Chapter 1

Introduction

1.1 History of Fracture Mechanics

For a long time, the people always had some idea about the role of a crack or notch. While breaking a stick a small notch is applied on it before bending. In the pre-industrial society swards played an important role. At that time good sward were made by folding a thin metal sheet at the centre line and then hammering it to make thin again and could be further folded. Therefore, a good sward contains many layers. Thus sward become tough because if in any layer of the sward a crack developed, then it is not likely to move from one layer to another.

First time Leonardo da Vinchi (1452-1519) makes a setup to measure the strength of a wire. He observed that the strength of the wire depends of its length. At that time a wire was not of high quality and this long wire was likely to posses many cracks.

The industrial revolution of the 19^{th} century led engineers to use iron and steel in place of traditional materials like stone and wood. Unlike stone, iron and steel had the advantage of being strong in tension, which meant that engineering structures could be made lighter at less cost.

The industrial revolution opened a new vista for us and as a result many machines and structures were designed and build. In nineteenth century many bridges, boilers, buildings, ships were failed caused by fracture. At that time locomotives used to face numerous accidents caused by failure of wheels and axels of the wheels. The fatigue of locomotive axels by applying controlled cyclic loads was investigated. This led to development of S-N diagram to find the endurance limit of steel. In between nineteenth and twentieth centuries, the entire industry was passionate with production while the failure theories were developed quite late. The industrial production accelerated at very rapid rate during World war II. Many of the aircrafts were made and improved dramatically within the six years of the war. Earlier the ships were made by joining plates together through the process of reviting.

During second World war United State was supplying ships and planes to the Great Britain then the greatest requirement of Britain was cargo ship to carry supplies. The German Navy was sinking cargo ships at three times the rate at which they could be replaced with existing ship building procedures. For fabricating ships quickly the united state developed a revolutionary procedure under the guidance of a famous construction engineer Henry Kaiser. The construction of new ships was changed to welded frames as opposed to the riveted construction of traditional ship design. The liberty ship program attained a resounding success with in a short time span. But the welded structure displayed some complications.

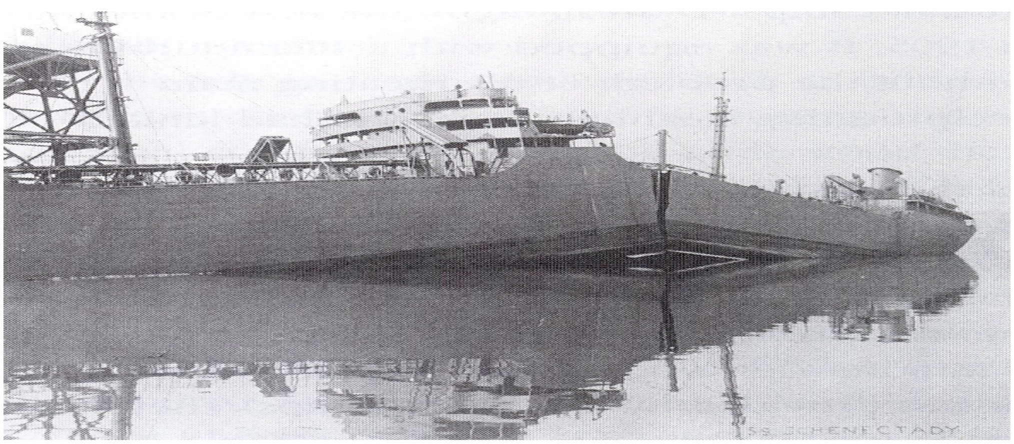

Fig. 1.1 Picture of cracked Liberty Ship

In 1943, one of vessels was broken into two parts in the cold temperature of North Atlantic Ocean (Fig. 1.1), while the ships made with modified construction did not show such failure. If a crack nucleated and grew in a plate there was possibility that plate had been into two parts but crack would not move to the another plate. In welded structure if the crack becomes critical then it will propagate through the entire hull of the ship because welded structure contains crack like flaws due to semi skilled work force and also steel of poor toughness was used to make liberty ships. Jet planes were developed initially as small fighter planes to fly at high altitudes but when this technology was extended for the construction of large body passenger plane, sometimes planes exploded in the air. The researchers of early fifties found that this is due to the discontinuity of the pressure at high altitudes as outside atmospheric pressure is high as compared to interior of the airplane and it becomes like high pressurized balloon under high tensile stress. Therefore at the high altitude a fatigue crack initiated near an opening in the fuselage ran through the entire body then unwanted event occurs (Fig. 1.2).

