3.3.13) as

$$A=\frac{f_1(\xi_s)-\alpha g_1(\xi_s)}{\alpha g_2(\xi_s)-f_2(\xi_s)}.$$

In figures, continuous and broken lines denote the function $f(\xi)$ and $g(\xi)$ respectively. Since the strength of the shock wave depends on $\vartheta_s^{(1)}$ i.e. $f(\xi_s)$, therefore $f(\xi_s)\approx 0$ shows that the strength of shock wave in a dusty gas changes due to applied gravity. An increasing nature of $f(\xi)$ near the piston shows that the kinetic energy of the dusty gas increases near the piston and decreasing nature of $f(\xi)$ at the shock $\xi=\xi_s$ shows that the kinetic energy of the dusty gas decreases at the shock. The monotonic decreasing nature of $g(\xi)$ shows that the internal energy between piston and shock wave of the dusty gas will exhaust to overcome the applied gravity. An increment in any one parameter among k_p, β and Ω causes to decrease the internal energy of the dusty gas between piston and shock wave. From fig.3.1 and fig.3.2, it is clear that an increase in the value of k_p at constant Ω and β causes to increase the kinetic energy

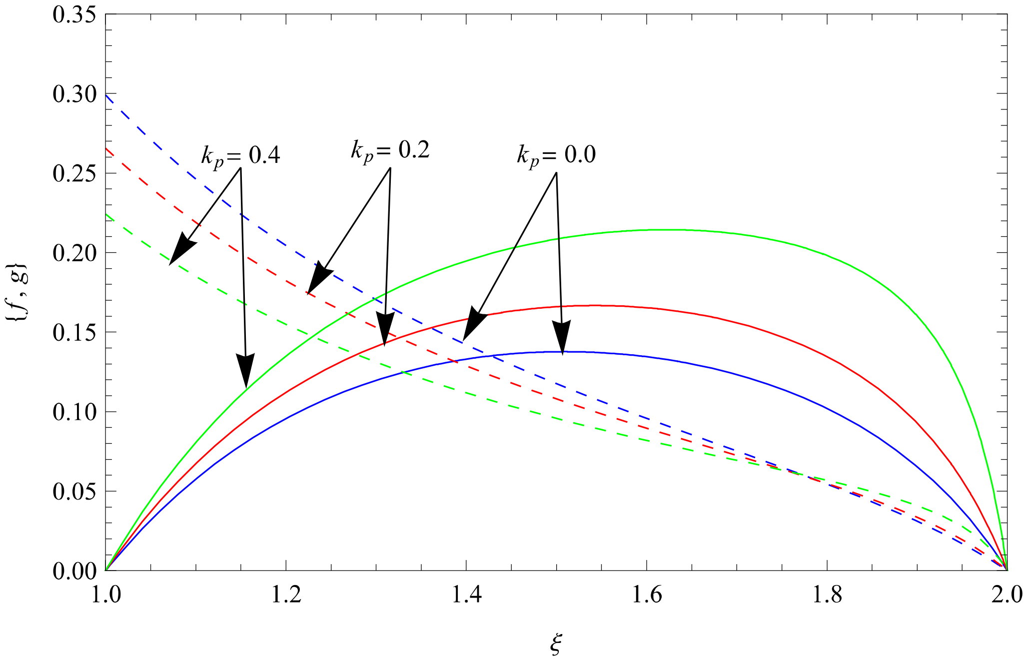

FIGURE 3.1: Profile of functions f and g for table–3.1.

