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Characterizations of certain Hankel transform involving Riemann–Liouville fractional derivatives

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Abstract
In this paper, the relation between two dimensional fractional Fourier transform and fractional Hankel transform is discussed in terms of radial functions. Various operational properties of Hankel transform and fractional Hankel transform are studied involving Riemann–Liouville fractional derivatives. The application of fractional Hankel transform is given in networks with time varying parameters.

Keywords Hankel transform · Fractional Hankel transform · Schwartz space · Fractional derivatives and integrals · Lizorkin space · Operational relations

Mathematics Subject Classification 26A33 · 42A50 · 42A38 · 46F05 · 46F12

1 Introduction

Riemann–Liouville fractional derivatives played an important role to solve many problems of applied mathematics, physics and engineering sciences. Yang et al. (2015) introduced a new fractional derivative without singular kernel and considered the potential application for modeling the steady heat-conduction problem. Yang et al. (2015) provided the method of integral transforms via local fractional calculus to solve various local fractional ordinary and local fractional partial differential equations. They presented the basics of the local fractional derivative operators and proved some new results of local integral transforms. Further, Yang (2016) proposed a class of the fractional derivatives of constant and variable orders and modeled fractional-order relaxation equations of constant and variable orders in the sense

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of Caputo type, from mathematical point of view. The results have much importance in description of the complex phenomenon arising in heat transfer. Yang et al. (2017) observed the new general fractional derivatives involving the kernels of the extended Mittag-Leffler type function and analyzed the mathematical models for the anomalous diffusion of fractional order. Further a new family of local fractional PDEs and linear, quasi-linear, semilinear and nonlinear local fractional PDEs were investigated by Yang et al. (2017). After that, Yang et al. (2017) discussed a family of special functions via the celebrated Mittag-Leffler function defined on Cantor sets and the obtained results were more useful for describing the characteristics of fractal special functions. Further Debbouche and Antonov (2017) presented non-differentiable analytical solutions of diffusion equations arising in fractal heat transfer.

Ortigueira and Machado (2015) discussed the concepts behind the formulation of operators, interpreted as fractional derivatives or fractional integrals. They accessed the Grünwald Letnikov, Riemann Liouville and Caputo fractional derivatives and the Riesz potential. They also obtained a Liebnitz rule for the Riesz potential. Abdeljawad and Torres (2017) considered symmetric duality of Caputo fractional derivatives in discrete version to relate left and right Riemann Liouville and Caputo fractional derivatives. They provide an evidence to the fact that in case of right fractional differences, one has to mix between nabla and delta operators. They derived right fractional summation by parts formula and left fractional difference Euler–Lagrange equations for discrete fractional variational problems, whose Lagrangians depend on right fractional differences. Baleanu et al. (2017) extended the definition of the fractional derivative operator of the Riemann Liouville and discussed its properties, by using generalized beta function. They also established some relatons to extended special functions of two or three variables, with the help of generating functions. Karite et al. (2018) investigated exact enlarged controllability for time fractional diffusion systems of Riemann Liouville type. By using the Hilbert uniqueness method, they proved exact enlarged controllability for both cases of zone and pointwise actuators. They gave a penalization method and characterized the minimum energy control. Baleanu et al. (2010) studied fractional systems and showed applications of fractional differentiation in nanotechnology.

Hankel transformation is an important tool to solve many problems of fractional derivatives and partial derivatives. Sneddon (1995) studied Fourier transform in terms of radial functions and found a relation between Fourier transform and Hankel transform in terms of radial functions. This relation played an important role for our present investigation. Torre (2008) proposed the fractionalisation of certain types of Hankel transform and proved the applicability of resulting transforms in connection with the evolution problems. Duffy (2004) used Hankel transform theory to solve many problems of partial differential equations like, elastic wave equation, heat equation, Laplace's equation, Poisson's equation etc. and some mixed boundary value problems. Luchko et al. (2008) introduced a new definition of fractional Fourier transform of real order $\alpha, 0 < \alpha \leq 1$ and studied its important properties, including the inversion formula and the operational relations for the fractional derivatives. They also showed the applications of the aforesaid transform for solving some model partial differential equations of fractional order.

Motivated from the above results, our main aim of this paper is to discuss the following objectives:

1. The relation between two dimensional fractional Fourier transform and fractional Hankel transform in terms of radial functions.
2. To introduce the fractional Hankel transform of order $\alpha, 0 < \alpha \leq 1$.
3. To study properties of Hankel transform and fractional Hankel transform with the help of Riemann Liouville fractional derivatives.

4. To find an application based on the theory of fractional Hankel transform, in networks with time varying parameters.

Now we give some definitions and properties, which are useful for our present work: From Luchko et al. (2008), let $V(\mathbb{R})$ be the set of all functions $v \in S$ satisfying

$$\left.\frac{\mathrm{d}^n v}{\mathrm{d}x^n}\right|_{x=0} = 0, \ (n = 0, 1, 2, \ldots).$$

The Lizorkin space $\Phi(\mathbb{R})$ is defined as the Fourier pre-image of the space $V(\mathbb{R})$ in the space S,

$$\Phi(\mathbb{R}) = \{\phi \in S : \mathcal{F}\phi \in V(\mathbb{R})\}.$$

The fractional Fourier transform of order α of a function u belonging to the Lizorkin space is defined by Kilbas et al. (2010) as :

$$(\mathcal{F}_\alpha \phi)(\omega) = \int_{-\infty}^{+\infty} e^{\mathrm{i}\,\mathrm{sign}(\omega)|\omega|^{\frac{1}{\alpha}} t} \phi(t)\mathrm{d}t. \tag{1.1}$$

A radial function is a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ satisfying $\phi(x, y) = \phi(|x - y|)$ for points y in discrete subset of $\mathbb{R}^2$.

The Hankel transformation of a function $\phi \in L^1(0, \infty)$ was defined by Zemanian (1968) to the distributions in the form:

$$h_\mu(\phi)(\omega) = \int_0^\infty (\omega x)^{\frac{1}{2}} J_\mu(\omega x)\phi(x)\mathrm{d}x, \ \omega \in (0, \infty), \ \mu \geq -\frac{1}{2}. \tag{1.2}$$

Let $\phi \in L^1(0, \infty)$ and $h_\mu \phi \in L^1(0, \infty)$. Then the inversion formula of Hankel transform is defined by

$$\phi(t) = \int_0^\infty (\omega t)^{\frac{1}{2}} J_\mu(\omega t)(h_\mu \phi)(\omega)\mathrm{d}\omega, \tag{1.3}$$

where J_μ is the Bessel's function of first kind of order μ.

Motivated from the work of Zemanian (1968), Altenburg (1982) introduced the space of type $\mathcal{H}$, consisting of all smooth complex functions ϕ on $(0, \infty)$ such that

$$\gamma_{p,q}(\phi) = \sup_{x \in (0,\infty)} (1 + x^2)^p \left| \left(\frac{1}{x} D\right)^q \phi(x)\right| < \infty, \tag{1.4}$$

for every $p, q \in \mathbb{N}$.

The topology on space $\mathcal{H}$ is considered to be associated with the family $\gamma_{p,q}$ of semi-norms.Thus h_μ is an automorphism on $\mathcal{H}$. Belhadj and Betancor (2002) showed that the space $\mathcal{H}$ coincides with the Schwartz space of all even functions $\mathcal{S}_{\text{even}}$.

From Zemanian (1968), Prudnikov et al. (1986), Samko et al. (1993), Luke (1969) and Luchko et al. (2008) we give some conceptual remarks about the Lizorkin space associated with Hankel transform, Riemann Liouville fractional derivatives and integrals and other properties, which are useful in our present paper:

Let $U(\mathbb{R}^+)$ be the space, defined as the space of all functions $u \in \mathcal{S}_{\text{even}}$, satisfying:

$$\frac{\mathrm{d}^n u}{\mathrm{d}x^n}|_{x=0} = 0, \quad n = 0, 1, 2, \ldots$$

The Lizorkin space $\Psi(\mathbb{R}^+)$ is defined as the space, which is the Hankel pre-image of the set $U(\mathbb{R}^+)$ in the space $\mathcal{S}_{\text{even}}$,

$$\Psi(\mathbb{R}^+) = \{\phi \in \mathcal{S}_{\text{even}} : h_\mu(\phi) \in U(\mathbb{R}^+)\}. \tag{1.5}$$

Further, for any function $\phi \in \Psi(\mathbb{R}^+)$,the following orthogonality condition is satisfied:

$$\int_0^\infty x^n \phi(x)\mathrm{d}x = 0, \quad n = 0, 1, 2, \dots \tag{1.6}$$

The other properties, which are useful in this paper are also given:
(I) For a function $\phi \in \Psi(\mathbb{R}^+)$ the fractional derivative D_β^α is defined as,

$$D_\beta^\alpha \phi(x) = (1-\beta)(D_{0+}^\alpha \phi)(x) - \beta(D_-^\alpha \phi)(x), \quad 0 < \alpha \le 1, \beta \in \mathbb{R}, \tag{1.7}$$

where D_{0+}^α and D_-^α are Riemann Liouville fractional derivatives on the positive real axis,

$$(D_{0+}^\alpha \phi)(x) = \frac{\mathrm{d}}{\mathrm{d}x}(I_{0+}^{1-\alpha}\phi)(x) \tag{1.8}$$

and

$$(D_-^\alpha \phi)(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)(I_-^{1-\alpha}\phi)(x), \tag{1.9}$$

where I_{0+}^α is the Riemann Liouville fractional integral operator

$$(I_{0+}^\alpha \phi)(x) = \frac{1}{\Gamma\alpha}\int_0^x (x-t)^{\alpha-1}\phi(t)\mathrm{d}t \tag{1.10}$$

and I_-^α is the Riemann Liouville fractional integral operator

$$(I_-^\alpha \phi)(x) = \frac{1}{\Gamma\alpha}\int_x^\infty (t-x)^{\alpha-1}\phi(t)\mathrm{d}t. \tag{1.11}$$

(II) Let $f, g \in L^1(0,\infty)$ then the following formulas for the fractional Riemann Liouville derivatives and integrals hold Samko et al. (1993),

$$\int_0^\infty f(x)(D_{0+}^\alpha g)(x)\mathrm{d}x = \int_0^\infty (D_-^\alpha f)(x)g(x)\mathrm{d}x \tag{1.12}$$

and

$$\int_0^\infty f(x)(I_{0+}^\alpha g)(x)dx = \int_0^\infty (I_-^\alpha f)(x)g(x)\mathrm{d}x. \tag{1.13}$$

(III) From Samko et al. (1993) I_-^α has the following property

$$(I_-^\alpha \phi)(x) = (\cos\alpha\pi\, I_{0+}^\alpha + \sin\alpha\pi\, S I_{0+}^\alpha)(\phi)(x), \tag{1.14}$$

where S is the singular operator, defined as

$$(S\phi)(x) = \frac{1}{\pi}\int_0^\infty \frac{\phi(t)}{t-x}\mathrm{d}t. \tag{1.15}$$