Fig. 1.2 Picture of an Airplane with fatigue crack

Therefore after the development of large ships made by welding plates and high capacity jet planes, a new direction of research was needed to study the reasons of failure and arrest of those. Upto second World war, engineers usually ensured that the maximum stress within a structure was limited to a certain percentage of the “tensile strength” of the material. Tensile strength for different materials could be conveniently measured in the laboratory for a different of materials are available in standard reference books. The failure of structural design in due to cause the effect of stress-raising corners and holes on the strength of a particular structure had never been appreciated by engineers. For this reason a new branch of engineering “Fracture Mechanics” is developed which characterizes the resistance of a material to fracture known as ‘toughness’.

1.2 Linear Elastic Fracture Mechanics

Fracture is the separation of body into pieces due to stress at the temperature below the melting point. Depending on the ability of the material to undergo plastic deformation before the fracture, two fracture modes can be defined as Brittle and Ductile.

In brittle fracture of elastic materials the singularity in the stress field at the crack tip is plays an important role. The order of this singularity is always $r^{-1/2}$ for the infinitesimal elasto-static analysis of a crack in a homogeneous isotropic material. The coefficient of this dominant term may depend in the elastic constants, the loading and geometry of the crack when the material is not homogeneous. It is known that the order of the singularity may be different from $r^{-1/2}$ and it may depends on the elastic constants as well. For example, in the plane problem of a crack lying in bonded interface of two isotropic and homogeneous elastic materials, the singularity is of $r^{-1/2}\cos(\gamma \log r)$, where γ depends on the elastic constants (Williams (1959)). When a crack terminates at right angle to a bonded interface, the singularity is of order $r^{-\lambda}$, where λ is a real number and it depends quite significantly on the elastic constants of the materials. But the order of the singularity at the interfacial crack tip becomes $r^{-\frac{1}{2}+i\gamma}$ in the stress field, where r is the distance from the point of consideration to the crack tip and γ is the material depending constant. The special attention of researcher was required to study the non-practical oscillatory stress singularity near the interfacial crack tip. The investigation of such behavior was studied by Comniou (1977). Itou (1988) had given a very important argument from the physical and plausible point of view by considering the weak points of previous study and obtained non-oscillatory stresses around the interfacial crack tip.

For instance, Brittle fracture is a low-energy process (low energy dissipation), which may lead to catastrophic failure without warning since the crack velocity is normally high. Therefore, little or no plastic deformation may be involved before separation of the solid. On the other hand, ductile fracture is a high-energy process in which a large amount of energy dissipation is associated with a large plastic deformation before instability of crack occurs. It is observed that slow crack growth occurs due to strain hardening at the crack tip region.

Fracture mechanics is based on the assumption that there exists crack in a work component. The crack may exist within the component either due to manufacturing defect or a crack during the service of a component may be nucleated and grown. Fracture mechanics deals with the query that under what loading condition crack may likely to grow? Fracture mechanics is also applied to the fatigue loading. Many industrial components are subjected to fluctuating loads and may fail through fatigue. Fatigue failure, which is a common phenomenon, may also occur due to fluctuating loads. Many scientists and engineers are working for the development of this field but till date no convenient and effective methods are adequately developed to control the fatigue failure. Initially, the fluctuating load gives to a crack, which grows slowly and then its growth rate picks up speed, and finally a situation comes when the crack length is long enough to be considered critical catastrophic failure.

Due to lack of accurate analysis for predicting the crack growth, reasonably high safety factors would be chosen to avoid unforeseen events. After the development of fracture mechanics, this ambiguity has been cleared through understanding the cause and effects of the fatigue failure. Then designers have started to use a much lower factor of safety to reduce the weight of the components to enhance their reliability and to reduce the cost of the structural components.

