Chapter 2

Interaction between interfacial and sub-interfacial cracks in composite media

2.1 Introduction

Recently, to the study of effect of imperfection or flaws at the interface of bonded dissimilar elastic materials due to their practical importance in designing engineering structures and machines has attracted the interest of researchers. When a crack is found in the materials used as a member of such structures, then it is extremely significant matter in engineering that one would regard the flaws or cracks as harmless or dangerous which seriously affect the corresponding structural integrity.

Various inter and intra-component defects contained in all existing materials. The presence of these defects decreases the strength and the lifetime of the structures with increase of the cost of exploitation. Therefore it becomes important to understand the mechanism of fracture in cracked materials and to predict that the residual strength of the structure. The previous studies were restricted by the case of cracked homogeneous solids, but the solutions of much more complex problems of composite structures with cracks subjected to normal loading are required in industries.

So the study of the stress field in presence of the crack at the interface dissimilar anisotropic materials for the practical importance. For solving these types of problems in anisotropic composite media involve mathematical complexities. The nature of the singularities is different for a crack embedded in a homogeneous medium and for an

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interfal crack. Furthermore, from a mathematical point of view, there is no smooth transition from one solution to the other, as the crack distance from the interface goes to zero. In the above context, this area, therefore, requires a great deal of attention.

For many potential applications the understanding of the mechanical properties of layered materials becomes challenging. But now a day such materials are widely used in the industries. So our job is to study the propagation of the interfacial cracks. It is believed that the propagation of the interfacial cracks depend on the asymptotic expansion of the stress in homogeneous media near the crack tip.

Recently sandwich structures are investigated by Zinno et al. (2010) for structural element of railway vehicle body. An important issue is raised by Altenbach and Krunch (2013), Altenbach et al. (2004), that group of laminated composites are sandwiched composite. It is also referred that sandwiched composites should be like two thin faces that will sandwich a core.

Due to uniform components, the interaction among the cracks occurs. A method was developed by Gorbatikh et al. (2007), through which stress intensity factors (SIF) at tips of two dimensional cracks were found at very small distance compared to crack length. Petrova (2000) proposed a method for modelling of interaction of different sets of cracks by approximation method. Das et al. (2004) made an effort to derive the analytical expression of SIFs of a pair of interfacial Griffith cracks situated in a composite media. In last few decades lot of research has been done in the area of composite media containing interfacial cracks (Das et al. (1996), Das and Patra (1998), Dhaliwal et al. (1990), Erdogan and Gupta (1971), Lowengrub and Sneddon (1973), Rice and Sih (1965), Sadowski et al. (2012), Itou (1988, 2010), Wu et al. (2003)). Due to lightweight and strength, Beryllium is of immense interest in spacecraft applications. Pure beryllium material is used in the manufacture of aircraft disc brakes, nuclear weapons and reactors, missile parts, heat shields, X-ray machine parts, mirrors and spacecrafts. Due to high-density property, Uranium is used as armour plate for tanks and counter weights in commercial airplanes. Thus during the formation of composite material very much useful in engineering purposes, orthotropic materials viz., Beryllium and Uranium would be of great interest. It is also found from the literature survey that Electrical resistivity, magnetic susceptibility and specific heat data reveal that Uranium–Beryllium-13 (UBe13) is superconducting below 0.85 K, while superconducting state appears to be extremely stable with an initial slope of the temperature derivative of the critical field. But to the best of my knowledge the interaction between an interfacial crack with a sub interfacial crack in orthotropic composite media has not yet been studied.

In this chapter an endeavour is made to calculate the stress intensity factors of an interfacial crack bonded between two orthotropic planes in presence of a sub interfacial crack. The effects of presence of sub-interfacial crack and also its position on the interfacial crack are calculated through stress magnification factors. The numerical values of possibilities of crack arrest and propagation of crack are displayed graphically for different particular cases.

2.2 Formulation of the Problem

Consider the electrostatic problem of two parallel cracks of finite length situated at the interfaces of an orthotropic strip 1 of thickness h and half plane 2 and also at the interface of strip 1 and the half plane 3. The geometry of the problem is shown in the Fig. 2.1 under the assumption of plane strain in an orthotropic medium, the displacement equations of motion is given by

$$C_{11}^{(i)} \frac{\partial^2 u^{(i)}}{\partial x^2} + C_{66}^{(i)} \frac{\partial^2 u^{(i)}}{\partial y^2} + (C_{12}^{(i)} + C_{66}^{(i)}) \frac{\partial^2 v^{(i)}}{\partial x \partial y} = 0, \tag{2.1}$$

$$C_{22}^{(i)}\frac{\partial^2 v^{(i)}}{\partial y^2}+C_{66}^{(i)}\frac{\partial^2 v^{(i)}}{\partial x^2}+(C_{12}^{(i)}+C_{66}^{(i)})\frac{\partial^2 u^{(i)}}{\partial x\partial y}=0, \tag{2.2}$$

where $C_{jk}^{(i)}$'s are the elastic constants. Here super scripts $i=1,2,3$ refer to the media 1, 2, 3 respectively.

It is assumed that crack defined by $|x|\le 1,\ y=h$ opened by internal normal and shearing tractions $p_1(x)$ and $p_2(x)$ respectively and $p_3(x)$ and $p_4(x)$ are those for crack defined by $|x|\le 1, y=0$.