(IV) From Luke (1969), the hypergeometric function ${}_pF_{p+1}$ has the following relation with ${}_0F_1$

$$
{}_pF_{p+1}\left(\begin{array}{l} a_p+r \\ b_p+r\ ,\ \beta+r \end{array};z\right) = {}_0F_1(-;\beta+r;z) + \frac{f_1 z}{r(\beta+r)} \\ {}_0F_1(-;\beta+r+1;z) + O(r^{-3}), \tag{1.16}
$$

where $f_1 = \sum_{j=1}^{p}(a_j - b_j)$, z is fixed and r is sufficiently large and positive.
(V) For $c, r, y > 0$ and $-Re\nu < Re\beta < r + \frac{3}{2}$, from Prudnikov et al. (1986), we have

$$
\int_0^\infty \frac{x^{\beta-1}}{x^r - y^r} J_\nu(cx)\mathrm{d}x = X(0), \tag{1.17}
$$

where

$$
X(\epsilon) = \frac{-\pi c^\nu y^{\beta+\nu-r}}{2^\nu r \Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{k!(\nu+1)_k(2k+\nu)^\epsilon} cot\left(\frac{2k+\beta+\nu}{r}\right)\pi \\ \times \left(-\frac{c^2y^2}{4}\right)^k + \frac{1}{2}\left(\frac{2}{c}\right)^{\beta-r} \sum_{k=0}^{\infty} \frac{1}{(r+rk-\beta)^\epsilon} \\ \times \frac{\Gamma\left(\frac{\nu+\beta-r-rk}{2}\right)}{\Gamma\left(\frac{\nu-\beta+r+rk}{2}+1\right)} \left(\frac{cy}{2}\right)^{rk}. \tag{1.18}
$$

(VI) The relation between J_ν and ${}_0F_1$ is given as

$$
\Gamma(\nu+1)J_\nu(t) = \left(\frac{t}{2}\right)^\nu {}_0F_1\left(-;\nu+1;-\frac{t^2}{4}\right). \tag{1.19}
$$

(VII) We have

$$
(\beta)_{2r} = \frac{\Gamma(\beta+2r)}{\Gamma\beta} \tag{1.20}
$$

and

$$
(\lambda)_{2r} = 2^{2r}\left(\frac{\lambda}{2}\right)_r \left(\frac{\lambda+1}{2}\right)_r, \tag{1.21}
$$

for $2r \in \mathbb{N}_0$.
(VIII) From Zemanian (1968), we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}(t^{-\mu}J_\mu(\omega t)) = -\omega t^{-\mu}J_{\mu+1}(\omega t). \tag{1.22}
$$

The entire paper is organized by the following way:

Section 1 is introductory, in which various definitions, properties and results are given.

In Sect. 2, the theory of fractional Hankel transform is introduced with the help of relation between two dimensional fractional Fourier transform and fractional Hankel transform, in terms of radial functions.

In Sect. 3, operational properties of Hankel transform and fractional Hankel transform are discussed, with the help of Riemann Liouville fractional derivatives.

In Sect. 4, the application of fractional Hankel transform in networks with time varying parameters is given.

In the last section, graphical representations of a signal in time domain and frequency domain are studied, by exploiting the theory of fractional Hankel transform.

2 Fractional Fourier transform of radial functions

In this section, we study two dimensional fractional Fourier transform of radial functions and obtain various results. With the help of these results we define the fractional Hankel transform.

Theorem 2.1 *The two dimensional fractional Fourier transform can be expressed in terms of fractional Hankel transform as*

$$(\mathcal{F}_\alpha f)(\omega_1, \omega_2) \leq \frac{\rho^{\frac{-1}{2\alpha}}}{\alpha} h_0^\alpha(F(r))(\rho), \tag{2.1}$$

where $F(r) = r^{\frac{1}{2}} f(r)$, *and* r, ρ *are radial functions.*

Proof

$$\begin{aligned}
&(\mathcal{F}_\alpha f)(\omega_1, \omega_2)\\
&= \frac{1}{2\pi\alpha} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(\operatorname{sign}(\omega_1)|\omega_1|^{\frac{1}{\alpha}} x_1 + \operatorname{sign}(\omega_2)|\omega_2|^{\frac{1}{\alpha}} x_2)} f\left(\sqrt{x_1^2 + x_2^2}\right) \mathrm{d}x_1 \mathrm{d}x_2\\
&= \frac{1}{2\pi\alpha} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(\omega_1|\omega_1|^{\frac{1}{\alpha}-1} x_1 + \omega_2|\omega_2|^{\frac{1}{\alpha}-1} x_2)} f\left(\sqrt{x_1^2 + x_2^2}\right) \mathrm{d}x_1 \mathrm{d}x_2.
\end{aligned}$$

If we put $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $\omega_1 = \rho\cos\phi$, $\omega_2 = \rho\sin\phi$, we obtain

$$\begin{aligned}
&(\mathcal{F}_\alpha f)(\omega_1, \omega_2)\\
&= \frac{1}{2\pi\alpha} \int_0^{2\pi}\int_0^{\infty} e^{i(r\cos\theta\rho\cos\phi|\rho\cos\phi|^{\frac{1}{\alpha}-1} + r\sin\theta\rho\sin\phi|\rho\sin\phi|^{\frac{1}{\alpha}-1})} f(r) r \mathrm{d}\theta \mathrm{d}r\\
&\leq \frac{1}{2\pi\alpha} \int_0^{2\pi}\int_0^{\infty} e^{i(r\cos\theta\rho\cos\phi\rho^{\frac{1}{\alpha}-1} + r\sin\theta\rho\sin\phi\rho^{\frac{1}{\alpha}-1})} f(r) r \mathrm{d}\theta \mathrm{d}r\\
&= \frac{1}{2\pi\alpha} \int_0^{2\pi}\int_0^{\infty} e^{ir\rho^{\frac{1}{\alpha}}\cos(\theta-\phi)} f(r) r \mathrm{d}\theta \mathrm{d}r.
\end{aligned}$$

Changing the order of integration by Fubini's theorem, we get

$$(\mathcal{F}_\alpha f)(\omega_1, \omega_2) \leq \frac{1}{2\pi\alpha} \int_0^{\infty} \left(\int_0^{2\pi} e^{ir\rho^{\frac{1}{\alpha}}\cos(\theta-\phi)} \mathrm{d}\theta \right) r f(r) \mathrm{d}r$$

$$(\mathcal{F}_\alpha f)(\omega_1, \omega_2) \leq \frac{1}{\alpha} \int_0^{\infty} r J_o(\rho^{\frac{1}{\alpha}} r) f(r) \mathrm{d}r$$

$$(\mathcal{F}_\alpha f)(\omega_1,\omega_2) \leq \frac{\rho^{\frac{-1}{2\alpha}}}{\alpha}\int_0^\infty (r\rho^{\frac{1}{\alpha}})^{\frac{1}{2}} J_o(\rho^{\frac{1}{\alpha}}r) r^{\frac{1}{2}} f(r)\mathrm{d}r.$$

Set $F(r) = r^{\frac{1}{2}} f(r)$, we have

$$(\mathcal{F}_\alpha f)(\omega_1,\omega_2) \leq \frac{\rho^{\frac{-1}{2\alpha}}}{\alpha} h_0^\alpha(F(r))(\rho).$$

□

Similarly, by using the technique of Sneddon (1995, pp. 63–65), we can prove the following relation

$$(\mathcal{F}_\alpha f)(\omega_1,\omega_2,\ldots,\omega_n) \leq \frac{\rho^{\frac{-1}{\alpha}(\frac{n}{2}-1)}}{\alpha^{\frac{n}{2}}}\int_0^\infty r(r^{\frac{n}{2}-1} f(r)) J_{\frac{n}{2}-1}(\rho^{\frac{1}{\alpha}}r)\mathrm{d}r.$$

Setting $\phi(r) = r^{\frac{n}{2}-1} f(r)$, $\nu = \frac{n}{2} - 1$, $\phi(\rho) = \rho^{\frac{1}{\alpha}(\frac{n}{2}-1)}(\mathcal{F}_\alpha f)(\omega_1,\omega_2,\ldots,\omega_n)$, we find

$$\phi(\rho) \leq \frac{1}{\alpha^{\frac{n}{2}}}\int_0^\infty r J_\nu(\rho^{\frac{1}{\alpha}}r)\phi(r)\mathrm{d}r. \tag{2.2}$$

From the above two expressions (2.1) and (2.2), we can define fractional Hankel transform in the following way:

The fractional Hankel transform of function $\phi \in \Psi(\mathbb{R}^+)$ of order α, $0 < \alpha \leq 1$ is defined by,

$$(h_\mu^\alpha \phi)(\omega) = \int_0^\infty (\omega^{\frac{1}{\alpha}}t)^{\frac{1}{2}} J_\mu(\omega^{\frac{1}{\alpha}}t)\phi(t)\mathrm{d}t, \quad \mu \geq -\frac{1}{2}. \tag{2.3}$$

For $\alpha = 1$,we get Eq. (1.2),which is the definition of conventional Hankel transform .

The corresponding inverse fractional Hankel transform is given by

$$\phi(t) = \frac{1}{\alpha}\int_0^\infty (\omega^{\frac{1}{\alpha}}t)^{\frac{1}{2}} J_\mu(\omega^{\frac{1}{\alpha}}t)(h_\mu^\alpha\phi(\omega))\omega^{\frac{1}{\alpha}-1}\mathrm{d}\omega. \tag{2.4}$$

Example Fractional Hankel transform of the function $f(t) = t^{\mu+\frac{1}{2}}e^{-at^2}$ can be evaluated as following:

From Erdélyi et al. (1954), using Hankel transform of the function $f(t)$,we have

$$(h_\mu f)(\omega) = \frac{\omega^{\mu+\frac{1}{2}}}{(2a)^{\mu+1}} e^{-\frac{\omega^2}{4a}}.$$

Thus

$$(h_\mu^\alpha f)(\omega) = \frac{\omega^{\frac{2\mu+1}{2\alpha}}}{(2a)^{\mu+1}} e^{-\frac{\omega^{\frac{2}{\alpha}}}{4a}}. \tag{2.5}$$

3 Operational relations for the fractional Hankel transform

In this section, various operational properties of Hankel transform and fractional Hankel transform are investigated and obtained important results.

Lemma 3.1 *For $\mu > 0, \omega \in (0, \infty)$ and $0 < \alpha < 1$, we have the following relation*

$$
\begin{aligned}
&{}_2F_3\left(\begin{matrix} \frac{\mu}{2}+\frac{3}{4}, & \frac{\mu}{2}+\frac{5}{4} & \\ \frac{\mu+\alpha}{2}+\frac{3}{4}, & \frac{\mu+\alpha}{2}+\frac{5}{4}, & \mu+1 \end{matrix}; -\left(\frac{\omega x}{2}\right)^2\right) \\
&= (\omega x)^{-\mu} J_\mu(\omega x) 2^\mu \Gamma(\mu+1) + \frac{\alpha}{\mu}(\omega x)^{1-\mu} J_{\mu+1}(\omega x) 2^\mu \Gamma(\mu+1) \\
&\quad + O\left(\left(\frac{\mu}{2}\right)^{-3}\right), \qquad (3.1)
\end{aligned}
$$

where $O\left(\left(\frac{\mu}{2}\right)^{-3}\right)$ are lower ordered terms.