In recent years, the study of the cracks and failure in solids has drawn much attention of several researchers due to its applications in industry particularly in the fabrication of electronic components, geophysics and earthquake engineering etc.

Those materials which are not able to release high stresses are usually found to have low toughness like diamond. The inter-atomic bonds of diamond are so strong that the material does not yield at the vicinity of the crack tip. Therefore lack of formation of plastic zone at the crack tip implies very low toughness of the material. To sustain against the high stress at the crack tip, material should be able to deform anelastically. The plastic zone plays an important role in Fracture mechanics. A designer who tries to avoid fracture failure in a component prefers that a large plastic zone is formed at the crack tip.

1.2.1 Stress Concentrations

The first quantitative evidence for the stress concentration effect of flaws was proposed by C. E. Inglis (1913). He showed that the local stresses around a corner or hole in a stressed plate could be many times higher than the average applied stress.

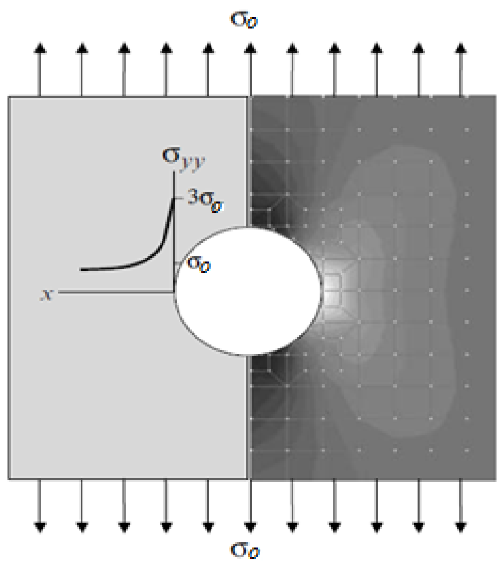

Fig. 1.3 Stress concentration around a hole in a uniformly stressed plate. The stress at the edge of the hole is three times the applied uniform stress.

He developed analytical solution through which the stress field around an elliptical hole in a large plate loaded under tensile stress σ_0 can be determined. He predicted that the presence of sharp corners, notches, or cracks serves to concentrate the applied stress at these points. He also showed that the degree of stress magnification at the edge of the hole in a stressed plate depends on the radius of curvature of the hole. Smaller the radius of curvature, greater the stress concentration. C. E. Inglis found that for an elliptical hole (Fig. 1.3), the stress concentration factor κ is equal to

$$\kappa = 1 + \sqrt{\frac{a}{\varpi}}, \tag{1.1}$$

where a is the radius of the hole and ϖ is the radius of curvature of the hole tip. For a circular hole, equation (1.1) gives $\kappa = 3$ (as shown in Fig. 1.1). For a very narrow elliptical hole, the stress concentration factor may be very much greater than one such that no real material can sustain the stress. So, a sharp crack in a body subjected to the small loading condition may cause to grow the crack for which the component may be broken. This is contrary to our observations. It should be noted that the stress concentration factor does not depend on the length of the hole but only on the ratio of the size to the radius of curvature.

1.3 Energy Balance Criterion

In fracture mechanics, it is important to study whether a crack in a component under given load conditions is likely to propagate under given load conditions in fracture mechanics for the sake of safety and security. Several methods were proposed to analyze this problem based on stress displacement or energy. Each method involves suitable parameter and limit value of the parameter for defining toughness of the material. Crack may propagate if the value of the parameter exceeds limiting value of the parameter for given load condition.