Boundary condition on *y=h* are

$$\sigma_{yy}^{(3)}(x,h)=\sigma_{yy}^{(1)}(x,h)=-p_1(x), \qquad |x|\le 1, \tag{2.3}$$

$$\sigma_{xy}^{(3)}(x,h)=\sigma_{xy}^{(1)}(x,h)=-p_2(x), \qquad |x|\le 1, \tag{2.4}$$

$$u^{(3)}(x,h)=u^{(1)}(x,h), \qquad |x|>1, \tag{2.5}$$

$$v^{(3)}(x,h)=v^{(1)}(x,h), \qquad |x|>1, \tag{2.6}$$

$$\sigma_{yy}^{(3)}(x,h)=\sigma_{yy}^{(1)}(x,h), \qquad |x|>1, \tag{2.7}$$

$$\sigma_{xy}^{(3)}(x,h)=\sigma_{xy}^{(1)}(x,h), \qquad |x|>1. \tag{2.8}$$

The boundary conditions at *y=0* are

$$\sigma_{yy}^{(1)}(x,0)=\sigma_{yy}^{(2)}(x,0)=-p_3(x), \qquad |x|\le 1, \tag{2.9}$$

$$\sigma_{xy}^{(1)}(x,0)=\sigma_{xy}^{(2)}(x,0)=-p_4(x), \qquad |x|\le 1, \tag{2.10}$$

$$u^{(1)}(x,0)=u^{(2)}(x,0), \qquad |x|>1, \tag{2.11}$$

$$v^{(1)}(x,0)=v^{(2)}(x,0), \qquad |x|>1, \tag{2.12}$$

$$\sigma_{yy}^{(1)}(x,0)=\sigma_{yy}^{(2)}(x,0), \qquad |x|>1, \tag{2.13}$$

$$\sigma_{xy}^{(1)}(x,0)=\sigma_{xy}^{(2)}(x,0), \qquad |x|>1. \tag{2.14}$$

If the elastic strip (1) and elastic half plane (3) are identical then the problem is reduced to our desired problem of an interfacial crack bonded between two half planes containing an embedded sub interfacial crack situated at the height h from the interface position (Fig. 2.2). In both the figures the half planes are represented by the inclined arrows.

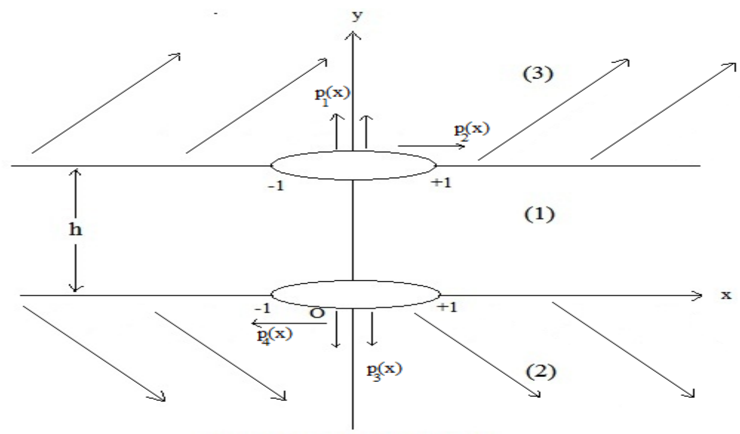

Fig. 2.1 Geometry of the problem

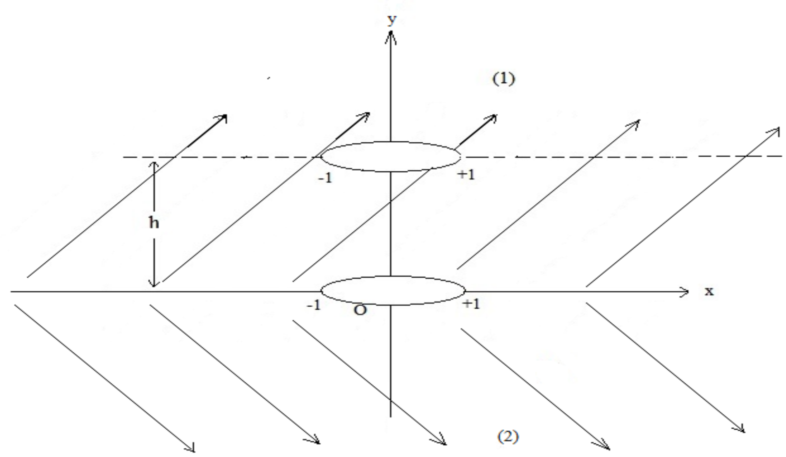

Fig. 2.2 Geometry of interfacial and sub-interfacial cracks

2.3 Solution of the problem

The appropriate integral solution of equations (2.1) and (2.2) can be taken as

$$u^{(i)}(x,y) = \int_0^\infty A^{(i)}(s,y)\,\sin sx\,ds, \tag{2.15}$$

$$v^{(i)}(x,y) = \int_0^\infty B^{(i)}(s,y)\,\cos sx\,ds. \tag{2.16}$$

For the strip 1, the solution of the above equations are given as

$$A^{(1)}(s,y) = A_1^{(1)}(s)ch(\gamma_1^{(1)}sy) + A_2^{(1)}(s)ch(\gamma_2^{(1)}sy)$$

$$+ C_1^{(1)}(s)sh(\gamma_1^{(1)}sy) + C_2^{(1)}(s)sh(\gamma_2^{(1)}sy), \qquad (2.17)$$

$$B^{(1)}(s,y) = B_1^{(1)}(s)sh(\gamma_1^{(1)}sy) + B_2^{(1)}(s)sh(\gamma_2^{(1)}sy)$$

$$+ D_1^{(1)}(s)ch(\gamma_1^{(1)}sy) + D_2^{(1)}(s)ch(\gamma_2^{(1)}sy), \qquad (2.18)$$

and for the half planes 2 and 3 solution of the above equations are given as

$$A^{(i)}(s,y) = A_1^{(i)}(s)e^{(-1)^i\gamma_1^{(i)}sy} + A_2^{(i)}(s)e^{(-1)^i\gamma_2^{(i)}sy}, \qquad (2.19)$$

$$B^{(i)}(s,y) = (-1)^i[B_1^{(i)}(s)e^{(-1)^i\gamma_1^{(i)}sy} + B_2^{(i)}(s)e^{(-1)^i\gamma_2^{(i)}sy}], \qquad (2.20)$$

where $\gamma_1^{(i)}$ and $\gamma_2^{(i)}(<\gamma_1^{(i)})$, $i = 1,2,3$ are the positive roots of the equation

$$C_{66}^{(i)}C_{22}^{(i)}\gamma^4 + [(C_{12}^{(i)} + C_{66}^{(i)})^2 - C_{22}^{(i)}C_{11}^{(i)} - (C_{66}^{(i)})^2]\gamma^2 + C_{11}^{(i)}C_{66}^{(i)} = 0, \qquad (2.21)$$

and $B_j^{(i)}(s), D_j^{(i)}(s)$ are related to the arbitrary functions $A_j^{(i)}(s)$ and $C_j^{(i)}(s)$ by

$$B_j^{(i)}(s) = -\alpha_j^{(i)}A_j^{(i)}(s)/\gamma_j^{(i)}, \quad D_j^{(i)}(s) = -\alpha_j^{(i)}C_j^{(i)}(s)/\gamma_j^{(i)}. \qquad (2.22)$$