Proof Taking $p = 2, a_1 = \frac{3}{4}, a_2 = \frac{5}{4}, b_1 = \frac{3}{4} + \frac{\alpha}{2}, b_2 = \frac{5}{4} + \frac{\alpha}{2}, r = \frac{\mu}{2}, \beta = \frac{\mu}{2} + 1, z = -(\frac{\omega x}{2})^2$ and $f_1 = (a_1 - b_1) + (a_2 - b_2) = -\alpha$ in Eq. (1.16), we get

$$
\begin{aligned}
&{}_2F_3\left(\begin{matrix} \frac{\mu}{2}+\frac{3}{4}, & \frac{\mu}{2}+\frac{5}{4} & \\ \frac{\mu+\alpha}{2}+\frac{3}{4}, & \frac{\mu+\alpha}{2}+\frac{5}{4}, & \mu+1 \end{matrix}; -\left(\frac{\omega x}{2}\right)^2\right) \\
&= {}_0F_1\left(-; \mu+1; -\left(\frac{\omega x}{2}\right)^2\right) + \frac{\alpha(\frac{\omega x}{2})^2}{\frac{\mu}{2}(\mu+1)} {}_0F_1\left(-; (\mu+1)+1; -\left(\frac{\omega x}{2}\right)^2\right) \\
&\quad + O\left(\left(\frac{\mu}{2}\right)^{-3}\right).
\end{aligned}
$$

Using the formula (1.19), we have

$$
\begin{aligned}
&{}_2F_3\left(\begin{matrix} \frac{\mu}{2}+\frac{3}{4}, & \frac{\mu}{2}+\frac{5}{4} & \\ \frac{\mu+\alpha}{2}+\frac{3}{4}, & \frac{\mu+\alpha}{2}+\frac{5}{4}, & \mu+1 \end{matrix}; -\left(\frac{\omega x}{2}\right)^2\right) \\
&= (\omega x)^{-\mu} J_\mu(\omega x) 2^\mu \Gamma(\mu+1) + \frac{\alpha}{\mu}(\omega x)^{1-\mu} J_{\mu+1}(\omega x) 2^\mu \Gamma(\mu+1) \\
&\quad + O\left(\left(\frac{\mu}{2}\right)^{-3}\right).
\end{aligned}
$$

□

Lemma 3.2 *Let $\mu > 0, \omega \in (0, \infty)$ and $0 < \alpha < 1$, then we find the result, which is given below:*

$$
\begin{aligned}
\left(I_{0+}^{\alpha}(\omega t)^{\frac{1}{2}} J_\mu(\omega t)\right)(x) \approx \frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}+\alpha)} \Big[& x^\alpha (\omega x)^{\frac{1}{2}} J_\mu(\omega x) \\
& + \frac{\alpha\omega}{\mu} x^{1+\alpha} (\omega x)^{\frac{1}{2}} J_{\mu+1}(\omega x) \Big], \qquad (3.2)
\end{aligned}
$$

for neglecting lower order terms.

Proof From Eq. (1.10),we have

$$\left(I_{0+}^{\alpha}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x)=\frac{1}{\Gamma\alpha}\int\limits_{0}^{x}(x-t)^{\alpha-1}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\mathrm{d}t$$
$$=\frac{1}{\Gamma\alpha}\int_{0}^{x}x^{\alpha-1}\left(1-\frac{t}{x}\right)^{\alpha-1}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\mathrm{d}t.$$

Let $\frac{t}{x}=s$, $\mathrm{d}t=x\mathrm{d}s$. Then the above expression becomes

$$\left(I_{0+}^{\alpha}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x)$$
$$=\frac{1}{\Gamma\alpha}x^{\alpha+\frac{1}{2}}\omega^{\frac{1}{2}}\int\limits_{0}^{1}(1-s)^{\alpha-1}s^{\frac{1}{2}}J_{\mu}(\omega xs)\mathrm{d}s$$
$$=\frac{1}{\Gamma\alpha}x^{\alpha+\frac{1}{2}}\omega^{\frac{1}{2}}\sum_{r=0}^{\infty}\frac{(-1)^{r}}{r!\Gamma(\mu+r+1)}\left(\frac{\omega}{2}\right)^{\mu+2r}\int\limits_{o}^{1}(1-s)^{\alpha-1}s^{\frac{1}{2}}(xs)^{\mu+2r}\mathrm{d}s.$$

By the property of beta function,

$$\left(I_{0+}^{\alpha}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x)$$
$$=\frac{1}{\Gamma\alpha}x^{\alpha+\frac{1}{2}}\omega^{\frac{1}{2}}\sum_{r=0}^{\infty}\frac{(-1)^{r}}{r!\Gamma(\mu+r+1)}\left(\frac{\omega x}{2}\right)^{\mu+2r}\frac{\Gamma(\mu+2r+\frac{3}{2})\Gamma\alpha}{\Gamma(\mu+2r+\frac{3}{2}+\alpha)}.$$

Using Eq. (1.20),we get

$$\left(I_{0+}^{\alpha}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x)$$
$$=x^{\alpha+\frac{1}{2}}\omega^{\frac{1}{2}}\sum_{r=0}^{\infty}\frac{(-1)^{r}}{r!}\left(\frac{\omega x}{2}\right)^{\mu+2r}\frac{\left(\mu+\frac{3}{2}\right)_{2r}\Gamma(\mu+\frac{3}{2})}{(\mu+1)_{r}\Gamma(\mu+1)\left(\mu+\frac{3}{2}+\alpha\right)_{2r}\Gamma(\mu+\frac{3}{2}+\alpha)}.$$

In view of Eq. (1.21),we have

$$\left(I_{0+}^{\alpha}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x)$$
$$=x^{\alpha+\frac{1}{2}}\omega^{\frac{1}{2}}\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+1)\Gamma(\mu+\frac{3}{2}+\alpha)}\sum_{r=0}^{\infty}\frac{(-1)^{r}}{r!}\left(\frac{\omega x}{2}\right)^{\mu+2r}\frac{1}{(\mu+1)_{r}}$$
$$\times\frac{2^{2r}\left(\frac{\mu}{2}+\frac{3}{4}\right)_{r}\left(\frac{\mu}{2}+\frac{5}{4}\right)_{r}}{2^{2r}\left(\frac{\mu+\alpha}{2}+\frac{3}{4}\right)_{r}\left(\frac{\mu+\alpha}{2}+\frac{5}{4}\right)_{r}}.$$

Thus

$$\begin{aligned}&\left(I_{0+}^{\alpha}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x)\\&=x^{\alpha}\frac{(\omega x)^{\mu+\frac{1}{2}}}{2^{\mu}}\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+1)\Gamma(\mu+\frac{3}{2}+\alpha)}\\&\times{}_2F_3\left(\begin{array}{c}\frac{\mu}{2}+\frac{3}{4}\ ,\ \frac{\mu}{2}+\frac{5}{4}\\ \frac{\mu+\alpha}{2}+\frac{3}{4}\ ,\ \frac{\mu+\alpha}{2}+\frac{5}{4}\ ,\ \mu+1\end{array};-\left(\frac{\omega x}{2}\right)^{2}\right).\end{aligned}$$

Applying Lemma 3.1 and neglecting lower ordered terms,

$$\begin{aligned}\left(I_{0+}^{\alpha}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x)\approxeq x^{\alpha}\frac{(\omega x)^{\mu+\frac{1}{2}}}{2^{\mu}}\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+1)\Gamma(\mu+\frac{3}{2}+\alpha)}\Big[(\omega x)^{-\mu}J_{\mu}(\omega x)\\\times2^{\mu}\Gamma(\mu+1)+\frac{\alpha}{\mu}(\omega x)^{1-\mu}J_{\mu+1}(\omega x)2^{\mu}\Gamma(\mu+1)\Big].\end{aligned}$$

So that

$$\begin{aligned}\left(I_{0+}^{\alpha}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x)\approxeq\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}+\alpha)}\Big[x^{\alpha}(\omega x)^{\frac{1}{2}}J_{\mu}(\omega x)\\+\frac{\alpha\omega}{\mu}x^{1+\alpha}(\omega x)^{\frac{1}{2}}J_{\mu+1}(\omega x)\Big].\end{aligned}$$

□

Lemma 3.3 *Let* $\omega\in(0,\infty)$, $0<\alpha<1$ *and* $\mu>0$. *Then we have*

$$\begin{aligned}&\left(I_{-}^{\alpha}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x)\\&\approxeq\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}+\alpha)}\Bigg[\left(\cos\alpha\pi+\sin\alpha\pi\tan(\alpha+\mu)\pi\right)\\&\times\left(x^{\alpha}(\omega x)^{\frac{1}{2}}J_{\mu}(\omega x)+\frac{\alpha\omega}{\mu}x^{\alpha+1}(\omega x)^{\frac{1}{2}}J_{\mu+1}(\omega x)\right)+\frac{1}{\pi}\left(\frac{2}{\omega}\right)^{\alpha}\\&\times\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{\mu+\alpha-k}{2}+\frac{1}{4}\right)}{\Gamma\left(\frac{\mu-\alpha+k}{2}+\frac{3}{4}\right)}\left(\frac{1}{\sqrt{2}}-\frac{\alpha\sqrt{2}}{\mu}\left(\frac{\mu+\alpha-k}{2}+\frac{1}{4}\right)\right)\left(\frac{\omega x}{2}\right)^{k}\Bigg].\end{aligned}\tag{3.3}$$

Proof From Eq. (1.14),we have

$$\left(I_{-}^{\alpha}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x)=\left(\cos\alpha\pi\,I_{0+}^{\alpha}+\sin\alpha\pi\,SI_{0+}^{\alpha}\right)\left((\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x),\tag{3.4}$$

where

$$(S\phi)(x)=\frac{1}{\pi}\int_{0}^{\infty}\frac{\phi(t)}{t-x}\mathrm{d}t.$$

Using Lemma 3.2, we have

$$(SI_{0+}^{\alpha}((\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)))(x) \approxeq \frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}+\alpha)}\left[\frac{\omega^{\frac{1}{2}}}{\pi}\int_{0}^{\infty}\frac{t^{\alpha+\frac{1}{2}}J_{\mu}(\omega t)}{t-x}\mathrm{d}t + \frac{\alpha\omega^{\frac{3}{2}}}{\mu\pi}\int_{0}^{\infty}\frac{t^{\alpha+\frac{3}{2}}J_{\mu+1}(\omega t)}{t-x}\mathrm{d}t\right](x). \tag{3.5}$$