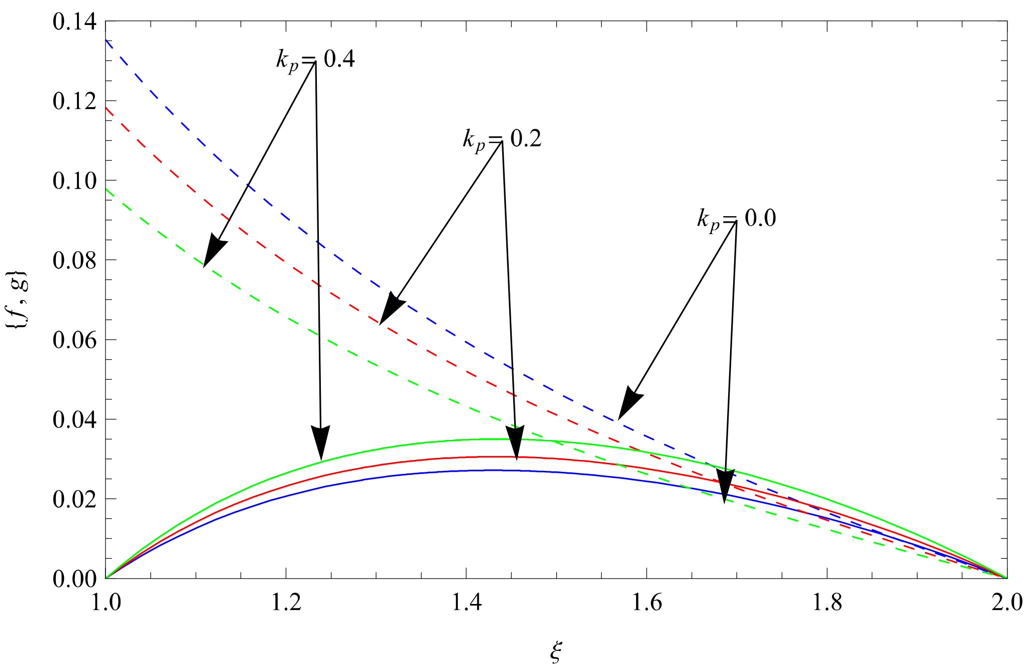

FIGURE 3.2: Profile of functions f and g for table table–3.2.

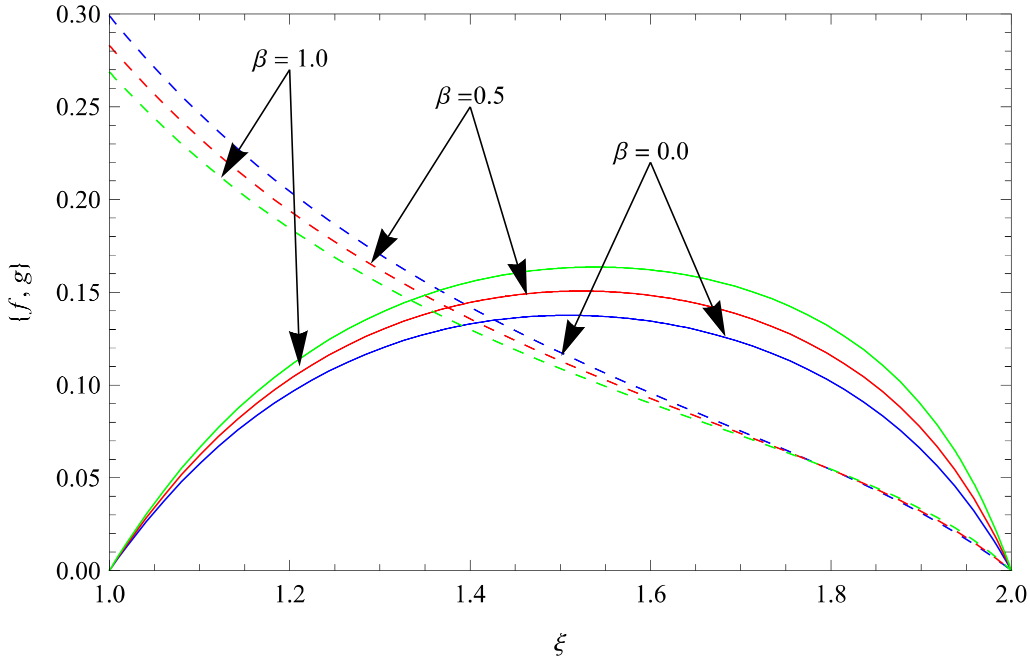

FIGURE 3.3: Profile of functions f and g for table–3.3.