The boundary conditions (2.3) and (2.7) in conjunction with (2.4) and (2.8) give the expressions of $A_1^{(1)}(s), A_2^{(1)}(s)$ as

$$A_1^{(1)}(s) = \delta_1(s)e^{-\gamma_1^{(3)}sh}A_1^{(3)}(s) + \delta_2(s)e^{-\gamma_1^{(3)}sh}A_2^{(3)}(s) - (1+\delta_3(3))C_1^{(1)}(s) + \delta_4(s)C_2^{(1)}(s), \qquad (2.23)$$

$$A_2^{(1)}(s) = -\delta_5(s)e^{-\gamma_1^{(3)}sh}A_1^{(3)}(s) - \delta_6(s)e^{-\gamma_2^{(3)}sh}A_2^{(3)}(s)$$

$$-\delta_7(s)C_1^{(1)}(s) - (1+\delta_8(s))C_2^{(1)}(s), \qquad (2.24)$$

where $\delta_1(s) = 2[(\mu_2^{(1)}\eta_1^{(3)} + \mu_1^{(3)}\eta_2^{(1)}) - (\mu_2^{(1)}\eta_1^{(3)} - \mu_1^{(3)}\eta_2^{(1)})e^{-2\gamma_2^{(1)}sh}]e^{-\gamma_1^{(1)}sh}/D, \qquad (2.25)$

$$\delta_2(s) = 2[(\mu_2^{(1)}\eta_2^{(3)} + \mu_2^{(3)}\eta_2^{(1)}) - (\mu_2^{(1)}\eta_2^{(3)} - \mu_2^{(3)}\eta_2^{(1)})e^{-2\gamma_2^{(1)}sh})]e^{-\gamma_1^{(1)}sh}/D, \qquad (2.26)$$

$$\delta_3(s) = [2(\mu_2^{(1)}\eta_1^{(1)} + \mu_1^{(1)}\eta_2^{(1)})e^{-2\gamma_1^{(1)}sh} + 2(\mu_2^{(1)}\eta_1^{(1)} - \mu_1^{(1)}\eta_2^{(1)})e^{-2(\gamma_1^{(1)}+\gamma_2^{(1)})sh}]/D, \qquad (2.27)$$

$$\delta_4(s) = 4\mu_2^{(1)}\eta_2^{(1)}e^{-(\gamma_1^{(1)}+\gamma_2^{(1)})sh}/D, \qquad (2.28)$$

$$\delta_5(s) = 2[(\mu_1^{(1)}\eta_1^{(3)} + \mu_1^{(3)}\eta_1^{(1)}) - (\mu_1^{(1)}\eta_1^{(3)} - \mu_1^{(3)}\eta_1^{(1)})e^{-2\gamma_1^{(1)}sh}]e^{-\gamma_2^{(1)}sh} / D, \tag{2.29}$$

$$\delta_6(s) = 2[(\mu_1^{(1)}\eta_2^{(3)} + \mu_2^{(3)}\eta_1^{(1)}) - (\mu_1^{(1)}\eta_2^{(3)} - \mu_2^{(3)}\eta_1^{(1)})e^{-2\gamma_1^{(1)}sh})]e^{-\gamma_2^{(1)}sh} / D, \tag{2.30}$$

$$\delta_7(s) = 4\mu_1^{(1)}\eta_1^{(1)}e^{-(\gamma_1^{(1)}+\gamma_2^{(1)})sh} / D, \tag{2.31}$$

$$\delta_8(s) = [2(\mu_2^{(1)}\eta_1^{(1)} + \mu_1^{(1)}\eta_2^{(1)})e^{-2\gamma_2^{(1)}sh} + 2(\mu_2^{(1)}\eta_1^{(1)} - \mu_1^{(1)}\eta_2^{(1)})e^{-2(\gamma_1^{(1)}+\gamma_2^{(1)})sh}] / D, \tag{2.32}$$

and $D = (\mu_2^{(1)}\eta_1^{(1)} - \mu_1^{(1)}\eta_2^{(1)}) + (\mu_2^{(1)}\eta_1^{(1)} + \mu_1^{(1)}\eta_2^{(1)})e^{-2\gamma_1^{(1)}sh} - (\mu_2^{(1)}\eta_1^{(1)} + \mu_1^{(1)}\eta_2^{(1)})e^{-2\gamma_2^{(1)}sh}$

$$- (\mu_2^{(1)}\eta_1^{(1)} - \mu_1^{(1)}\eta_2^{(1)})e^{-2(\gamma_1^{(1)}+\gamma_2^{(1)})sh}.$$

The boundary conditions (2.9) and (2.13) in conjunction with equations (2.10) and (2.14) yield

$$A_1^{(2)}(s) = [\mu_2^{(2)}\eta_1^{(1)}A_1^{(1)}(s) + \mu_2^{(2)}\eta_2^{(1)}A_2^{(1)}(s) - \mu_1^{(1)}\eta_2^{(2)}C_1^{(1)}(s) - \mu_2^{(1)}\eta_2^{(2)}C_2^{(1)}(s)] / D', \tag{2.33}$$

$$A_2^{(2)}(s) = [-\mu_1^{(2)}\eta_1^{(1)}A_1^{(1)}(s) - \mu_1^{(2)}\eta_2^{(1)}A_2^{(1)}(s) + \mu_1^{(1)}\eta_1^{(2)}C_1^{(1)}(s) + \mu_2^{(1)}\eta_1^{(2)}C_2^{(1)}(s)] / D', \tag{2.34}$$

with $D' = \mu_2^{(2)}\eta_1^{(2)} - \mu_1^{(2)}\eta_2^{(2)}$, $\beta_j^{(i)} = \alpha_j^{(i)} + (\gamma_j^{(i)})^2$, $\eta_j^{(i)} = C_{12}^{(i)} - C_{22}^{(i)}\alpha_j^{(i)}$,

$\mu_j^{(i)} = C_{66}^{(i)}\dfrac{\beta_j^{(i)}}{\gamma_j^{(i)}}$, $i = 1, 2, 3$; $j = 1, 2$.