From Eqs. (1.17) and (1.18), we find the value of following integrals:

$$\int_{0}^{\infty}\frac{t^{\alpha+\frac{1}{2}}J_{\mu}(\omega t)}{t-x}\mathrm{d}t = \pi x^{\alpha+\frac{1}{2}}J_{\mu}(\omega x)\tan(\alpha+\mu)\pi + \frac{1}{2}\left(\frac{2}{\omega}\right)^{\alpha+\frac{1}{2}} \times\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{\mu+\alpha-k}{2}+\frac{1}{4}\right)}{\Gamma\left(\frac{\mu-\alpha+k}{2}+\frac{3}{4}\right)}\left(\frac{\omega x}{2}\right)^{k} \tag{3.6}$$

and

$$\int_{0}^{\infty}\frac{t^{\alpha+\frac{3}{2}}J_{\mu+1}(\omega t)}{t-x}\mathrm{d}t = \pi x^{\alpha+\frac{3}{2}}J_{\mu+1}(\omega x)\tan(\alpha+\mu)\pi + \frac{1}{2}\left(\frac{2}{\omega}\right)^{\alpha+\frac{3}{2}} \times\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{\mu+\alpha-k}{2}+\frac{5}{4}\right)}{\Gamma\left(\frac{\mu-\alpha+k}{2}+\frac{3}{4}\right)}\left(\frac{\omega x}{2}\right)^{k}. \tag{3.7}$$

Putting the value of the integrals from Eqs. (3.6) and (3.7) in Eq. (3.5), we obtain

$$(SI_{0+}^{\alpha}((\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)))(x) \approxeq \frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}+\alpha)}\Bigg[x^{\alpha}(\omega x)^{\frac{1}{2}}J_{\mu}(\omega x)\tan(\alpha+\mu)\pi + \frac{2^{\alpha-\frac{1}{2}}}{\pi\omega^{\alpha}}\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{\mu+\alpha-k}{2}+\frac{1}{4}\right)}{\Gamma\left(\frac{\mu-\alpha+k}{2}+\frac{3}{4}\right)}\left(\frac{\omega x}{2}\right)^{k} + \frac{\alpha\omega}{\mu}x^{1+\alpha} \times(\omega x)^{\frac{1}{2}}J_{\mu+1}(\omega x)\tan(\mu+\alpha)\pi + \frac{\alpha 2^{\alpha+\frac{1}{2}}}{\mu\pi\omega^{\alpha}} \times\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{\mu+\alpha-k}{2}+\frac{5}{4}\right)}{\Gamma\left(\frac{\mu-\alpha+k}{2}+\frac{3}{4}\right)}\left(\frac{\omega x}{2}\right)^{k}\Bigg].$$

Thus Eq. (3.4) becomes

$$\left(I_{-}^{\alpha}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x) \approxeq \frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}+\alpha)}\Bigg[\left(\cos\alpha\pi+\sin\alpha\pi\tan(\alpha+\mu)\pi\right) \times\left(x^{\alpha}(\omega x)^{\frac{1}{2}}J_{\mu}(\omega x)+\frac{\alpha\omega}{\mu}x^{\alpha+1}(\omega x)^{\frac{1}{2}}J_{\mu+1}(\omega x)\right)+\frac{1}{\pi}\left(\frac{2}{\omega}\right)^{\alpha} \times\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{\mu+\alpha-k}{2}+\frac{1}{4}\right)}{\Gamma\left(\frac{\mu-\alpha+k}{2}+\frac{3}{4}\right)}\left(\frac{1}{\sqrt{2}}-\frac{\alpha\sqrt{2}}{\mu}\left(\frac{\mu+\alpha-k}{2}+\frac{1}{4}\right)\right)\left(\frac{\omega x}{2}\right)^{k}\Bigg].$$

□

Lemma 3.4 *For $\omega \in (0,\infty)$, $0 < \alpha < 1$ and $\mu > 0$, we prove the following:*

$$\textit{(I)}\quad h_\mu(I^\alpha_{0+}\phi)(\omega) \approxeq \frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}+\alpha)}\Bigg[\Big(\cos\alpha\pi + \sin\alpha\pi\tan(\alpha+\mu)\pi\Big) \times\Big(h_\mu(x^\alpha\phi) + \frac{\alpha\omega}{\mu}h_{\mu+1}(x^{1+\alpha}\phi)\Big)\Bigg](\omega). \tag{3.8}$$

$$\textit{(II)}\quad h_\mu(I^\alpha_{-}\phi)(\omega) \approxeq \frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}+\alpha)}\Bigg[h_\mu(x^\alpha\phi) + \frac{\alpha\omega}{\mu}h_{\mu+1}(x^{1+\alpha}\phi)\Bigg](\omega). \tag{3.9}$$

Proof **(I)** We take

$$h_\mu(I^\alpha_{0+}\phi)(\omega) = \int_0^\infty (\omega x)^{\frac{1}{2}} J_\mu(\omega x)(I^\alpha_{0+}\phi)(x)\mathrm{d}x.$$

In view of Eq. (1.13),we get

$$h_\mu(I^\alpha_{0+}\phi)(\omega) = \int_0^\infty \Big(I^\alpha_{-}(\omega t)^{\frac{1}{2}} J_\mu(\omega t)\Big)(x)\phi(x)\mathrm{d}x.$$

By using Lemma 3.3, the above expression becomes

$$\begin{aligned} &h_\mu(I^\alpha_{0+}\phi)(\omega)\\ &\approxeq \frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}+\alpha)}\Bigg[\Big(\cos\alpha\pi + \sin\alpha\pi\tan(\alpha+\mu)\pi\Big)\\ &\times\Bigg(\int_0^\infty (\omega x)^{\frac{1}{2}} J_\mu(\omega x)x^\alpha\phi(x)\mathrm{d}x + \frac{\alpha\omega}{\mu}\int_0^\infty (\omega x)^{\frac{1}{2}} J_{\mu+1}(\omega x)x^{\alpha+1}\phi(x)\mathrm{d}x\Bigg)\\ &+\frac{1}{\pi}\Big(\frac{2}{\omega}\Big)^\alpha\sum_{k=0}^\infty \frac{\Gamma\big(\frac{\mu+\alpha-k}{2}+\frac{1}{4}\big)}{\Gamma\big(\frac{\mu-\alpha+k}{2}+\frac{3}{4}\big)}\Bigg(\frac{1}{\sqrt{2}} - \frac{\alpha\sqrt{2}}{\mu}\Big(\frac{\mu+\alpha-k}{2}+\frac{1}{4}\Big)\Bigg)\\ &\times\Big(\frac{\omega}{2}\Big)^k\int_0^\infty x^k\phi(x)\mathrm{d}x\Bigg]. \end{aligned}$$

Since the function ϕ is in Lizorkin space $\Psi(\mathbb{R}^+)$, then from Eq. (1.6) $\int_0^\infty x^k\phi(x)\mathrm{d}x = 0$.

Thus

$$h_\mu(I^\alpha_{0+}\phi)(\omega) \approxeq \frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}+\alpha)}\Bigg[\Big(\cos\alpha\pi + \sin\alpha\pi\tan(\alpha+\mu)\pi\Big) \times\Big(h_\mu(x^\alpha\phi) + \frac{\alpha\omega}{\mu}h_{\mu+1}(x^{1+\alpha}\phi)\Big)\Bigg](\omega).$$

□

(II) We have

$$h_\mu(I_-^\alpha\phi)(\omega)=\int_0^\infty(\omega x)^{\frac{1}{2}}J_\mu(\omega x)(I_-^\alpha\phi)(x)\mathrm{d}x.$$

By the virtue of Eq. (1.13),

$$h_\mu(I_-^\alpha\phi)(\omega)=\int_0^\infty\Big(I_{0+}^\alpha(\omega t)^{\frac{1}{2}}J_\mu(\omega t)\Big)(x)\phi(x)\mathrm{d}x.$$

Applying Lemma 3.2, we obtain

$$h_\mu(I_-^\alpha\phi)(\omega)\approxeq\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}+\alpha)}\Bigg[\int_0^\infty(\omega x)^{\frac{1}{2}}J_\mu(\omega x)x^\alpha\phi(x)\mathrm{d}x+\frac{\alpha\omega}{\mu}\times\int_0^\infty(\omega x)^{\frac{1}{2}}J_{\mu+1}(\omega x)x^{\alpha+1}\phi(x)\mathrm{d}x\Bigg].$$

Thus

$$h_\mu(I_-^\alpha\phi)(\omega)\approxeq\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}+\alpha)}\Bigg[h_\mu(x^\alpha\phi)+\frac{\alpha\omega}{\mu}h_{\mu+1}(x^{1+\alpha}\phi)\Bigg](\omega).$$

□

Lemma 3.5 *Let* $\omega\in(0,\infty)$, $\mu>0$ *and* $0<\alpha<1$. *Then*

$$\begin{aligned}&\Big(D_{0+}^\alpha(\omega t)^{\frac{1}{2}}J_\mu(\omega t)\Big)(x)\\&\approxeq\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}-\alpha)}\Bigg[(\omega x)^{\frac{1}{2}}J_\mu(\omega x)x^{-\alpha}+\frac{\omega\Big(\big(\frac{7}{2}-\alpha\big)\big(1-\alpha\big)-\mu\alpha\Big)}{\mu\big(\mu+\frac{3}{2}-\alpha\big)}\\&\times(\omega x^{\frac{1}{2}})x^{1-\alpha}J_{\mu+1}(\omega x)-\frac{(1-\alpha)\omega^2}{\mu\big(\mu+\frac{3}{2}-\alpha\big)}(\omega x)^{\frac{1}{2}}J_{\mu+2}(\omega x)x^{2-\alpha}\Bigg].\end{aligned}\tag{3.10}$$

Proof From Eq. (1.8),we have

$$\Big(D_{0+}^\alpha(\omega t)^{\frac{1}{2}}J_\mu(\omega t)\Big)(x)=\frac{\mathrm{d}}{\mathrm{d}x}\bigg(\Big(I_{0+}^{1-\alpha}(\omega t)^{\frac{1}{2}}J_\mu(\omega t)\Big)(x)\bigg).$$

Using Lemma 3.2,

$$\begin{aligned}\Big(D_{0+}^\alpha(\omega t)^{\frac{1}{2}}J_\mu(\omega t)\Big)(x)\approxeq\frac{\mathrm{d}}{\mathrm{d}x}\Bigg[\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{5}{2}-\alpha)}\bigg(&x^{1-\alpha}(\omega x)^{\frac{1}{2}}J_\mu(\omega x)\\&+\frac{(1-\alpha)\omega}{\mu}x^{2-\alpha}(\omega x)^{\frac{1}{2}}J_{\mu+1}(\omega x)\bigg)\Bigg].\end{aligned}$$