In 1920's, the right idea for the growth of crack and estimation of structural material was developed using atomistic model. At that time engineers was busy in production and earn more money therefore his work was ignored. An energy balance approach was adopted by Griffith to determine the strength of cracked solid i.e., the work done, must be equal to the surface energy stored in the newly created surfaces, during a crack extension. A. A. Griffith studied the effect of scratches and surface finish on the strength of machine parts subjected to alternating loads. Although Inglis's theory showed that the stress increases at the tip of a crack or flaw depended only on the geometrical shape of the crack and not its absolute size. This seemed contrary to the well known fact that larger cracks are propagated more easily than smaller ones. Griffith suggested that the fracture strength observed in experiments, as well as the dependence of strength on the size of the material was due to the presence of microscopic flaws in the bulk material.

Griffith theory implies that the fracture strength of a cracked solid with a crack is proportional to the square root of the surface energy and inversely proportional to the square root of the crack size i.e., $\sigma_c \propto \sqrt{(\gamma_c E)/a}$, where σ_c is the critical stress, E is the Young's modulus and γ is specific surface energy and 2a is the length of the crack. This relationship shows a specific functional form between the failure stress and crack size.

Griffith proposed that the reduction in strain energy due to the formation of a crack must be equal to or greater than the increase in surface energy required by the new crack faces. According to Griffith, there are two conditions necessary for crack growth:

(i) The bonds at the crack tip must be stressed to the point of failure. The stress at the crack tip is a function of the stress concentration factor, which depends on the ratio of its radius of curvature to its length.

(ii) For an increment of crack extension, the amount of strain energy released must be greater than or equal to that required for the surface energy of the two new crack faces.

The second condition may be given as

$$\frac{dU_s}{da} \geq \frac{dU_\gamma}{da}, \tag{1.2}$$

where U_s is the strain energy, U_γ is the surface energy, and da is the crack length increment.

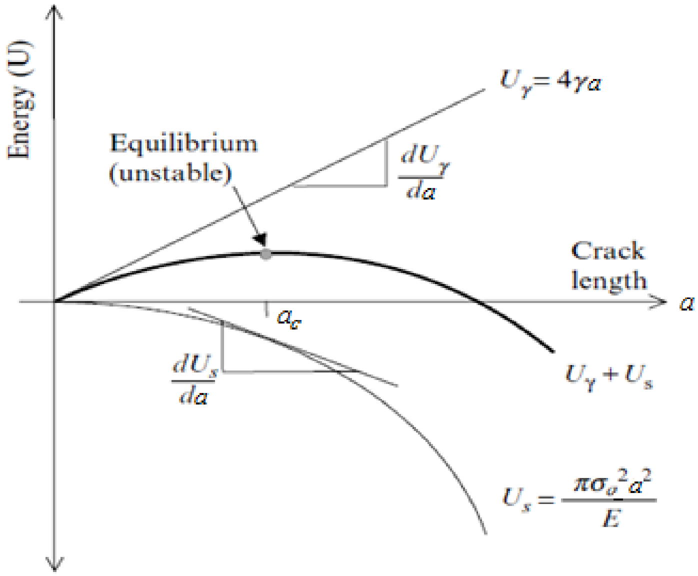

Fig. 1.4 Energy versus crack length showing strain energy released and surface energy required as crack length increases for a uniformly applied stress.

Equation (1.2) predicts that for a crack to extend, the rate of strain energy release per unit of crack extension must be at least equal to the rate of surface energy requirement. A crack will not extend until the strain energy release rate becomes equal to the surface energy requirement. Beyond this point, more energy becomes available by the released strain energy than is required by the newly created crack surfaces which leads to unstable crack growth and fracture of the specimen. Another words, fracture occurs when the strain energy change that results from an increment of crack growth is sufficient to provide for the increase in the surface energy of the new crack surfaces. The equilibrium condition shown in Fig. 1.4 is unstable, and fracture of the specimen will occur at the equilibrium condition. The presence of instability may be given in terms of surface energy U_s. If $\frac{d^2U_s}{da^2} < 0$, then equilibrium condition is unstable and for stable equilibrium condition $\frac{d^2U_s}{da^2} > 0$.