With considerable computational effort, equations (2.3), (2.4), (2.9) and (2.10) are reduced to the following singular integral equations for the determination of unknown functions $f_i(x)$ satisfying the conditions

$$\int_{-1}^{1} f_i(t)dt = 0, \qquad i = 1, 2, 3, 4. \tag{2.35}$$

$$\phi_l(x) + \frac{1}{\pi i e_l r_l}\int_{-1}^{1}\frac{\phi_l(t)}{t-x}dt + \int_{-1}^{1}[k_{l1}(x,t)\phi_1(t)dt + k_{l2}(x,t)\phi_2(t) + k_{l3}(x,t)\phi_3(t) + k_{l4}(x,t)\phi_4(t)]dt$$

$$= g_l(x), \; -1 < x < 1, \quad l = 1,2,3,4, \tag{2.36}$$

where $\phi_l(x) = \sqrt{a_1 b_1}\, f_1(x) + i r_l \sqrt{c_1 d_1}\, f_2(x), \qquad l = 1,2$

$$= \sqrt{a_2 b_2}\, f_3(x) + i r_l \sqrt{c_2 d_2}\, f_4(x), \qquad l = 3,4$$

$$e_l = \sqrt{a_1 b_1 c_1 d_1}, \qquad l = 1, 2,$$

$$= \sqrt{a_2 b_2 c_2 d_2}, \qquad l = 3, 4,$$

$$r_l = (-1)^l, \quad l = 1, 2, 3, 4,$$

$$g_l(x) = \frac{2}{\pi}[\sqrt{b_1 / a_1}\, p_1(x) - r_l \sqrt{d_1 / c_1}\, p_2(x)],$$

$$= \frac{2}{\pi}[\sqrt{b_2 / a_2}\, p_3(x) - r_l \sqrt{d_2 / c_2}\, p_4(x)].$$

$$2\pi K_{lm} = (\frac{1}{a_1} k_{11}(x,t) + r_l r_m \frac{1}{c_1} k_{22}(x,t)) + i e_l^{-1}(r_l d_1 k_{21}(x,t) - r_m b_1 k_{12}(x,t)), \qquad l = 1, 2,$$

$$= (\frac{1}{a_2} k_{33}(x,t) + r_l r_m \frac{1}{c_2} k_{44}(x,t)) + i e_l^{-1}(r_l d_2 k_{43}(x,t) - r_m b_2 k_{34}(x,t)), \qquad l = 3, 4,$$

where

$$a_1 = -[\eta_1^{(3)} w_{22}(\infty) - \eta_2^{(3)} w_{21}(\infty)], \quad \frac{1}{b_1} = [\eta_1^{(3)} w_{12}(\infty) - \eta_2^{(3)} w_{11}(\infty)],$$

$$c_1 = -[\mu_1^{(3)} w_{12}(\infty) - \mu_2^{(3)} w_{11}(\infty)], \quad \frac{1}{d_1} = [\mu_1^{(3)} w_{22}(\infty) - \mu_2^{(3)} w_{21}(\infty)],$$

$$a_2 = -[\eta_1^{(1)} w_{44}(\infty) - \eta_2^{(1)} w_{43}(\infty)], \quad \frac{1}{b_2} = [\eta_1^{(1)} w_{34}(\infty) - \eta_2^{(1)} w_{33}(\infty)],$$

$$c_2 = -[\mu_1^{(1)} w_{34}(\infty) - \mu_2^{(1)} w_{33}(\infty)], \quad \frac{1}{d_2} = [\mu_1^{(1)} w_{44}(\infty) - \mu_2^{(1)} w_{43}(\infty)],$$

$$k_{ij}(x,t) = \int_0^\infty d_{ij}(s) \cos s(t-x), \quad i = 1, 3; \quad j = 1, 3,$$

$$i = 2, 4; \; j = 2, 4, \tag{2.37}$$

$$k_{ij}(x,t) = \int_0^\infty d_{ij}(s) \sin s(t-x), \quad i = 1, 3; \quad j = 2, 4,$$

$$i = 2, 4; \; j = 1, 3. \tag{2.38}$$

The expressions of $d_{ij}(s)$ are given in Appendix-I.

2.4 Solution of the integral equations

The solution of above integral equations in (2.36) may be assumed as

$$\phi_l(x) = \omega_l(x) \sum_{n=0}^{\infty} C_{\ln} P_n^{(\alpha_l, \beta_l)}(x), \qquad l = 1, 2, 3, 4, \tag{2.39}$$

where $\omega_l(x) = (1-x)^{\alpha_l}(1+x)^{\beta_l}$, $\alpha_l = -\frac{1}{2} + i\omega_l$, $\beta_l = -\frac{1}{2} - i\omega_l$, $\omega_l = r_l\omega$

with $\omega = \frac{1}{2\pi}\ln\left|\frac{1+e_l}{1-e_l}\right|$.