Consider

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1-\alpha}(\omega x)^{\frac{1}{2}}J_{\mu}(\omega x)\right)$$
$$=\omega^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-\mu}J_{\mu}(\omega x)x^{\mu+\frac{3}{2}-\alpha}\right)$$
$$=\omega^{\frac{1}{2}}\left(x^{-\mu}J_{\mu}(\omega x)\left(\mu+\frac{3}{2}-\alpha\right)x^{\mu-\alpha+\frac{1}{2}}+x^{\mu+\frac{3}{2}-\alpha}\frac{\mathrm{d}}{\mathrm{d}x}(x^{-\mu}J_{\mu}(\omega x))\right).$$

In view of Eq. (1.22),

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1-\alpha}(\omega x)^{\frac{1}{2}}J_{\mu}(\omega x)\right)$$
$$=\omega^{\frac{1}{2}}\left(x^{-\alpha+\frac{1}{2}}\left(\mu+\frac{3}{2}-\alpha\right)J_{\mu}(\omega x)-\omega x^{-\alpha+\frac{3}{2}}J_{\mu+1}(\omega x)\right). \tag{3.11}$$

Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{2-\alpha}(\omega x)^{\frac{1}{2}}J_{\mu+1}(\omega x)\right)$$
$$=\omega^{\frac{1}{2}}\left(x^{-\alpha+\frac{3}{2}}\left(\mu+\frac{7}{2}-\alpha\right)J_{\mu+1}(\omega x)-\omega x^{-\alpha+\frac{5}{2}}J_{\mu+2}(\omega x)\right). \tag{3.12}$$

Thus

$$\left(D^{\alpha}_{0+}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x)$$
$$\approx\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}-\alpha)}\Bigg[(\omega x)^{\frac{1}{2}}J_{\mu}(\omega x)x^{-\alpha}+\frac{\omega\left(\left(\frac{7}{2}-\alpha\right)\left(1-\alpha\right)-\mu\alpha\right)}{\mu\left(\mu+\frac{3}{2}-\alpha\right)}$$
$$\times(\omega x^{\frac{1}{2}})x^{1-\alpha}J_{\mu+1}(\omega x)-\frac{(1-\alpha)\omega^{2}}{\mu\left(\mu+\frac{3}{2}-\alpha\right)}(\omega x)^{\frac{1}{2}}J_{\mu+2}(\omega x)x^{2-\alpha}\Bigg].$$

□

Lemma 3.6 *Let* $\omega\in(0,\infty)$, $\mu>0$ *and* $0<\alpha<1$. *Then*

$$\left(D^{\alpha}_{-}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x)$$
$$\approx\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}-\alpha)}\Bigg[\left(\cos\alpha\pi-\sin\alpha\pi\tan(\mu-\alpha)\pi\right)$$
$$\times\Bigg((\omega x)^{\frac{1}{2}}x^{-\alpha}J_{\mu}(\omega x)+\frac{\omega\left((\frac{7}{2}-\alpha)(1-\alpha)-\mu\alpha\right)}{\mu\left(\mu+\frac{3}{2}-\alpha\right)}(\omega x)^{\frac{1}{2}}x^{1-\alpha}J_{\mu+1}(\omega x)-\frac{(1-\alpha)\omega^{2}}{\mu\left(\mu+\frac{3}{2}-\alpha\right)}$$
$$\times(\omega x)^{\frac{1}{2}}x^{2-\alpha}J_{\mu+2}(\omega x)\Bigg)-\frac{1}{\left(\mu+\frac{3}{2}-\alpha\right)}\frac{1}{\pi}\left(\frac{2}{\omega}\right)^{1-\alpha}\sum_{k=1}^{\infty}\left(\frac{k\omega}{2}\right)\frac{\Gamma\left(\frac{\mu-\alpha-k}{2}+\frac{3}{4}\right)}{\Gamma\left(\frac{\mu+\alpha+k}{2}+\frac{1}{4}\right)}$$
$$\times\left(\frac{1}{\sqrt{2}}-\frac{(1-\alpha)\sqrt{2}}{\mu}\left(\frac{\mu-\alpha-k}{2}+\frac{3}{4}\right)\right)\left(\frac{\omega x}{2}\right)^{k-1}\Bigg]. \tag{3.13}$$

Proof From Eq. (1.9),we have

$$\left(D^{\alpha}_{-}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x)=-\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(I^{1-\alpha}_{-}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x)\right).$$

Applying Lemma 3.3,

$$\begin{aligned}&\left(D_{-}^{\alpha}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x)\\&\approxeq-\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{5}{2}-\alpha)}\left[\left(\cos(1-\alpha)\pi+\sin(1-\alpha)\pi\tan(1-\alpha+\mu)\pi\right)\right.\right.\\&\quad\times\left(x^{1-\alpha}(\omega x)^{\frac{1}{2}}J_{\mu}(\omega x)+\frac{(1-\alpha)\omega}{\mu}x^{2-\alpha}(\omega x)^{\frac{1}{2}}J_{\mu+1}(\omega x)\right)+\frac{1}{\pi}\left(\frac{2}{\omega}\right)^{1-\alpha}\\&\quad\left.\left.\times\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{\mu+1-\alpha-k}{2}+\frac{1}{4}\right)}{\Gamma\left(\frac{\mu-1+\alpha+k}{2}+\frac{3}{4}\right)}\left(\frac{1}{\sqrt{2}}-\frac{(1-\alpha)\sqrt{2}}{\mu}\left(\frac{\mu+1-\alpha-k}{2}+\frac{1}{4}\right)\right)\left(\frac{\omega x}{2}\right)^{k}\right]\right].\end{aligned}$$

Thus

$$\begin{aligned}&\left(D_{-}^{\alpha}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x)\\&\approxeq\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{5}{2}-\alpha)}\left[\left(\cos\alpha\pi-\sin\alpha\pi\tan(\mu-\alpha)\pi\right)\right.\\&\quad\times\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1-\alpha}(\omega x)^{\frac{1}{2}}J_{\mu}(\omega x)+\frac{(1-\alpha)\omega}{\mu}x^{2-\alpha}(\omega x)^{\frac{1}{2}}J_{\mu+1}(\omega x)\right)-\frac{1}{\pi}\left(\frac{2}{\omega}\right)^{1-\alpha}\\&\quad\left.\times\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{\mu-\alpha-k}{2}+\frac{3}{4}\right)}{\Gamma\left(\frac{\mu+\alpha+k}{2}+\frac{1}{4}\right)}\left(\frac{1}{\sqrt{2}}-\frac{(1-\alpha)\sqrt{2}}{\mu}\left(\frac{\mu-\alpha-k}{2}+\frac{3}{4}\right)\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\omega x}{2}\right)^{k}\right].\end{aligned}$$

Taking into account Eqs. (3.11) and (3.12), the above expression yields

$$\begin{aligned}&\left(D_{-}^{\alpha}(\omega t)^{\frac{1}{2}}J_{\mu}(\omega t)\right)(x)\\&\approxeq\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}-\alpha)}\left[\left(\cos\alpha\pi-\sin\alpha\pi\tan(\mu-\alpha)\pi\right)\right.\\&\times\left((\omega x)^{\frac{1}{2}}x^{-\alpha}J_{\mu}(\omega x)+\frac{\omega\left((\frac{7}{2}-\alpha)(1-\alpha)-\mu\alpha\right)}{\mu\left(\mu+\frac{3}{2}-\alpha\right)}(\omega x)^{\frac{1}{2}}x^{1-\alpha}J_{\mu+1}(\omega x)-\frac{(1-\alpha)\omega^{2}}{\mu\left(\mu+\frac{3}{2}-\alpha\right)}\right.\\&\left.\times(\omega x)^{\frac{1}{2}}x^{2-\alpha}J_{\mu+2}(\omega x)\right)-\frac{1}{\left(\mu+\frac{3}{2}-\alpha\right)}\frac{1}{\pi}\left(\frac{2}{\omega}\right)^{1-\alpha}\sum_{k=1}^{\infty}\left(\frac{k\omega}{2}\right)\frac{\Gamma\left(\frac{\mu-\alpha-k}{2}+\frac{3}{4}\right)}{\Gamma\left(\frac{\mu+\alpha+k}{2}+\frac{1}{4}\right)}\\&\left.\times\left(\frac{1}{\sqrt{2}}-\frac{(1-\alpha)\sqrt{2}}{\mu}\left(\frac{\mu-\alpha-k}{2}+\frac{3}{4}\right)\right)\left(\frac{\omega x}{2}\right)^{k-1}\right].\end{aligned}$$

□

Lemma 3.7 *The following relations hold for* $\omega\in(0,\infty)$, $0<\alpha<1$ *and* $\mu>0$:

$$\begin{aligned}\textbf{\textit{(I)}}\quad h_{\mu}(D_{0+}^{\alpha}\phi(x))(\omega)\approxeq&\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}-\alpha)}\left[\left(\cos\alpha\pi-\sin\alpha\pi\tan(\mu-\alpha)\pi\right)\right.\\&\times\left(h_{\mu}(x^{-\alpha}\phi)+\frac{\omega\left((\frac{7}{2}-\alpha)(1-\alpha)-\mu\alpha\right)}{\mu(\mu+\frac{3}{2}-\alpha)}h_{\mu+1}(x^{1-\alpha}\phi)\right.\\&\left.\left.-\frac{(1-\alpha)\omega^{2}}{\mu(\mu+\frac{3}{2}-\alpha)}h_{\mu+2}(x^{2-\alpha}\phi)\right)\right](\omega).\end{aligned}\tag{3.14}$$

$$\textbf{\textit{(II)}}\quad h_\mu(D_-^\alpha\phi(x))(\omega) \approxeq \frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}-\alpha)}\Bigg[h_\mu(x^{-\alpha}\phi) + \frac{\omega\big((\frac{7}{2}-\alpha)(1-\alpha)-\mu\alpha\big)}{\mu(\mu+\frac{3}{2}-\alpha)} \times h_{\mu+1}(x^{1-\alpha}\phi) - \frac{(1-\alpha)\omega^2}{\mu(\mu+\frac{3}{2}-\alpha)}h_{\mu+2}(x^{2-\alpha}\phi)\Bigg](\omega). \quad (3.15)$$

Proof **(I)** We take

$$h_\mu(D_{0+}^\alpha\phi)(\omega) = \int_0^\infty (\omega x)^{\frac{1}{2}} J_\mu(\omega x)(D_{0+}^\alpha\phi)(x)\mathrm{d}x.$$

In view of Eq. (1.12),we get

$$h_\mu(D_{0+}^\alpha\phi)(\omega) = \int_0^\infty \Big(D_-^\alpha(\omega t)^{\frac{1}{2}} J_\mu(\omega t)\Big)(x)\phi(x)\mathrm{d}x.$$