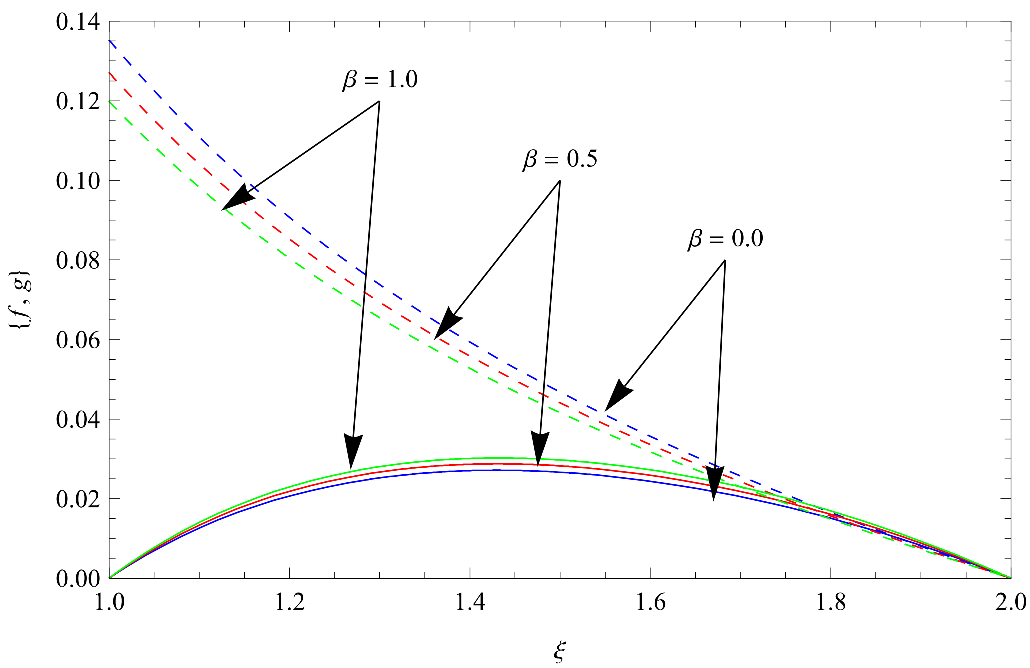

FIGURE 3.4: Profile of functions f and g for table–3.4.

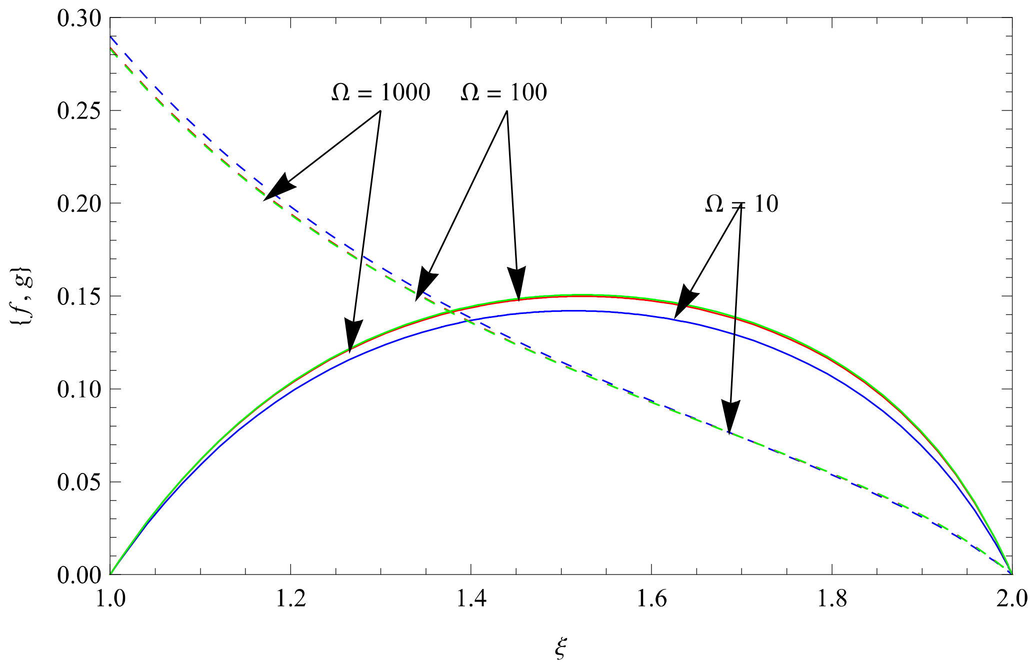

FIGURE 3.5: Profile of functions f and g for table–3.5.

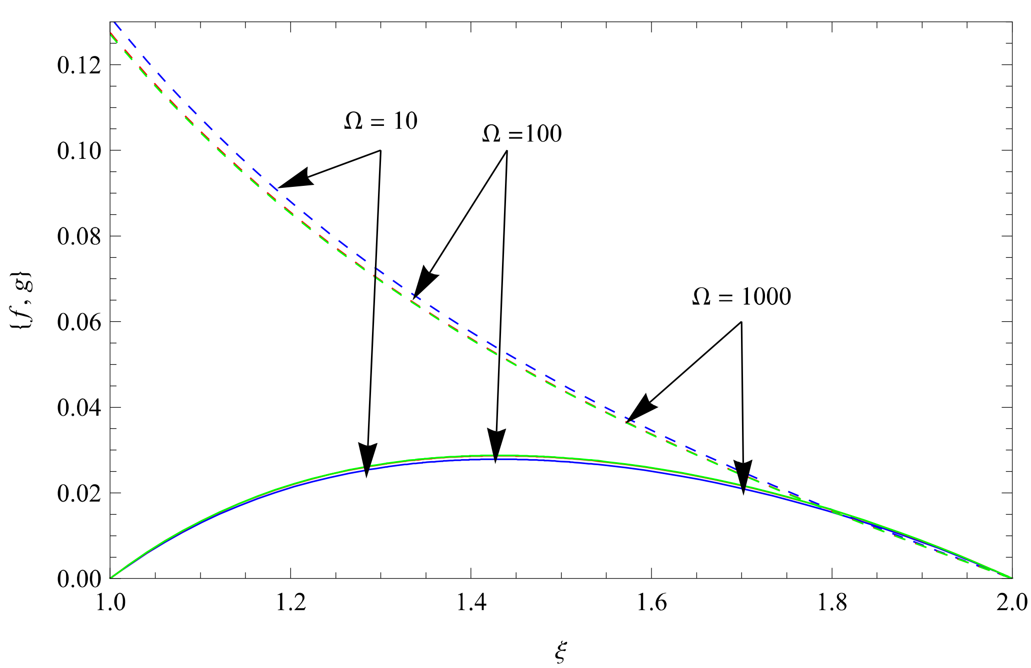

FIGURE 3.6: Profile of functions f and g for table–3.6.