Now using equation (2.35), we get

$$\int_{-1}^{1}\phi_i(t)dt = 0, \qquad i = 1,2,3,4,$$

which implies $C_{l0} = 0, \quad l = 1,2,3,4.$

Substituting equation (2.36) in the equation (2.39) and integrating with respect to x from -1 to 1, we get the following system of simultaneous algebraic equations for the determination of unknown constants C_{lj} as

$$\frac{\sqrt{1-e_l^2}}{2ie_l r_l} C_{lj+1}\ \theta_j^{(-\alpha_l,-\beta_l)} + \sum_{n=1}^{\infty}\sum_{m=1}^{4} C_{mn} L^{\bullet}{}_{lmnj}(x) = F_{lj}(x), \tag{2.40}$$

where

$$L^{\bullet}{}_{lmnj}(x) = \int_{-1}^{1}\omega_l^{-1}(x)P_j^{(-\alpha_l,-\beta_l)}(x)\,dx\left(\int_{-1}^{1}k_{lm}(x,t)\omega_m(t)P_n^{(\alpha_m,\beta_m)}(t)dt\right),$$

$$F_{lj}(x) = \int_{-1}^{1}g_l(x)\omega_l^{-1}(x)P_j^{(-\alpha_l,-\beta_l)}(x)d(x), \qquad l = 1,2,3,4, \quad j = 0,1,2,.....,$$

$$\theta_j^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{\Gamma(j+1)(2j+\alpha+\beta+1)\Gamma(j+\alpha+\beta+1)}.$$

Finally, the stress intensity factors at the tips of the crack at $y = 0$ are given by

$$\sqrt{b_2/a_2}\ K_I + ir_l\sqrt{d_2/c_2}\ K_{II} = \frac{i\pi\sqrt{1-e_l^2}}{2e_l r_l}\sum_{n=1}^{\infty}C_{\ln}P_n^{(\alpha_l,\beta_l)}(1), \quad l = 3,4. \tag{2.41}$$

Now stress magnification factors (SMF) are defined by $M_I = \frac{K_I}{K_I^*}$ and $M_{II} = \frac{K_{II}}{K_{II}^*}$ (Saha et al. (1999), Das (2006), Kobayashi and Moss (1969), Rose (1986), Sneddon and Lowengrub (1969), Nisitani and Murakami (1974)), where K_I^* and K_{II}^* are the stress intensity factors of mode II crack situated at the interface of two half planes in the absence of sub-interfacial crack and these are given by Das et al. (2004) as

$$K_I^* = -p, \quad K_{II}^* = -\frac{p}{\pi}\sqrt{\frac{b_1 c_1}{a_1 d_1}} \ln\left|\frac{1+\sqrt{a_1 b_1 c_1 d_1}}{1-\sqrt{a_1 b_1 c_1 d_1}}\right|,$$

where $p_1(x) = p$ and $p_2(x) = 0$.

2.5 Results and discussions

In this section the strip 1 and half plane 3 are considered to be identical. The orthotropic materials 1 and 2 are considered as α-Uranium and Beech wood respectively. The elastic constants in *GPa* unit are taken as

Materials	$C_{11}^{(i)}$	$C_{22}^{(i)}$	$C_{12}^{(i)}$	$C_{66}^{(i)}$
α-Uranium *(i=1)*	148.03	133.48	32.06	51.22
Beech Wood *(i=2)*	1.172	10.89	1.03	0.71

The numerical values of stress magnification factors M_I and M_{II} are depicted through Figs. 2.3 and 2.4 respectively for different values of h when $p_1(x) = p$, $p_2(x) = 0$, $p_3(x) = p$ and $p_4(x) = 0$. It is seen from the Fig. 2.3 that there is a possibility of shielding. But from Fig. 2.4, it is observed that at *h=33*, there is a possibility of amplification. Thus after *h=33*, the propagation tendency of interfacial crack increases. It is clear from the graphical representations that when the distance between the sub - interfacial and interfacial cracks are of moderate value, then there is a possibility of arrest of the propagation of interfacial crack. But as the distance between the cracks increases, there is a possibility of crack propagation. This is due to the fact that as a limiting case when h becomes very large, the effect of sub - interfacial crack disappears and it causes propagation of interfacial crack.

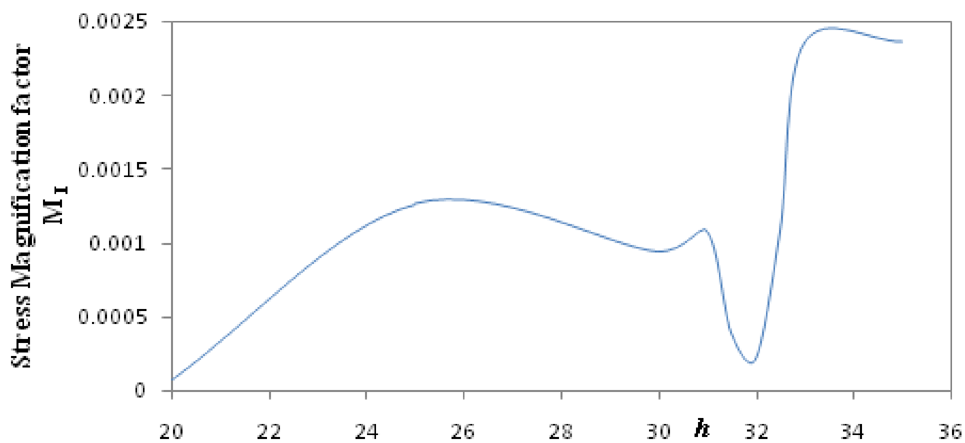

Fig. 2.3 Plot of stress magnification factor M_I vs. h

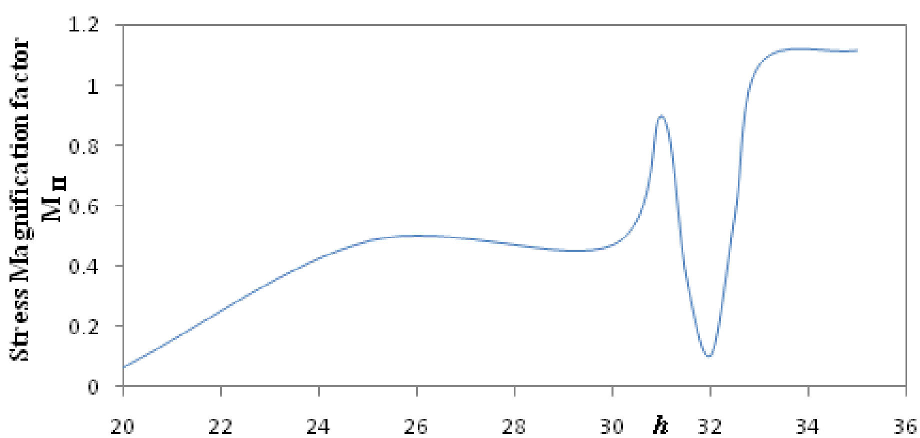

Fig. 2.4 Plot of Stress Magnification factor M_{II} vs. h

2.6 Conclusion

In this chapter two important goals have been achieved. First one is finding the expression of SMF for the interfacial crack due to the presence of sub-interfacial crack. Second one is the graphical presentation of possibilities of crack arrest and also crack propagation for various values of the distance between the cracks.