The energy balance criterion indicates whether crack growth is possible. It's occurance depends on the state of stress at the crack tip. A crack will not extend until the bonds at the crack tip are loaded to their tensile strength, even if there is sufficient strain energy stored to permit crack growth. For example, if the crack tip is blunted or rounded, then the crack may not extend because of an insufficient stress concentration. The energy balance criterion is a necessary, but not a sufficient condition for fracture. Fracture only occurs when the stress at the crack tip is sufficient to break the bonds there. This does not imply that all solids fail upon the immediate application of a load. In practice, stress singularities that arise due to an "infinitely sharp" crack tip are avoided by plastic deformation of the material. However, if such an infinitely sharp crack tip could be obtained, then the crack would not extend unless it attain it's critical energy for propagation.

Although engineers in the nineteenth century had a concept of stress concentration at the crack tip, the breakthrough came with a research article by Griffith (1921). The Griffith theory represents a breakthrough in the strength theory of solids. It successfully explains why there is an order of magnitude difference between the theoretical strength and experimentally measured failure load for a solid. In particular, it provides a well-defined physical mechanism that controls the failure process, which is lacking in the classical phenomenological failure theories. The original work of Griffith deals with fracture of brittle glass.

In brittle type of the material small amount of the energy is required for advancing the crack, therefore once crack propagation starts then catastrophic failure occurs, like glass. But it was observed that crack propagation will not occur in most of the materials in spite of energy release is greater than the crack energy. Finally Griffith theory was restricted upto the Brittle type of the material and fails to derive a limiting parameter for the prediction of a crack growth subjected to the loading conditions.

Later on, another significant contribution was made by J. R. Rice based on the energy approach. To describe the energy flow into the crack tip per unit fractured area, Rice (1968) introduced J integral which is regarded as elegant and powerful mathematical tool.

1.3.1 Energy Release Rate and Crack energy

Griffith theory not only holds good for the brittle materials but also holds for some of ductile materials such as steel. But the surface energy predicted by Griffith's theory was very high. A group of researchers working under G. R. Irwin at the U. S. Naval research Laboratory during second World war realized that plasticity must play the significant role in the fracture of ductile materials.

George Irwin (1948) was known as one of the pioneers of the modern fracture mechanics due to his invention of convenient parameters of crack growth like stress intensity factor and crack energy. Later that many investigators and engineers were attracted towards fracture mechanics due to which it becomes important branch of engineering. The other parameter likes crack opening displacement (wells (1961, 1963) and J-integral (Rice (1968)) had been developed to measure the large plastic zone at the crack tip because Irwin's analysis was restricted up to less ductile materials.

Irwin postulated that the energy due to plastic deformation must be added to the surface energy associated with the creation of new surfaces. He recognized that for ductile materials, the surface energy term is often negligible compared to energy associated with plastic deformation. Further he defined a quantity G, the strain energy-release rate or “crack driving force”, which is the total energy absorbed during cracking per unit increase in crack length and per unit thickness.

The assumption that all the strain energy available for surface energy of new crack faces can not be applied to ductile solids where other energy dissipative mechanisms exist. For example, in crystalline solids, considerable energy is consumed in the movement of dislocations in the crystal lattice and this may happen at applied stresses well below the ultimate strength of the material. Dislocation movement in a ductile material is an indication of yield or plastic deformation.

Irwin (1948) and Orowan (1955) modified Griffith’s equation to take into account the non-reversible energy mechanisms associated with the plastic zone by simply including this term in the original Griffith equation as

$$\frac{dU_s}{da} = \frac{dU_\gamma}{da} + \frac{dU_p}{da} . \tag{1.3}$$

The right-hand side of above equation is symbolized by R and is called the crack resistance. The crack resistance indicates the minimum amount of energy required for crack extension in J/m^2 (i.e., J/m per unit crack width) with the condition that Griffith crack energy is met. This energy is measure of toughness. Ductile materials are tougher than brittle materials because they can absorb energy in the plastic zone, which is usually called as plastic strain energy (U_p).

Irwin (1948) suggested that during crack growth up to a certain depth in the surface of the crack an anelastic deformation will occur. Thus R is not only the surface energy but also it is the sum of energy required to create two new surfaces and to cause anelastic deformation. For this important phenomena of the prediction of a crack growth, it is important to study both the parameters energy release rate and crack energy. The crack will grow when the energy release rate exceeds the crack resistance. Then crack attains kinetic energy and may grow faster than the speed of supersonic wave.