of the dusty gas near the piston and to decrease at the shock wave. From fig.3.3 and fig.3.4, we infer that the increasing values of β at constant Ω and k_p causes to increase the kinetic energy of the dusty gas near the piston and decrease at the shock wave. From fig.3.5 and fig.3.6, it may be noted that the increasing values of Ω at constant k_p and β results in an increase in the kinetic energy of the dusty gas near the piston and to decrease at the shock wave.

3.4 Strong Shock Wave Approximation

In case of strong shock wave, the flow region becomes narrow and $\xi - 1 \leq \xi_s - 1 \ll 1$, as a result equations (3.2.23) and (3.2.24) may be written as

$$\frac{\Gamma - 1 + 2Z_0}{2(1 - Z_0)} a^{(0)} f'(\xi) - g(\xi) = 0, \tag{3.4.1}$$

$$\frac{2(1 - Z_0)}{\Gamma - 1 + 2Z_0} a^{(0)} g'(\xi) - f(\xi) = -\frac{1}{\xi^2}. \tag{3.4.2}$$

From equations (3.4.1) and (3.4.2), we have

$$\frac{\partial^2 f}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial f}{\partial \xi} - \frac{1}{{a^{(0)}}^2 \xi^2} f = -\frac{1}{{a^{(0)}}^2 \xi^4}. \tag{3.4.3}$$

The general solution of equation (3.4.3) may be written as

$$f(\xi) = c_1 \xi^{\omega} + c_2 \xi^{-\omega} + \frac{\omega^2}{(\omega^2 - 4)} \frac{1}{\xi^2}. \tag{3.4.4}$$

Using above equation in equation (3.4.1), we have

$$g(\xi) = \frac{(\Gamma - 1 + 2Z_0)}{2(1 - Z_0)\xi} \left[c_1 \xi^{\omega} + c_2 \xi^{-\omega} - \frac{2\omega}{(\omega^2 - 4)} \frac{1}{\xi^2} \right], \tag{3.4.5}$$

TABLE 3.7: Value of function $f(\xi_s)$ for varying parameters of dust particles

k_p	β	Ω	$f(\xi_s)$
0.0	0.5	1000	-0.497847
0.2	0.5	1000	-0.623254
0.4	0.5	1000	-0.841998
0.1	0.0	1000	-0.499713
0.1	0.5	1000	-0.553561
0.1	1.0	1000	-0.606005
0.1	0.5	10	-1.0468
0.1	0.5	100	-0.577153
0.1	0.5	1000	-0.553561

where $\omega = 1/a^{(0)}$ and c_1, c_2 are constants, which are determined with the help of equations (3.2.22) and (3.3.13) as

$$c_1 = \frac{\omega}{(4-\omega^2)}\left[\frac{\omega\left[\frac{\Gamma-1+2Z_0}{2}\alpha+(1-Z_0)\,\xi_s\right]\xi_s^{-\omega}-\left[(\Gamma-1+2Z_0)\,\alpha+(1-Z_0)\,\omega\xi_s\right]\xi_s^{-2}}{\left[\frac{\Gamma-1+2Z_0}{2}\alpha-(1-Z_0)\,\xi_s\right]\xi_s^{\omega}+\left[\frac{\Gamma-1+2Z_0}{2}\alpha+(1-Z_0)\,\xi_s\right]\xi_s^{-\omega}}\right], \tag{3.4.6}$$

$$c_2 = \frac{\omega}{(4-\omega^2)}\left[\frac{\omega\left[\frac{\Gamma-1+2Z_0}{2}\alpha-(1-Z_0)\,\xi_s\right]\xi_s^{\omega}-\left[(\Gamma-1+2Z_0)\,\alpha+(1-Z_0)\,\omega\xi_s\right]\xi_s^{-2}}{\left[\frac{\Gamma-1+2Z_0}{2}\alpha-(1-Z_0)\,\xi_s\right]\xi_s^{\omega}+\left[\frac{\Gamma-1+2Z_0}{2}\alpha+(1-Z_0)\,\xi_s\right]\xi_s^{-\omega}}\right]. \tag{3.4.7}$$