Appendix - A

$d_{11}(s) = -\eta_1^{(3)} w'_{22}(s) + \eta_2^{(3)} w'_{21}(s)$, $\quad d_{12}(s) = \eta_1^{(3)} w'_{12}(s) - \eta_2^{(3)} w'_{11}(s)$,

$d_{13}(s) = [-\eta_1^{(3)}(\omega_{13}(s)\omega_{22}(s) - \omega_{23}(s)\omega_{12}(s)) + \eta_2^{(3)}(\omega_{13}(s)\omega_{21}(s) - \omega_{23}(s)\omega_{11}(s))]/D_1(s)$,

$d_{14}(s) = [-\eta_1^{(3)}(\omega_{14}(s)\omega_{22}(s) - \omega_{24}(s)\omega_{12}(s)) - \eta_2^{(3)}(\omega_{14}(s)\omega_{21}(s) - \omega_{24}(s)\omega_{11}(s))]/D_1(s)$,

$d_{21}(s) = -\mu_1^{(3)} w'_{22}(s) + \mu_2^{(3)} w'_{21}(s)$, $\quad d_{22}(s) = -\mu_1^{(3)} w'_{12}(s) + \mu_2^{(3)} w'_{11}(s)$,

$d_{23}(s) = [\mu_1^{(3)}(\omega_{13}(s)\omega_{22}(s) - \omega_{23}(s)\omega_{12}(s)) - \mu_2^{(3)}(\omega_{13}(s)\omega_{21}(s) - \omega_{23}(s)\omega_{11}(s))]/D_1(s)$,

$d_{24}(s) = [\mu_1^{(3)}(\omega_{14}(s)\omega_{22}(s) - \omega_{24}(s)\omega_{12}(s)) - \mu_2^{(3)}(\omega_{14}(s)\omega_{21}(s) - \omega_{24}(s)\omega_{11}(s))]/D_1(s)$,

$D_1(s) = \omega_{11}(s)\omega_{22}(s) - \omega_{12}(s)\omega_{21}(s)$,

$d_{31}(s) = [M_{11}(s)(\omega_{31}(s)\omega_{44}(s) - \omega_{41}(s)\omega_{34}(s)) + M_{12}(s)(\omega_{32}(s)\omega_{43}(s) - \omega_{41}(s)\omega_{33}(s))]/D_2(s)$,

$d_{32}(s) = [M_{11}(s)(\omega_{32}(s)\omega_{44}(s) - \omega_{42}(s)\omega_{34}(s)) - M_{12}(s)(\omega_{32}(s)\omega_{43}(s) - \omega_{42}(s)\omega_{33}(s))]/D_2(s)$,

$d_{33}(s) = M_{11}(\infty)w'_{44}(s) + M'_{11}(s)w'_{44}(\infty) + M'_{11}(s)w'_{44}(s) - M_{12}(\infty)w'_{43}(s) - M'_{12}(s)w_{43}(\infty) - M'_{12}(s)w'_{43}(s)$,

$d_{34}(s) = -M_{11}(\infty)w'_{34}(s) - M'_{11}(s)w'_{34}(\infty) - M'_{11}(s)w'_{34}(s) + M_{12}(\infty)w'_{33}(s) + M'_{12}(s)w_{33}(\infty) + M'_{12}(s)w'_{33}(s)$,

$d_{41}(s) = [\mu_1^{(1)}(\omega_{31}(s)\omega_{44}(s) - \omega_{41}(s)\omega_{34}(s)) - \mu_2^{(1)}(\omega_{31}(s)\omega_{43}(s) - \omega_{41}(s)\omega_{33}(s))]/D_2(s)$,

$d_{42}(s) = [\mu_1^{(1)}(\omega_{32}(s)\omega_{44}(s) - \omega_{42}(s)\omega_{34}(s)) - \mu_2^{(1)}(\omega_{32}(s)\omega_{43}(s) - \omega_{42}(s)\omega_{33}(s))]/D_2(s)$,

$d_{43}(s) = -\mu_1^{(1)} w'_{44}(s) + \mu_2^{(1)} w'_{43}(s)$, $\quad d_{44}(s) = -\mu_1^{(1)} w'_{34}(s) + \mu_2^{(1)} w'_{33}(s)$,

$M_{11}(s) = \eta_1^{(1)}\delta_2(s) - \eta_2^{(1)}\delta_6(s)$, $\quad M_{12}(s) = \eta_1^{(1)}\delta_1(s) - \eta_2^{(1)}\delta_5(s)$,

$D_2(s) = \omega_{33}(s)\omega_{44}(s) - \omega_{34}(s)\omega_{43}(s)$,

$w'_{ij}(s) = w_{ij}(s) - w_{ij}(\infty)$, $\quad i, j = 1,2$; $i, j = 3,4$

and $M'_{1j}(s) = M_{1j}(s) - M_{1j}(\infty)$, $\quad j = 1,2$,

$w_{ij}(s) = \omega_{ij}(s)/[\omega_{11}(s)\omega_{22}(s) - \omega_{12}(s)\omega_{21}(s)]$, $\quad i, j = 1,2$