1.4 Stress intensity factor

During the second World war, George R. Irwin became interested in the fracture of steel armor plating during penetration by ammunition. His experimental work at the U.S. Naval Research Laboratory in Washington, D. C. led, in 1957, to a theoretical formulation of fracture that continues to find wide application. In engineering field the problem of single variable is easier to solve as compared to problem of two variables. If it is possible to combine the independent variables to form a new independent variable then the solution of the problem becomes much simpler like wave propagation problem. There are two main variables field stress and crack length for a given geometry in fracture mechanics.

Irwin (1958) combines these two variables and define a new variable called stress intensity factor denoted by K after the name of his collaborator Kies. Stress intensity factor (SIF) is one of the most important physical quantities in fracture mechanics. It plays an important role in fracture process and in determining the critical value of strain energy density, which is an intrinsic material property. The stress intensity factor (K) measures the stress intensity near the crack tip due to remote loading or residual stresses. The magnitude of K depends on geometry, size and location of the crack, magnitude and distribution of load. The stress intensity factor is a single-parameter characterization of the crack tip stress field. In Fracture Mechanics, the stresses just ahead of the crack tips are quantites worthy of investigation. Hence correct determination of SIF value is a necessary step towards safe design of structures. The knowledge of the stress field in the vicinity of crack is prerequisite for the prediction of fracture and failure of the material. In an elastic state, the crack tip stress have singularities inverse square root type (i.e., of the type $r^{-1/2}$). A singularity of type of $r^{-1/2}$ appears to be inherent in the nature of the stresses arising at the crack tip in an elastic state both under mechanical and thermal stress field.

The stress intensity factors for fundamental Modes-I, II and III are defined in Cartesian co-ordinates as

$$K_I = \lim_{x \to a^+} \sqrt{2(x-a)}\, \sigma_{yy}(x,0),$$

$$K_{II} = \lim_{x \to a^+} \sqrt{2(x-a)}\, \sigma_{xy}(x,0),$$

$$K_{III} = \lim_{x \to a^+} \sqrt{2(x-a)}\, \sigma_{yz}(x,0). \tag{1.4}$$

An important property of the stress intensity factors is that these are additive for the same type of loading. This means that the stress intensity factor for a complicated system of loads may be derived from the addition of the stress intensity factors determined for each load considered individually.

The main advantage of stress intensity factor is that Griffith criterion is a necessary but not sufficient condition for crack growth. However, using stress intensity factor a necessary and sufficient condition for crack growth may be determined more easily.

The basic energy-concept of fracture mechanics was proposed by Griffith (1924), but it was accepted as an engineering science during 1950 with successful practical applications mainly as a result of Irwin's work (1957a, 1957b). Irwin first introduced the energy release rate to establish a fracture criterion. He then defined the stress intensity factor K and derived the relationship between the energy release rate G based on Westergaard's solution (1939) for the stress and displacement fields in a cracked plate.

$$G_I = \frac{K_I^2}{E}, G_{II} = \frac{K_{II}^2}{E} \text{ and } G_{III} = \frac{K_{III}^2}{E}. \tag{1.5}$$

Due to the *G*-*K* relationship, Irwin proposed to use the stress intensity factor as a fracture parameter, which is more direct approach for fracture mechanics applications.

1.4.1 The Critical stress intensity factor

The value K_I at the point of crack extension is called the critical value denoted as K_{IC}. It does not necessarily indicate the fracture of the specimen. It is usually regarded as a material property and can be used to characterize toughness. In plane stress, the critical value of K_I for fracture depends on the thickness of the plate. Hence, K_{IC} is often called the "plane strain fracture toughness" and has units $MPa\, m^{1/2}$. Low values of K_{IC} implies that, for a given stress, a material can only sustend a small length of crack before a crack propagation.