Using above results in (3.2.21) the first order solutions are given as

$$\vartheta^{(1)} = \frac{1}{x}\left[c_1\left(\frac{x}{t}\right)^{1+\omega}+c_2\left(\frac{x}{t}\right)^{1-\omega}+\frac{\omega^2}{(\omega^2-4)}\left(\frac{t}{x}\right)^2\right], \tag{3.4.8}$$

$$a^{(1)} = \frac{(\Gamma-1+2Z_0)}{2\,(1-Z_0)\,x}\left[c_1\left(\frac{x}{t}\right)^{\omega}-c_2\left(\frac{x}{t}\right)^{-\omega}-\frac{2\omega}{(\omega^2-4)}\left(\frac{t}{x}\right)^2\right]. \tag{3.4.9}$$

From the above table it is clear that $f(\xi_s)$ is negative for the dusty gas in which volume fraction of dust particles is less than five percentage of the total volume of gas. Hence strength of shock wave becomes weak. it is also observed here that by increasing the value of any one parameter among β, k_p and Ω causes to further weaken the strength of shock wave. From equation (3.4.5), we have $g'(\xi) < 0$ therefore $g(\xi)$ is monotonic decreasing function of ξ, hence internal energy will exhaust. Also it is observed that in case of strong shock wave the effect of presence of dust particles in the gas have similar behaviour as in case of weak shock wave. From equation (3.4.1), we have

$$f'(1) = \frac{2(1-Z_0)}{(\Gamma-1+2Z_0)a^{(0)}}g(1),$$

which shows that the effect of presence of dust particles is to accelerate the wave motion if the sign of $g(1)$ is positive and deceleration will occur if $g(1)$ is negative.

From equation (3.4.2), we have

$$g'(1) = -\frac{(\Gamma-1+2Z_0)}{2(1-Z_0)a^{(0)}}(1-f(1)).$$

Negative sign of $g\prime(1)$ shows that the internal energy of the dusty gas will exhaust and an increment in the value of any one parameters among β, k_p and Ω will contribute in rapid decrease of internal energy of the dusty gas.

3.5 Results and Discussion

In the present section we discuss the structure of the shock wave front. The characteristic lines for the system of equations $(3.2.1-3.2.3)$ are given as

$$\frac{dx}{dt} = \vartheta - a, \qquad \vartheta, \qquad \vartheta + a. \tag{3.5.1}$$

Substituting the value of ϑ and a from equations (3.2.9) and (3.2.11), we have

$$\frac{dx}{dt} = \vartheta^{(0)} - a^{(0)} + \varepsilon\left(\vartheta^{(1)} - a^{(1)}\right), \tag{3.5.2}$$

$$\frac{dx}{dt} = \vartheta^{(0)} + \varepsilon\vartheta^{(1)}, \tag{3.5.3}$$

$$\frac{dx}{dt} = \vartheta^{(0)} + a^{(0)} + \varepsilon\left(\vartheta^{(1)} + a^{(1)}\right). \tag{3.5.4}$$

From the equations $(3.5.2 - 3.5.4)$ it is clear that the characteristics are not straight lines in the presence of applied gravity. The inclination of second and third characteristics near the piston increases in absence of gravity whereas tendency of first characteristics is opposite to second and third characteristics.
We now discuss the effect of presence of dust particles in the gas on the nature of shock front. In the presence of weak gravitational field in a dusty gas, the position of shock front is given as

$$\frac{x}{t} = s + \varepsilon s', \tag{3.5.5}$$

which shows that the inclination of shock front decreases due to presence of gravitational field because the strength of shock wave becomes weak due to applied gravity. Since increment in the dust particle parameters will participate in the strength of shock wave and causes to weaken it, so increment in the dust particle parameter will contribute in decreasing the inclination of shock front. To discuss the piston problem in a dusty gas with boundary conditions at the piston

$$\vartheta|_{\xi=1} = \vartheta_p, \qquad a|_{\xi=1} = a_p,$$

which requires the perturbation state boundary conditions given as

$$f(1) = 0, \qquad g(1) = 0. \tag{3.5.6}$$

The solution profiles of equations $(3.2.23 - 3.2.24)$ together with boundary condition (3.5.6) are shown in figures $(3.7 - 3.12)$. The result shows that the internal energy of shock wave front in a dusty gas will exhaust more rapidly with an increase in the value of k_p, β and Ω and shock wave becomes weak due to applied gravity.