$\quad = \omega_{ij}(s)/[\omega_{33}(s)\omega_{44}(s) - \omega_{34}(s)\omega_{43}(s)]$, $\quad i, j = 3,4$, where

$$\omega_{i1}(s) = L_{i1}(s) + [L_{i3}(s)(L_{31}(s)L_{44}(s) - L_{34}(s)L_{41}(s)) - L_{i4}(s)(L_{31}(s)L_{43}(s) - L_{33}(s)L_{41}(s))]/L_1(s),$$

$$\omega_{i2}(s) = L_{i2}(s) + [L_{i3}(s)(L_{32}(s)L_{44}(s) - L_{34}(s)L_{42}(s)) - L_{i4}(s)(L_{32}(s)L_{43}(s) - L_{33}(s)L_{42}(s))]/L_1(s),$$

$$\omega_{i3}(s) = [L_{i3}(s)L_{44}(s) - L_{i4}(s)L_{43}(s)]/L_1(s),$$

$$\omega_{i4}(s) = [L_{i3}(s)L_{34}(s) - L_{i4}(s)L_{33}(s)]/L_1(s),$$

$$L_1(s) = L_{33}(s)L_{44}(s) - L_{34}(s)L_{43}(s), \qquad i = 1,2,$$

and

$$\omega_{i3}(s) = L_{i3}(s) + [L_{i1}(s)(L_{13}(s)L_{22}(s) - L_{12}(s)L_{23}(s)) - L_{i2}(s)(L_{13}(s)L_{21}(s) - L_{11}(s)L_{23}(s))]/L_2(s),$$

$$\omega_{i4}(s) = L_{i4}(s) + [L_{i1}(s)(L_{14}(s)L_{22}(s) - L_{12}(s)L_{24}(s)) - L_{i2}(s)(L_{14}(s)L_{21}(s) - L_{11}(s)L_{24}(s))]/L_2(s)$$

$$L_2(s) = L_{11}(s)L_{22}(s) - L_{12}(s)L_{21}(s), \qquad i = 3,4,$$

where

$$L_{1i}(s) = 1 + \{[\mu_i^{(3)}(\eta_1^{(1)} - \eta_2^{(1)}) + \eta_i^{(3)}(\mu_1^{(1)} - \mu_2^{(1)})] - [\mu_i^{(3)}(\eta_1^{(1)} - \eta_2^{(1)}) - \eta_i^{(3)}(\mu_1^{(1)} + \mu_2^{(1)})]e^{-2\gamma_1^{(1)}sh}$$

$$+[\mu_i^{(3)}(\eta_1^{(1)} - \eta_2^{(1)}) + \eta_i^{(3)}(\mu_1^{(1)} + \mu_2^{(1)})]e^{-2\gamma_2^{(1)}sh} - [\mu_i^{(3)}(\eta_1^{(1)} - \eta_2^{(1)}) - \eta_i^{(3)}(\mu_1^{(1)} - \mu_2^{(1)})]e^{-2(\gamma_1^{(1)}+\gamma_2^{(1)})sh}\}, \; i = 1,2$$

$$L_{13}(s) = e^{-\gamma_1^{(1)}sh} + \frac{1}{D}[2\mu_1^{(1)}\eta_1^{(1)} - \mu_1^{(1)}\eta_2^{(1)} - \mu_2^{(1)}\eta_1^{(1)}]e^{-\gamma_1^{(1)}sh} + \frac{1}{D}[2\mu_1^{(1)}\eta_1^{(1)} - \mu_1^{(1)}\eta_2^{(1)} + 2\mu_2^{(1)}\eta_1^{(1)}]e^{-(\gamma_1^{(1)}+2\gamma_2^{(1)})sh},$$

$$L_{14}(s) = e^{-\gamma_2^{(1)}sh} - \frac{1}{D}[2\mu_2^{(1)}\eta_2^{(1)} - \mu_1^{(1)}\eta_2^{(1)} - \mu_2^{(1)}\eta_1^{(1)}]e^{-\gamma_2^{(1)}sh} - \frac{1}{D}[2\mu_2^{(1)}\eta_2^{(1)} - \mu_1^{(1)}\eta_2^{(1)} + 2\mu_2^{(1)}\eta_1^{(1)}]e^{-(2\gamma_1^{(1)}+\gamma_2^{(1)})sh},$$

$$L_{2i}(s) = \frac{\alpha_i^{(3)}}{\gamma_i^{(3)}} + \frac{1}{D}\{[\mu_i^{(3)}(\frac{\alpha_1^{(1)}\eta_2^{(1)}}{\gamma_1^{(1)}} - \frac{\alpha_2^{(1)}\eta_2^{(1)}}{\gamma_2^{(1)}}) + \eta_i^{(3)}(\frac{\alpha_1^{(1)}\mu_2^{(1)}}{\gamma_1^{(1)}} - \frac{\alpha_2^{(1)}\mu_2^{(1)}}{\gamma_2^{(1)}})]$$

$$-[\mu_i^{(3)}(\frac{\alpha_1^{(1)}\eta_2^{(1)}}{\gamma_1^{(1)}} + \frac{\alpha_2^{(1)}\eta_2^{(1)}}{\gamma_2^{(1)}}) + \eta_i^{(3)}(\frac{\alpha_1^{(1)}\mu_2^{(1)}}{\gamma_1^{(1)}} - \frac{\alpha_2^{(1)}\mu_2^{(1)}}{\gamma_2^{(1)}})]e^{-2\gamma_1^{(1)}sh}$$

$$+[\mu_i^{(3)}(\frac{\alpha_1^{(1)}\eta_2^{(1)}}{\gamma_1^{(1)}} + \frac{\alpha_2^{(1)}\eta_2^{(1)}}{\gamma_2^{(1)}}) - \eta_i^{(3)}(\frac{\alpha_1^{(1)}\mu_2^{(1)}}{\gamma_1^{(1)}} - \frac{\alpha_2^{(1)}\mu_2^{(1)}}{\gamma_2^{(1)}})]e^{-2\gamma_2^{(1)}sh}$$

$$-[\mu_i^{(3)}(\frac{\alpha_1^{(1)}\eta_2^{(1)}}{\gamma_1^{(1)}} - \frac{\alpha_2^{(1)}\eta_2^{(1)}}{\gamma_2^{(1)}}) - \eta_i^{(3)}(\frac{\alpha_1^{(1)}\mu_2^{(1)}}{\gamma_1^{(1)}} - \frac{\alpha_2^{(1)}\mu_2^{(1)}}{\gamma_2^{(1)}})]e^{-2(\gamma_1^{(1)}+\gamma_2^{(1)})sh}\}, \qquad i = 1,2,$$