Using Lemma 3.6,

$$\begin{aligned} &h_\mu(D_{0+}^\alpha\phi)(\omega)\\ &\approxeq \frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}-\alpha)}\Bigg[\Big(\cos\alpha\pi - \sin\alpha\pi\tan(\mu-\alpha)\pi\Big)\Bigg(\int_0^\infty(\omega x)^{\frac{1}{2}}J_\mu(\omega x)x^{-\alpha}\phi(x)\mathrm{d}x\\ &+\frac{\omega\big((\frac{7}{2}-\alpha)(1-\alpha)-\mu\alpha\big)}{\mu(\mu+\frac{3}{2}-\alpha)}\int_0^\infty(\omega x)^{\frac{1}{2}}J_{\mu+1}(\omega x)x^{1-\alpha}\phi(x)\mathrm{d}x - \frac{(1-\alpha)\omega^2}{\mu(\mu+\frac{3}{2}-\alpha)}\\ &\times\int_0^\infty(\omega x)^{\frac{1}{2}}J_{\mu+2}(\omega x)x^{2-\alpha}\phi(x)\mathrm{d}x\Bigg) - \frac{1}{\mu+\frac{3}{2}-\alpha}\left(\frac{1}{\pi}\right)\left(\frac{2}{\omega}\right)^{1-\alpha}\\ &\times\sum_{k=1}^\infty\left(\frac{k\omega}{2}\right)\frac{\Gamma\big(\frac{\mu-\alpha-k}{2}+\frac{3}{4}\big)}{\Gamma\big(\frac{\mu+\alpha+k}{2}+\frac{1}{4}\big)}\left(\frac{1}{\sqrt{2}} - \frac{(1-\alpha)\sqrt{2}}{\mu}\left(\frac{\mu-\alpha-k}{2}+\frac{3}{4}\right)\right)\\ &\times\left(\frac{\omega}{2}\right)^{k-1}\int_0^\infty x^{k-1}\phi(x)\mathrm{d}x\Bigg]. \end{aligned}$$

Since the function ϕ is in Lizorkin space $\Psi(\mathbb{R}^+)$, therefore using Eq. (1.6),

$$\int_0^\infty x^{k-1}\phi(x)\mathrm{d}x = 0.$$

Thus

$$\begin{aligned} h_\mu(D_{0+}^\alpha\phi(x))(\omega) &\approxeq \frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}-\alpha)}\Bigg[\Big(\cos\alpha\pi - \sin\alpha\pi\tan(\mu-\alpha)\pi\Big)\\ &\times\Bigg(h_\mu(x^{-\alpha}\phi) + \frac{\omega\big((\frac{7}{2}-\alpha)(1-\alpha)-\mu\alpha\big)}{\mu(\mu+\frac{3}{2}-\alpha)}h_{\mu+1}(x^{1-\alpha}\phi) - \frac{(1-\alpha)\omega^2}{\mu(\mu+\frac{3}{2}-\alpha)}h_{\mu+2}(x^{2-\alpha}\phi)\Bigg)\Bigg](\omega). \end{aligned}$$

□

(II) We have

$$h_\mu(D_-^\alpha\phi)(\omega)=\int_0^\infty(\omega x)^{\frac{1}{2}}J_\mu(\omega x)(D_-^\alpha\phi)(x)\mathrm{d}x.$$

By the virtue of Eq. (1.12),

$$h_\mu(D_-^\alpha\phi)(\omega)=\int_0^\infty\left(D_{0+}^\alpha(\omega t)^{\frac{1}{2}}J_\mu(\omega t)\right)(x)\phi(x)\mathrm{d}x.$$

Applying Lemma 3.5, we obtain

$$\begin{aligned}&h_\mu(D_-^\alpha\phi)(\omega)\\&\approxeq\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}-\alpha)}\Bigg[\int_0^\infty(\omega x)^{\frac{1}{2}}J_\mu(\omega x)x^{-\alpha}\phi(x)\mathrm{d}x+\frac{\omega\big((\frac{7}{2}-\alpha)(1-\alpha)-\mu\alpha\big)}{\mu(\mu+\frac{3}{2}-\alpha)}\\&\times\int_0^\infty(\omega x)^{\frac{1}{2}}J_{\mu+1}(\omega x)x^{1-\alpha}\phi(x)\mathrm{d}x-\frac{(1-\alpha)\omega^2}{\mu(\mu+\frac{3}{2}-\alpha)}\int_0^\infty(\omega x)^{\frac{1}{2}}J_{\mu+2}(\omega x)x^{2-\alpha}\phi(x)\mathrm{d}x\Bigg].\end{aligned}$$

Thus

$$\begin{aligned}h_\mu(D_-^\alpha\phi(x))(\omega)\approxeq\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2}-\alpha)}\Bigg[&h_\mu(x^{-\alpha}\phi)+\frac{\omega\big((\frac{7}{2}-\alpha)(1-\alpha)-\mu\alpha\big)}{\mu(\mu+\frac{3}{2}-\alpha)}\\&\times h_{\mu+1}(x^{1-\alpha}\phi)-\frac{(1-\alpha)\omega^2}{\mu(\mu+\frac{3}{2}-\alpha)}h_{\mu+2}(x^{2-\alpha}\phi)\Bigg](\omega).\end{aligned}$$

□

Theorem 3.8 *Let $0<\alpha\leq 1$, $\mu>0$. Then the following operational relation holds for any value of parameter β:*

$$\begin{aligned}(h_\mu^\alpha D_\beta^\alpha\phi)(\omega)\approxeq\frac{\Gamma\big(\mu+\frac{3}{2}\big)}{\Gamma\big(\mu+\frac{3}{2}-\alpha\big)}&\Big[\big(1-\beta\big)\big(\cos\alpha\pi-\sin\alpha\pi\tan(\mu-\alpha)\pi\big)-\beta\Big]\\&\times\Bigg[h_\mu^\alpha(x^{-\alpha}\phi)+\frac{\omega^{\frac{1}{\alpha}}\big((\frac{7}{2}-\alpha)(1-\alpha)-\mu\alpha\big)}{\mu\big(\mu+\frac{3}{2}-\alpha\big)}h_{\mu+1}^\alpha(x^{1-\alpha}\phi)\\&-\frac{(1-\alpha)\omega^{\frac{2}{\alpha}}}{\mu\big(\mu+\frac{3}{2}-\alpha\big)}h_{\mu+2}^\alpha(x^{2-\alpha}\phi)\Bigg](\omega).\end{aligned}\tag{3.16}$$

Proof By the definition of fractional Hankel transform,

$$\begin{aligned}(h_\mu^\alpha D_\beta^\alpha\phi)(\omega)&=\int_0^\infty(\omega^{\frac{1}{\alpha}}x)^{\frac{1}{2}}J_\mu(\omega^{\frac{1}{\alpha}}x)(D_\beta^\alpha\phi)(x)\mathrm{d}x\\&=\int_0^\infty(\omega^{\frac{1}{\alpha}}x)^{\frac{1}{2}}J_\mu(\omega^{\frac{1}{\alpha}}x)[(1-\beta)(D_{0+}^\alpha\phi)(x)-\beta(D_-^\alpha\phi)(x)]\mathrm{d}x.\end{aligned}$$

By the virtue of Eq. (1.12),we get

$$(h^{\alpha}_{\mu} D^{\alpha}_{\beta}\phi)(\omega) = (1-\beta)\int_0^\infty \Big(D^{\alpha}_{-}(\omega^{\frac{1}{\alpha}}t)^{\frac{1}{2}} J_\mu(\omega^{\frac{1}{\alpha}}t)\Big)(x)\phi(x)\mathrm{d}x$$
$$-\beta\int_0^\infty \Big(D^{\alpha}_{0+}(\omega^{\frac{1}{\alpha}}t)^{\frac{1}{2}} J_\mu(\omega^{\frac{1}{\alpha}}t)\Big)(x)\phi(x)\mathrm{d}x.$$

Using Lemmas 3.5 and 3.6 and then replacing ω by $\omega^{\frac{1}{\alpha}}$, we have

$$\begin{aligned}
&(h^{\alpha}_{\mu} D^{\alpha}_{\beta}\phi)(\omega)\\
&\approxeq (1-\beta)\frac{\Gamma\big(\mu+\frac{3}{2}\big)}{\Gamma\big(\mu+\frac{3}{3}-\alpha\big)}\Big[\big(\cos\alpha\pi-\sin\alpha\pi\tan(\mu-\alpha)\pi\big)\\
&\times\Big(\int_0^\infty (\omega^{\frac{1}{\alpha}}x)^{\frac{1}{2}} J_\mu(\omega^{\frac{1}{\alpha}}x)x^{-\alpha}\phi(x)\mathrm{d}x+\frac{\omega^{\frac{1}{\alpha}}\Big((\frac{7}{2}-\alpha)(1-\alpha)-\mu\alpha\Big)}{\mu\big(\mu+\frac{3}{2}-\alpha\big)}\\
&\times\int_0^\infty (\omega^{\frac{1}{\alpha}}x)^{\frac{1}{2}} J_{\mu+1}(\omega^{\frac{1}{\alpha}}x)x^{1-\alpha}\phi(x)\mathrm{d}x-\frac{(1-\alpha)\omega^{\frac{2}{\alpha}}}{\mu\big(\mu+\frac{3}{2}-\alpha\big)}\\
&\times\int_0^\infty (\omega^{\frac{1}{\alpha}}x)^{\frac{1}{2}} J_{\mu+2}(\omega^{\frac{1}{\alpha}}x)x^{2-\alpha}\phi(x)\mathrm{d}x\Big)-\frac{1}{\mu+\frac{3}{2}-\alpha}\Big(\frac{1}{\pi}\Big)\Big(\frac{2}{\omega^{\frac{1}{\alpha}}}\Big)^{1-\alpha}\\
&\times\sum_{k=1}^{\infty}\Big(\frac{k\omega^{\frac{1}{\alpha}}}{2}\Big)\frac{\Gamma\big(\frac{\mu-\alpha-k}{2}+\frac{3}{4}\big)}{\Gamma\big(\frac{\mu+\alpha+k}{2}+\frac{1}{4}\big)}\Big(\frac{1}{\sqrt{2}}-\frac{(1-\alpha)\sqrt{2}}{\mu}\Big(\frac{\mu-\alpha-k}{2}+\frac{3}{4}\Big)\Big)\Big(\frac{\omega^{\frac{1}{\alpha}}}{2}\Big)^{k-1}\\
&\times\int_0^\infty x^{k-1}\phi(x)dx\Big]-\beta\frac{\Gamma\big(\mu+\frac{3}{2}\big)}{\Gamma\big(\mu+\frac{3}{3}-\alpha\big)}\Big[\int_0^\infty (\omega^{\frac{1}{\alpha}}x)^{\frac{1}{2}} J_\mu(\omega^{\frac{1}{\alpha}}x)x^{-\alpha}\phi(x)\mathrm{d}x\\
&+\frac{\omega^{\frac{1}{\alpha}}\Big((\frac{7}{2}-\alpha)(1-\alpha)-\mu\alpha\Big)}{\mu\big(\mu+\frac{3}{2}-\alpha\big)}\int_0^\infty (\omega^{\frac{1}{\alpha}}x)^{\frac{1}{2}} J_{\mu+1}(\omega^{\frac{1}{\alpha}}x)x^{1-\alpha}\phi(x)dx\\
&-\frac{(1-\alpha)\omega^{\frac{2}{\alpha}}}{\mu\big(\mu+\frac{3}{2}-\alpha\big)}\int_0^\infty (\omega^{\frac{1}{\alpha}}x)^{\frac{1}{2}} J_{\mu+2}(\omega^{\frac{1}{\alpha}}x)x^{2-\alpha}\phi(x)\mathrm{d}x\Big].
\end{aligned}\tag{3.17}$$

Since the function ϕ is an element of Lizorkin space $\Psi(\mathbb{R}^+)$, therefore using Eq. (1.6), $\int\limits_0^\infty x^{k-1}\phi(x)dx = 0$.