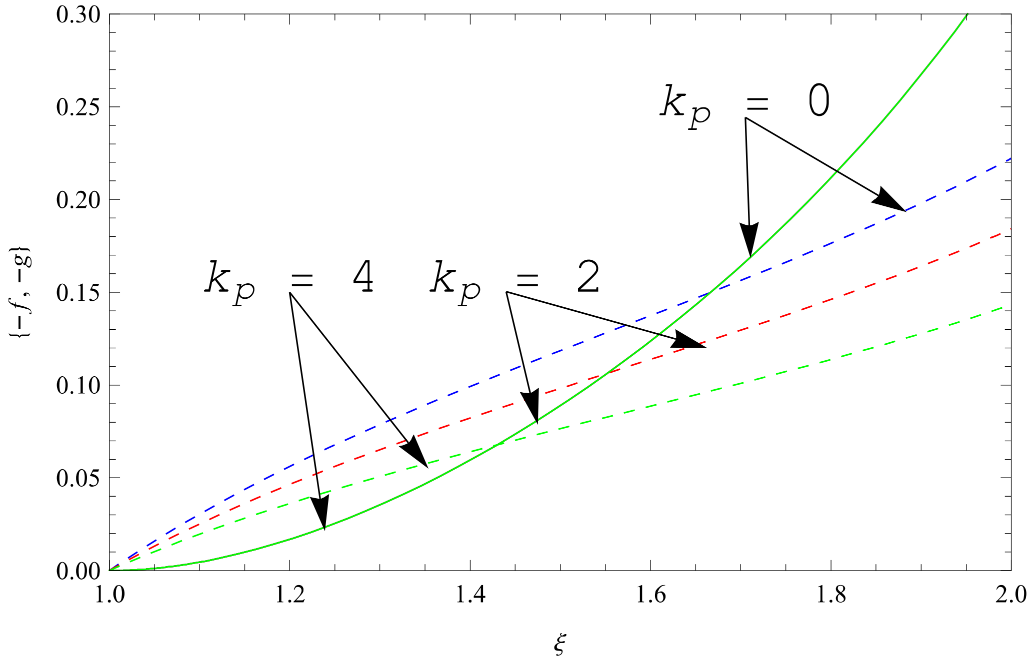

FIGURE 3.7: Profile of functions f and g for different values of k_p.

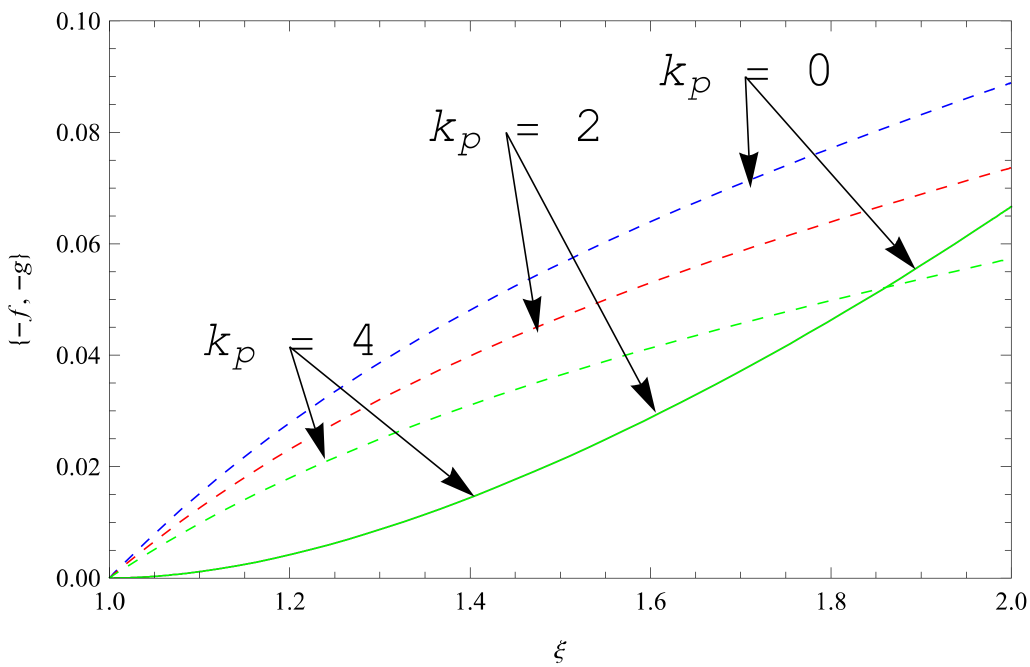

FIGURE 3.8: Profile of functions f and g for different values of k_p.