The condition $K_I = K_{IC}$ does not necessarily correspond to fracture or failure of the specimen. K_{IC} describes the beginning of crack extension. Whether this is a stable or unstable condition depends upon the crack system. Catastrophic fracture occurs when the equilibrium condition is unstable. In terms of stress intensity factor condition for a crack is stable when $\frac{dK_I}{da} < 0$ and unstable when $\frac{dK_I}{da} > 0$. The condition $K_I = K_{IC}$ for the stable configuration means that the crack is on the point of extension but will not extend unless the applied stress is increased. If this happens, a new stable equilibrium crack length will result. For the unstable configuration, the crack will immediately extend rapidly throughout the specimen and lead to failure. Under these conditions, for each increment of crack extension there is insufficient surface energy to account for the release in strain potential energy.

When $K_I = K_{IC}$, then G_C becomes the critical value of the rate of release in strain energy for the material which leads to crack extension and possibly fracture of the specimen. The relationship between K_I and G is significant because it means that the K_{IC} condition is a necessary and sufficient criterion for crack growth since it have both the stress and energy balance criteria. The value of K_{IC} describes the stresses (indirectly) at the crack tip as well as the strain energy release rate at the beginning of crack extension.

1.5 Crack tip opening displacement

Crack tip opening displacement (CTOD) is a parameter unlike the strain energy release rate and stress intensity factor to characterize a crack in both linear and elastic-plastic fracture mechanics. The approach CTOD was formulated by Wells (1961, 1963). The material can not sustain very high stresses within the plastic zone, and the usual stress field no longer exists of the square root singularity. The rigorous analysis is more complicated and therefore a simple model is required. The formation of plastic zone at the vicinity of the crack tip does not allow the material to withstand high stresses predicted by the elastic analysis. it is observed that material is soft in front of the crack tip when the effective crack length is longer than actual i.e., the size of the plastic zone in front of the crack tip determines the effective crack length.

1.6 Cracking Modes

A crack in solid consists of disjoint upper and lower faces. The joint of the two crack faces forms the front. The two crack faces are usually assumed to lie in the same surface before deformation. When the cracked body is subjected to the external loads, the two crack faces move with respect to each other and these movement may be described by the differences in displacements between the upper and lower crack surfaces. There are three independent movements corresponding to three fundamental fracture modes as pointed out by Irwin and denoted by Mode-I, Mode-II, Mode-III as shown in Fig. 1.5 and any fracture mode in a cracked body may be described by one of the three basic modes or their combinations. Mode-I is called opening mode and in this case cracked surfaces moving apart i.e., the displacement is normal to the cracked surface. In Mode II cracked surfaces slides over one another and the relative displacement is normal to the crack front, it is also called sliding mode. Mode III is called tearing mode and displacement is parallel to the crack front.

Cracks can be considered as surfaces of discontinuity of the material. The fracture process zone is a small region surrounding the crack tip where fracture develops through the successive stage of inhomogeneous slip, void growth, and bond breaking on the atomic scale. The crack front is the line connecting all adjacent sites where separation may occur subsequently. During continued separation, this line will move along a geometric surface termed as the fracture surface. The area of this surface, i.e., the developed crack area will increase as the crack grows. The type of the crack discussed in the thesis is Griffith crack, which is represented by a line segment in a two dimensional diagram, is in reality a long flat ribbon shaped cavity in a solid. It is stretched in such a way that the stress pattern remains unaltered.

The theory of cracks in two dimensional medium was first studied by Griffith (1921). Two dimensional Griffith crack problems are very important engineering problems. A Griffith crack in Mechanics of fracture broadly deals with the science of strength of materials relating to the study of the load bearing capacity of a body with or without the presence of cracks and various principals governing the crack development.