Thus Eq. (3.17) becomes

$$(h_\mu^\alpha D_\beta^\alpha \phi)(\omega) \approxeq \frac{\Gamma\big(\mu+\frac{3}{2}\big)}{\Gamma\big(\mu+\frac{3}{2}-\alpha\big)}\Big[\big(1-\beta\big)\big(\cos\alpha\pi - \sin\alpha\pi\tan(\mu-\alpha)\pi\big) - \beta\Big]$$
$$\times\Big[h_\mu^\alpha(x^{-\alpha}\phi) + \frac{\omega^{\frac{1}{\alpha}}\big((\frac{7}{2}-\alpha)(1-\alpha)-\mu\alpha\big)}{\mu\big(\mu+\frac{3}{2}-\alpha\big)} h_{\mu+1}^\alpha(x^{1-\alpha}\phi)$$
$$-\frac{(1-\alpha)\omega^{\frac{2}{\alpha}}}{\mu\big(\mu+\frac{3}{2}-\alpha\big)} h_{\mu+2}^\alpha(x^{2-\alpha}\phi)\Big](\omega).$$

Verification: Putting $\alpha = 1$ in Eq. (3.16), we get

$$h_\mu(D\phi(x))(\omega) \approxeq \frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{1}{2})}\big(-(1-\beta)-\beta\big)\Big(h_\mu(x^{-1}\phi) - \frac{\omega}{\mu+\frac{1}{2}} h_{\mu+1}(\phi)\Big)(\omega).$$

Therefore,

$$h_\mu(D\phi(x))(\omega) \approxeq \Big(-\Big(\mu+\frac{1}{2}\Big)h_\mu(x^{-1}\phi) + \omega h_{\mu+1}(\phi)\Big)(\omega), \tag{3.18}$$

which is true in the case of ordinary derivative of order 1.

Case (I) If $\omega = 0$:
Putting $\omega = 0$ in Eq. (3.16), we obtain

$$(h_\mu^\alpha D_\beta^\alpha \phi)(0) \approxeq \frac{\Gamma\big(\mu+\frac{3}{2}\big)}{\Gamma\big(\mu+\frac{3}{2}-\alpha\big)}\Big[\big(1-\beta\big)\big(\cos\alpha\pi - \sin\alpha\pi\tan(\mu-\alpha)\pi\big) - \beta\Big]$$
$$\times\Big[h_\mu^\alpha(x^{-\alpha}\phi)(0) + \frac{\omega^{\frac{1}{\alpha}}\big((\frac{7}{2}-\alpha)(1-\alpha)-\mu\alpha\big)}{\mu\big(\mu+\frac{3}{2}-\alpha\big)} h_{\mu+1}^\alpha(x^{1-\alpha}\phi)(0)$$
$$-\frac{(1-\alpha)\omega^{\frac{2}{\alpha}}}{\mu\big(\mu+\frac{3}{2}-\alpha\big)} h_{\mu+2}^\alpha(x^{2-\alpha}\phi(0))\Big]$$
$$\approxeq 0. \tag{3.19}$$

Case (II) If $\beta = \frac{1}{2}$:
Putting $\beta = \frac{1}{2}$ in Eq. (3.16), we get

$$(h_\mu^\alpha D_{\frac{1}{2}}^\alpha \phi)(\omega) \approxeq \frac{\Gamma\big(\mu+\frac{3}{2}\big)}{2\Gamma\big(\mu+\frac{3}{2}-\alpha\big)}\Big[\big(\cos\alpha\pi - \sin\alpha\pi\tan(\mu-\alpha)\pi\big) - 1\Big]$$
$$\times\Big[h_\mu^\alpha(x^{-\alpha}\phi) + \frac{\omega^{\frac{1}{\alpha}}\big((\frac{7}{2}-\alpha)(1-\alpha)-\mu\alpha\big)}{\mu\big(\mu+\frac{3}{2}-\alpha\big)} h_{\mu+1}^\alpha(x^{1-\alpha}\phi)$$
$$-\frac{(1-\alpha)\omega^{\frac{2}{\alpha}}}{\mu\big(\mu+\frac{3}{2}-\alpha\big)} h_{\mu+2}^\alpha(x^{2-\alpha}\phi)\Big](\omega).$$

Case (III) If $\beta = 0$, $D_0^\alpha \equiv D_{0+}^\alpha$:
From Eq. (3.16), we have

$$
(h_{\mu}^{\alpha} D_0^{\alpha} \phi)(\omega) \approxeq \frac{\Gamma\big(\mu+\frac{3}{2}\big)}{\Gamma\big(\mu+\frac{3}{2}-\alpha\big)}\Big[\Big(\cos\alpha\pi - \sin\alpha\pi\,\tan(\mu-\alpha)\pi\Big)\Big]
\times\Bigg[h_{\mu}^{\alpha}(x^{-\alpha}\phi) + \frac{\omega^{\frac{1}{\alpha}}\big((\frac{7}{2}-\alpha)(1-\alpha)-\mu\alpha\big)}{\mu\big(\mu+\frac{3}{2}-\alpha\big)} h_{\mu+1}^{\alpha}(x^{1-\alpha}\phi)
-\frac{(1-\alpha)\omega^{\frac{2}{\alpha}}}{\mu\big(\mu+\frac{3}{2}-\alpha\big)} h_{\mu+2}^{\alpha}(x^{2-\alpha}\phi)\Bigg](\omega).
$$

Case (IV) If $\beta = 1$, $D_1^{\alpha} \equiv -D_{-}^{\alpha}$:
From Eq. (3.16), we find

$$
(h_{\mu}^{\alpha} D_1^{\alpha} \phi)(\omega) \approxeq -\frac{\Gamma\big(\mu+\frac{3}{2}\big)}{\Gamma\big(\mu+\frac{3}{2}-\alpha\big)}\Bigg[h_{\mu}^{\alpha}(x^{-\alpha}\phi) + \frac{\omega^{\frac{1}{\alpha}}\big((\frac{7}{2}-\alpha)(1-\alpha)-\mu\alpha\big)}{\mu\big(\mu+\frac{3}{2}-\alpha\big)}
\times h_{\mu+1}^{\alpha}(x^{1-\alpha}\phi) - \frac{(1-\alpha)\omega^{\frac{2}{\alpha}}}{\mu\big(\mu+\frac{3}{2}-\alpha\big)} h_{\mu+2}^{\alpha}(x^{2-\alpha}\phi)\Bigg](\omega).
$$

□

4 Application of fractional Hankel transform

In this section, we give the application of fractional Hankel transform in networks with time varying parameters.

From Eq. (2.3), the fractional Hankel transform is given by,

$$
(h_{\mu}^{\alpha} g)(\omega) = \int_0^{\infty} (\omega^{\frac{1}{\alpha}} t)^{\frac{1}{2}} J_{\mu}(\omega^{\frac{1}{\alpha}} t) g(t) \mathrm{d}t.
$$

If we put $g(t) = t^{\frac{1}{2}} f(t)$, then above equation yields

$$
(h_{\mu}^{\alpha} g)(\omega) = (\omega^{\frac{1}{\alpha}})^{\frac{1}{2}} \int_0^{\infty} t J_{\mu}(\omega^{\frac{1}{\alpha}} t) f(t) \mathrm{d}t.
$$

Then, we have

$$
(h_{\mu}^{\alpha} g)(\omega) = (\omega^{\frac{1}{\alpha}})^{\frac{1}{2}} (h_{\mu}^{\alpha} f)(\omega), \tag{4.1}
$$

where

$$
(h_{\mu}^{\alpha} f)(\omega) = \int_0^{\infty} t J_{\mu}(\omega^{\frac{1}{\alpha}} t) f(t) \mathrm{d}t. \tag{4.2}
$$

Now we find the inversion formula of fractional Hankel transform (4.2) of a function f as in Eq. (2.4) ,

$$
f(t) = \frac{1}{\alpha} \int_0^{\infty} \omega^{\frac{2}{\alpha}-1} J_{\mu}(\omega^{\frac{1}{\alpha}} t)(h_{\mu}^{\alpha} f)(\omega) \mathrm{d}\omega. \tag{4.3}
$$

In Gerardi (1959, p. 201) replacing a by $a^{\frac{1}{\alpha}}$, $0 < \alpha \leq 1$ in the Bessel's differential equation, we get

$$\frac{d^2x}{dt^2} + \frac{1}{t}\frac{dx}{dt} + \left((a^{\frac{1}{\alpha}})^2 - \frac{\mu^2}{t^2}\right)x = \frac{f(t)}{t^2}. \tag{4.4}$$

If $f(t) = 0$; the complementary solution is,

$$X_c = AJ_\mu(a^{\frac{1}{\alpha}}t) + BY_\mu(a^{\frac{1}{\alpha}}t),$$

where A and B are arbitrary constants, $J_\mu(a^{\frac{1}{\alpha}}t)$ and $Y_\mu(a^{\frac{1}{\alpha}}t)$ are Bessel's functions of first kind and second kind respectively.