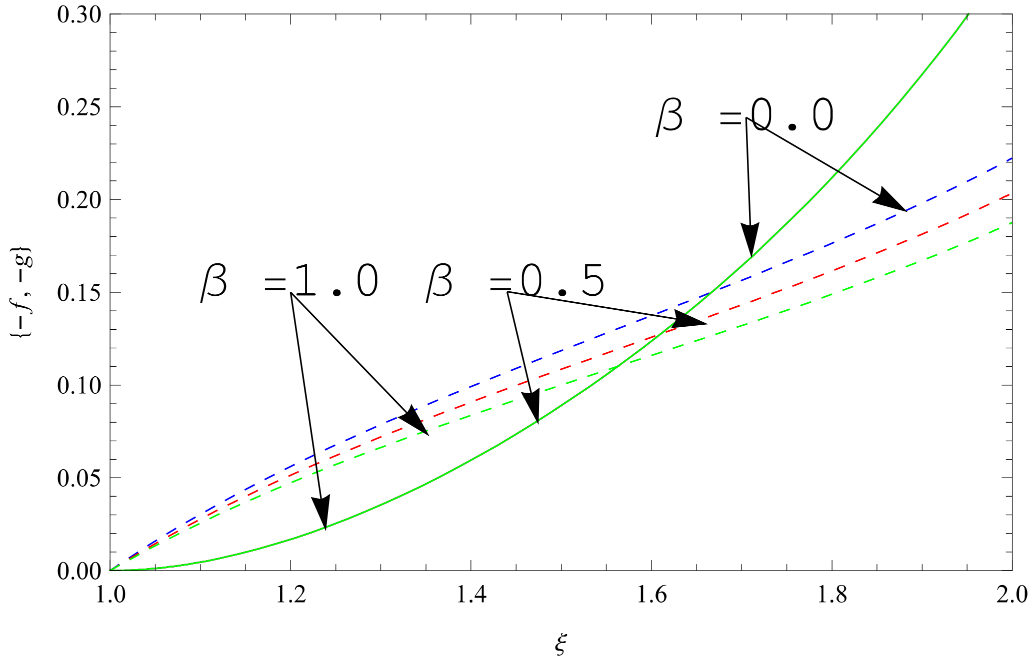

FIGURE 3.9: Profile of functions f and g for different values of β.

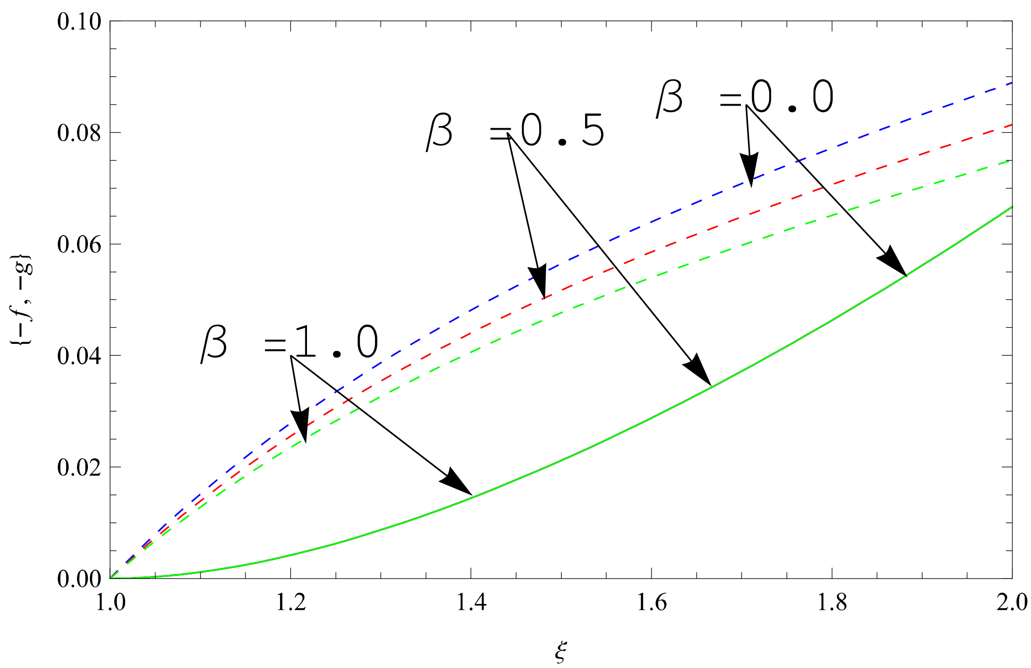

FIGURE 3.10: Profile of functions f and g for different values of β.