$$L_{23}(s)=\{\frac{\alpha_1^{(1)}}{\gamma_1^{(1)}}+\frac{1}{D}[\frac{\alpha_1^{(1)}}{\gamma_1^{(1)}}(\mu_1^{(1)}\eta_2^{(1)}+\mu_2^{(1)}\eta_1^{(1)})-2\mu_1^{(1)}\eta_1^{(1)}\frac{\alpha_2^{(1)}}{\gamma_2^{(1)}}]\}e^{-\gamma_1^{(1)}sh}$$

$$+\frac{1}{D}[\frac{\alpha_1^{(1)}}{\gamma_1^{(1)}}(\mu_1^{(1)}\eta_2^{(1)}-\mu_2^{(1)}\eta_1^{(1)})+2\mu_1^{(1)}\eta_1^{(1)}\frac{\alpha_2^{(1)}}{\gamma_2^{(1)}}]\}e^{-(\gamma_1^{(1)}+2\gamma_2^{(1)})sh},$$

$$L_{24}(s)=\{\frac{\alpha_2^{(1)}}{\gamma_2^{(1)}}+\frac{1}{D}[-\frac{\alpha_2^{(1)}}{\gamma_2^{(1)}}(\mu_1^{(1)}\eta_2^{(1)}+\mu_2^{(1)}\eta_1^{(1)})+2\mu_2^{(1)}\eta_2^{(1)}\frac{\alpha_1^{(1)}}{\gamma_1^{(1)}}]\}e^{-\gamma_2^{(1)}sh}$$

$$-\frac{1}{D}[\frac{\alpha_2^{(1)}}{\gamma_2^{(1)}}(\mu_1^{(1)}\eta_2^{(1)}-\mu_2^{(1)}\eta_1^{(1)})+2\mu_2^{(1)}\eta_2^{(1)}\frac{\alpha_1^{(1)}}{\gamma_1^{(1)}}]\}e^{-(2\gamma_1^{(1)}+\gamma_2^{(1)})sh},$$

$$L_{31}(s)=l_1\delta_1(s)-l_2\delta_5(s)\ , \qquad L_{32}(s)=l_1\delta_2(s)-l_2\delta_6(s),$$

$$L_{33}(s)=l_1(1-\delta_3(s))+l_2\delta_7(s)+l_3, \quad L_{34}(s)=-l_1\delta_4(s)+l_2(1+\delta_8(s))+l_4,$$

with

$$l_i=1+\eta_i^{(1)}(\mu_1^{(2)}-\mu_2^{(2)})/D',\ l_{i+2}=\mu_i^{(1)}(\eta_1^{(2)}-\eta_2^{(2)})/D'\ ,\ \ i=1,2$$

$$L_{41}(s)=m_1\delta_1(s)-m_2\delta_5(s), \qquad L_{42}(s)=m_1\delta_2(s)-m_2\delta_6(s),$$

$$L_{43}(s)=m_1(1-\delta_3(s))-m_2\delta_7(s)+m_3, \quad L_{44}(s)=-m_1\delta_4(s)+m_2(1+\delta_8(s))+m_4,$$

$$m_i=\frac{\eta_i^{(1)}}{D'}(\frac{\alpha_1^{(2)}\mu_2^{(2)}}{\gamma_1^{(2)}}-\frac{\alpha_2^{(2)}\mu_1^{(2)}}{\gamma_2^{(2)}}),\ \ m_{i+2}=\frac{\alpha_i^{(1)}}{\gamma_i^{(1)}}+\frac{\mu_i^{(1)}}{D'}(\frac{\alpha_1^{(2)}\eta_2^{(2)}}{\gamma_1^{(2)}}-\frac{\alpha_2^{(2)}\eta_1^{(2)}}{\gamma_2^{(2)}}),\ i=1,2$$

$$\omega_{1i}(\infty)=1+[\mu_i^{(3)}(\eta_1^{(1)}-\eta_2^{(1)})+\eta_i^{(3)}(\mu_1^{(1)}-\mu_2^{(1)})]/[\mu_2^{(1)}\eta_1^{(1)}-\mu_1^{(1)}\eta_2^{(1)}],$$

$$\omega_{2i}(\infty)=\frac{\alpha_i^{(3)}}{\gamma_i^{(3)}}+[\mu_i^{(3)}(\frac{\alpha_1^{(1)}\eta_2^{(1)}}{\gamma_1^{(1)}}-\frac{\alpha_2^{(1)}\eta_1^{(1)}}{\gamma_2^{(1)}})+\eta_i^{(3)}(\frac{\alpha_1^{(1)}\mu_2^{(1)}}{\gamma_1^{(1)}}-\frac{\alpha_2^{(1)}\eta_1^{(1)}}{\gamma_2^{(1)}})]/[\mu_2^{(1)}\eta_1^{(1)}-\mu_1^{(1)}\eta_2^{(1)}],\ \ i=1,2$$

$$\omega_{3i}(\infty)=1+[\mu_i^{(1)}(\eta_1^{(2)}-\eta_2^{(2)})+\eta_i^{(1)}(\mu_1^{(2)}-\mu_2^{(2)})]/[\mu_2^{(2)}\eta_1^{(2)}-\mu_1^{(2)}\eta_2^{(2)}],$$

$$\omega_{4i}(\infty)=\frac{\alpha_i^{(1)}}{\gamma_i^{(1)}}+[\mu_i^{(1)}(\frac{\alpha_1^{(2)}\eta_2^{(2)}}{\gamma_1^{(2)}}-\frac{\alpha_2^{(2)}\eta_1^{(2)}}{\gamma_2^{(2)}})+\eta_i^{(1)}(\frac{\alpha_1^{(2)}\mu_2^{(2)}}{\gamma_1^{(2)}}-\frac{\alpha_2^{(1)}\eta_1^{(1)}}{\gamma_2^{(2)}})]/[\mu_2^{(2)}\eta_1^{(2)}-\mu_1^{(2)}\eta_2^{(2)}],\ \ i=3,4.$$