Now proceeding as in Gerardi (1959, p. 201) we find

$$h^\alpha_\mu\left(\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \frac{\mu^2}{t^2} + (a^{\frac{1}{\alpha}})^2\right)x(t) = \left((a^{\frac{1}{\alpha}})^2 - (\omega^{\frac{1}{\alpha}})^2\right)(h^\alpha_\mu x)(\omega). \tag{4.5}$$

For finding the charge $q(t)$ in a positive feedback circuit at time t [Gerardi (1959), p. 203], we take the Bessel's differential equation

$$\frac{d^2q}{dt^2} + \frac{1}{t}\frac{dq}{dt} + \frac{q}{C_oL_o} - \frac{kq}{C_oL_ot^2} = \frac{K_1k}{L_ot^{(\mu+1)}}, \tag{4.6}$$

where C_o and L_o denote initial capacitance and initial inductance respectively, K_1, k are constants and

$$K_1 = L_o\frac{2^{(\mu+1)}}{k\pi}\frac{\Gamma\left(\mu+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}.$$

Putting $(a^{\frac{1}{\alpha}})^2 = \dfrac{1}{C_oL_o}$ and $\mu^2 = \dfrac{k}{C_oL_o}$ in Eq. (4.6), then

$$\frac{d^2q}{dt^2} + \frac{1}{t}\frac{dq}{dt} - \frac{\mu^2q}{t^2} + (a^{\frac{1}{\alpha}})^2q = \frac{K_1k}{L_ot^{(\mu+1)}}. \tag{4.7}$$

Taking fractional Hankel transform of Eq. (4.7) and using Eq. (4.5), we have

$$(h^\alpha_\mu q)(\omega)\left(a^{\frac{2}{\alpha}} - \omega^{\frac{2}{\alpha}}\right) = h^\alpha_\mu\left(\frac{K_1k}{L_ot^{\mu+1}}\right)(\omega). \tag{4.8}$$

From Gerardi (1959, p. 203), set

$$h^\alpha_\mu\left(\frac{K_1k}{L_ot^{\mu+1}}\right) = \frac{2}{\pi}\omega^{\frac{\mu-1}{\alpha}},$$

then Eq. (4.8) becomes

$$(h^\alpha_\mu q)(\omega) = \frac{2}{\pi}\frac{\omega^{\frac{\mu-1}{\alpha}}}{\left(a^{\frac{2}{\alpha}} - \omega^{\frac{2}{\alpha}}\right)}. \tag{4.9}$$

Taking the inverse fractional Hankel transform of Eq. (4.9) and using the technique of Gerardi (1959, p. 205), we get

$$q(t) = (a^{\frac{1}{\alpha}})^\mu\left(\tan\mu\pi\, J_\mu(a^{\frac{1}{\alpha}}t) - \sec\mu\pi\, H_{-\mu}(a^{\frac{1}{\alpha}}t)\right), \tag{4.10}$$

where $H_{-\mu}(a^{\frac{1}{\alpha}}t)$ is Struve's function of order μ and $J_\mu(a^{\frac{1}{\alpha}}t)$ is Bessel function of first kind of order μ.

5 Conclusions and observations

In the present paper, the authors have introduced the fractional Hankel transform in a new way and analyzed their properties, with the help of Riemann–Liouville fractional derivatives and integrals. These results are very useful to solve the problems of partial differential equations and ordinary differential equations. This theory will be also applicable in engineering sciences and theoretical physics. In Sect. 4, we have done an application of fractional Hankel transformation in ordinary differential equation for the networks with time varying parameters.

In this section, the graphical representations of a signal in time domain as well as in frequency domain due to fractional Hankel transform are shown with some remarkable observations.

For example, for $\mu > -1, a > 0$, the classical Hankel transform of the function

$$f(t) = t^{\mu+\frac{1}{2}} e^{-at^2} \tag{5.1}$$

is given in Erdélyi et al. (1954) as

$$(h_\mu f)(\omega) = \frac{\omega^{\mu+\frac{1}{2}}}{(2a)^{\mu+1}} e^{-\frac{\omega^2}{4a}}.$$

The fractional Hankel transform of the function $f(t)$ is given as

$$(h_\mu^\alpha f)(\omega) = \frac{\omega^{\frac{2\mu+1}{2\alpha}}}{(2a)^{\mu+1}} e^{-\frac{\omega^{\frac{2}{\alpha}}}{4a}}. \tag{5.2}$$

For $\mu = \frac{1}{2}$ and $a = 1$, we can compare the results between classical Hankel transform and fractional Hankel transform, for different values of α.

Case (I) For $\alpha = 1$, the classical Hankel transform of $f(t)$ is given by

$$(h_{\frac{1}{2}} f)(\omega) = \frac{\omega}{(2)^{\frac{3}{2}}} e^{-\frac{\omega^2}{4}}. \tag{5.3}$$

Case (II) For $\alpha = \frac{1}{2}$, the fractional Hankel transform of $f(t)$ is,

$$(h_{\frac{1}{2}}^{\frac{1}{2}} f)(\omega) = \frac{\omega^2}{(2)^{\frac{3}{2}}} e^{-\frac{\omega^4}{4}}. \tag{5.4}$$

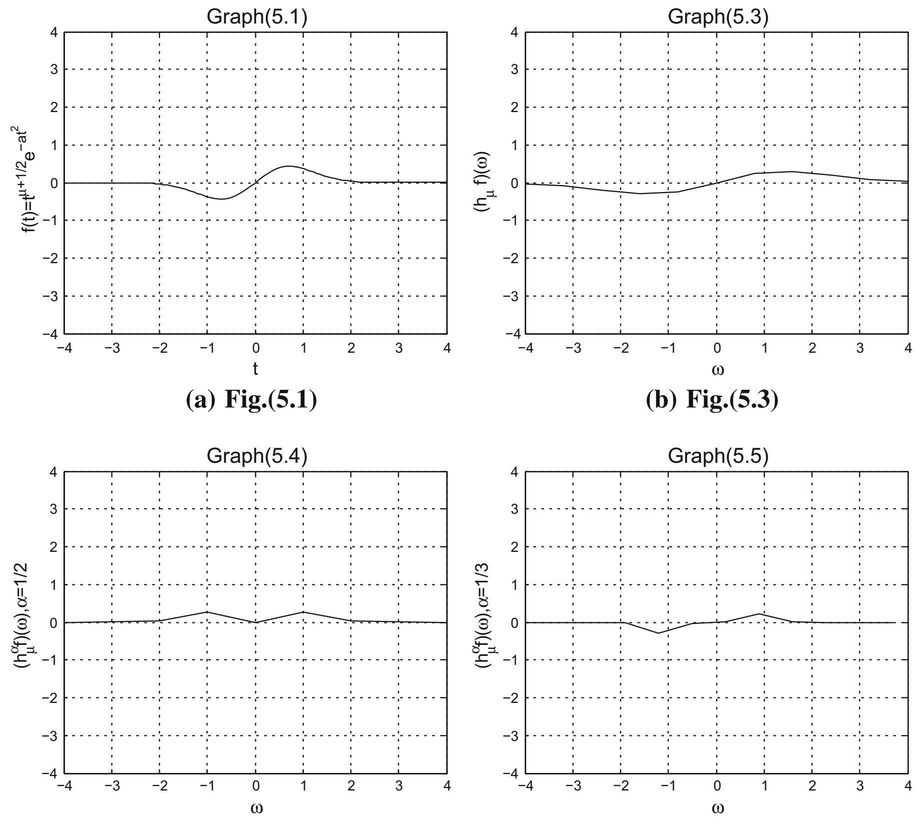
Case (III) For $\alpha = \frac{1}{3}$, the fractional Hankel transform of $f(t)$ is,

$$(h_{\frac{1}{2}}^{\frac{1}{3}} f)(\omega) = \frac{\omega^3}{(2)^{\frac{3}{2}}} e^{-\frac{\omega^6}{4}}. \tag{5.5}$$

Using MATLAB the following graphical representations are obtained:

With the help of graphical representations Fig. 1a–d, it is shown that the peaks obtained in frequency domain, due to fractional Hankel transform are more clear and area bounded by them is lesser than that of classical Hankel transform. The above analysis indicates that the whole information of the signal in case of fractional Hankel transform is more accurate than the classical Hankel transform, for different values of α, such as $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{3}$.

Further, it is observed that the result obtained in Theorem 3.8 has a nice graphical representation. For the function $f(t) = t^{\mu+\frac{1}{2}} e^{-at^2}$, taking $\mu = \frac{1}{2}$ and $a = 1$, the results have



(a) Fig.(5.1) (b) Fig.(5.3)

(c) Fig.(5.4) (d) Fig.(5.5)

Fig. 1 Graphical representations of a signal in time domain as well as in frequency domain, due to classical Hankel transform and fractional Hankel transform for $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{3}$, respectively

been compared between classical Hankel transform of ordinary derivative of the function f and fractional Hankel transform of fractional derivative of the function f.

The following approximate results obtained:

Case (I) Classical Hankel transform of ordinary derivative of $f(t)$ is given by

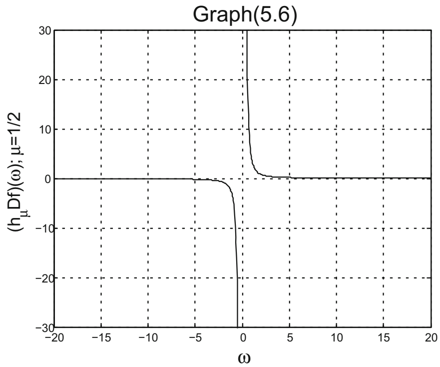
$$(h_{\frac{1}{2}} Df)(\omega) \approxeq \sqrt{\frac{2}{\pi}} \left(\frac{1}{\omega} + \frac{4}{\omega^3} \right). \tag{5.6}$$

Case (II) Taking $\alpha = \frac{1}{2}$ and $\beta = 1$, fractional Hankel transform of fractional derivative of $f(t)$ is given by

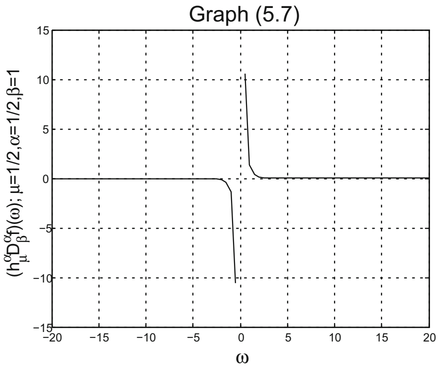
$$\left(h^{\frac{1}{2}}_{\frac{1}{2}} D^{\frac{1}{2}}_1 f \right)(\omega) \approxeq \frac{7}{3\sqrt{\pi}\omega^3}. \tag{5.7}$$

Case (III) Taking $\alpha = \frac{1}{3}$ and $\beta = 1$, fractional Hankel transform of fractional derivative of $f(t)$ is given by

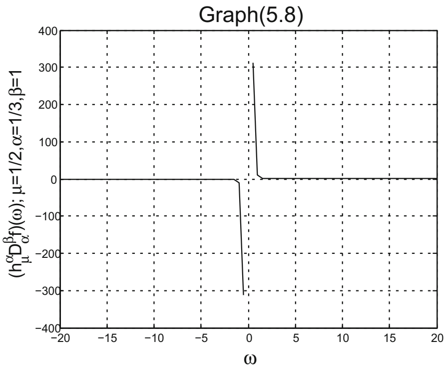
$$\left(h^{\frac{1}{3}}_{\frac{1}{2}} D^{\frac{1}{3}}_1 f \right)(\omega) \approxeq \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{6})\Gamma(\frac{5}{3})} \frac{2^{\frac{7}{6}}}{\omega^5} \left(\frac{283}{135} \right). \tag{5.8}$$



(a) Fig.(5.6)



(b) Fig.(5.7)



(c) Fig.(5.8)

Fig. 2 Graphical representations of $h_{\mu}^{\alpha}(D_{\alpha}^{\beta} f)(\omega)$ for a signal f, in classical sense as well as in fractional sense for $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{3}$, respectively

The results have been found quite similar to that of classical Hankel transform, which is clear by the graphical representations of Fig. 2a–c respectively:

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