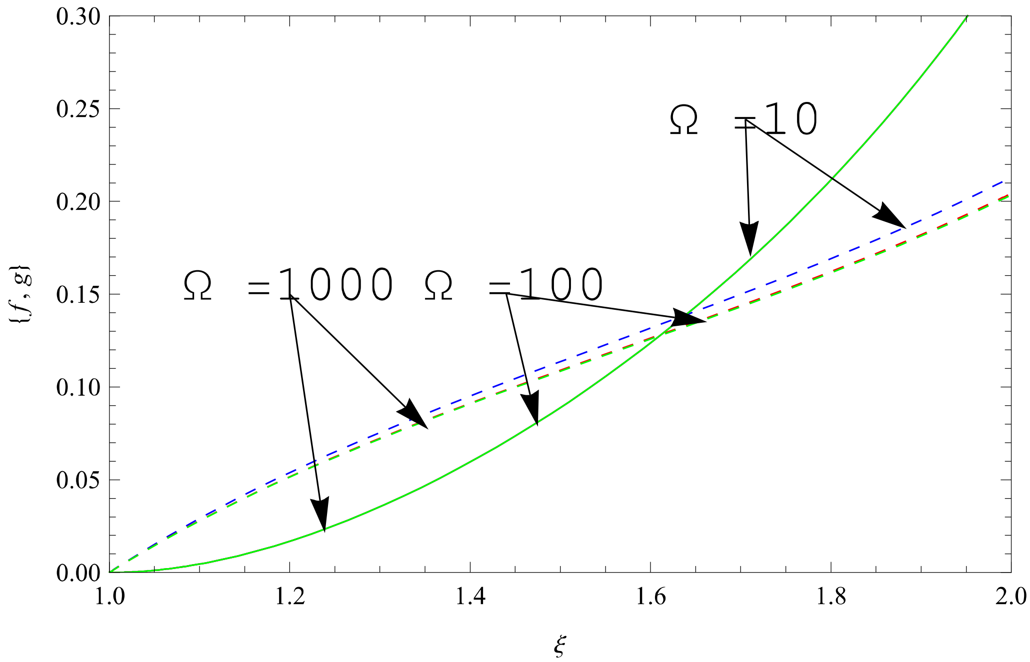

FIGURE 3.11: Profile of functions f and g for different values of Ω at $a^{(0)}/\vartheta_p = 2$.

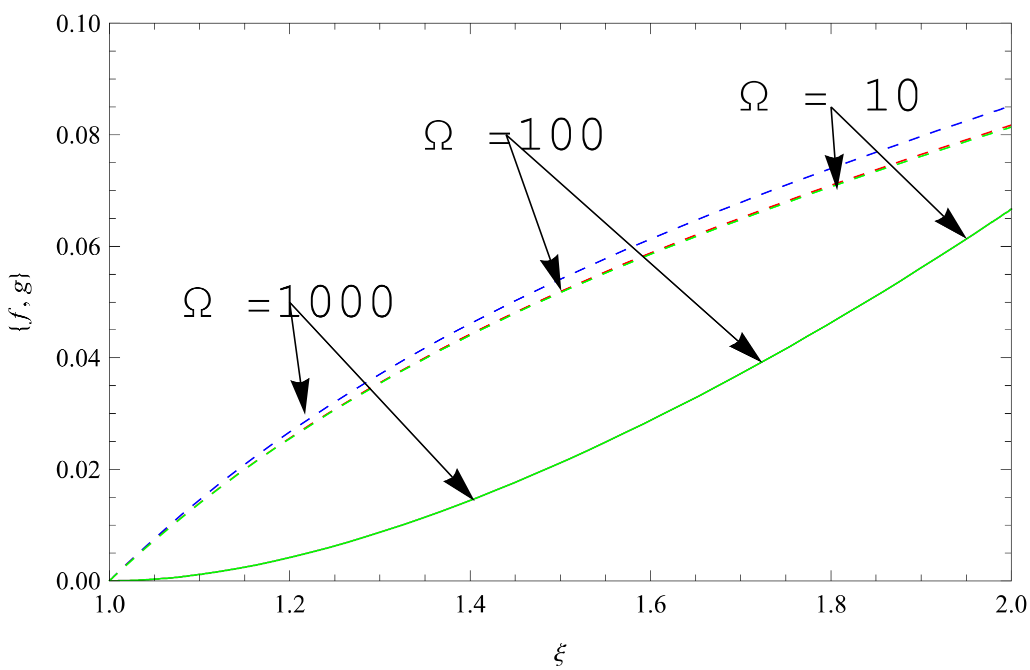

FIGURE 3.12: Profile of functions f and g for different values of Ω at $a^{(0)}/\vartheta_p = 4$.

3.6 Conclusions

In the present work, the motion of plane piston in a dusty gas under the influence of weak gravitational field is discussed and following conclusions may be drawn from the above discussion.

1. In case of weak shock wave, the internal energy of the dusty gas between piston and shock wave decreases. An increment in the value of any one parameter among k_p, β and Ω causes to decrease the internal energy of the dusty gas between piston and shock wave.

2. In case of weak shock wave, the kinetic energy of the dusty gas increases near the piston and decreases at the shock wave.

3. In case of weak shock wave, an increment in the value of any one parameter among k_p, β and Ω causes to increase the kinetic energy of the dusty gas near the piston and to decrease at the shock.

4. An increment in the value of any one parameter among k_p, β and Ω results to further weaken the strong shock wave.

5. Internal energy of strong shock wave will exhaust due to applied gravity and an increment among any one parameter k_p, β and Ω will contribute in rapid decrease in internal energy of the dusty gas.

6. It is observed that the solution of the plane piston problem with weak gravitational field in a dusty gas reduces to the solution presented by Wen-rui (1985) for $\theta = 0$.

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