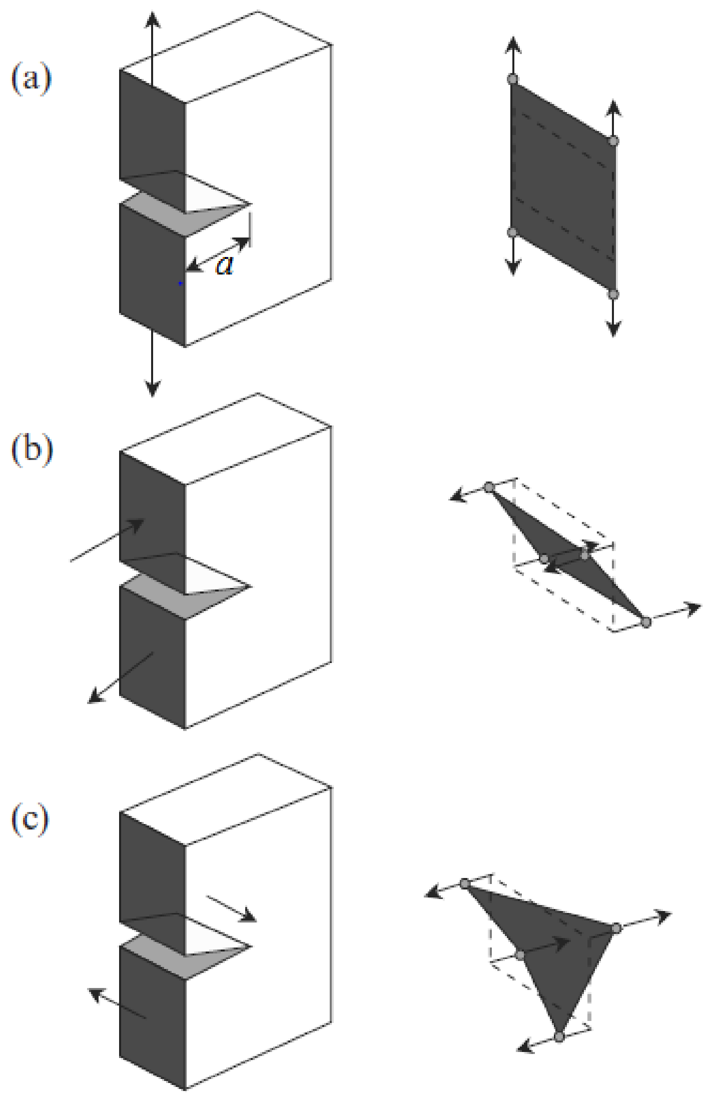

Fig. 1.5 Fundamental Irwin's cracking modes of fracture. (a) Mode I (b) Mode II and (c) Mode III. The displacement of atoms near the crack tip are indicated by right side of the figures.

The development of crack existing in a body may depend on the basic parameters like material, shape and dimension of the body, the mode of applying an external load, time and number of cycles of load, temperature, degree of environmental reactivity, strain rate and deformation history. The main object of this thesis is to present the method of analysis and solutions of some two dimensional interfacial problem towards finding the quantities of physical interest e.g., stress intensity factor (SIF), strain energy release rate and stress magnification factor. This discussion centers round the frame work of the classical linear theory of elasticity which takes account of small displacement and strains. Furthermore, the parameters of the solid are considered to be independent of temperature and its state of stress. Study of crack problems is extremely important for determining the intensity and safety of structural components. Due to fabrication, joining and design requirements, geometric discontinuities like cracks and faults often appear in structural materials. Researchers like Inglis (1913), Griffith (1921), Muskhelishvili (1953), Orowan (1955), Irwin (1957a, 1958, 1962), Williams (1959), Barn-blatt and Cherepanoe (1961), England and Green (1963) and others have shown a new dimension of a detailed study of problems relating to the equilibrium of elastic bodies with cracks to find the criteria for growth of macroscopic cracks.

This study aims at considering two dimensional Griffith crack problems represented by a line segment which is in realty a long flat ribbon-shaped cavity in a solid and is stressed in such a way that the stress pattern remains unaltered while passing in a direction parallel to the plane of the crack. The fact that plastic strain develops at tips of Griffith cracks is not considered in the present analysis. Irwin (1957b) supported that this will not result insignificant loss of accuracy in the calculation of relevant quantities in the theory of fracture.

Composite materials are most often an exceptional technological choice to make more competitive products for industrial uses. Properties of those products offer many advantages compared to those of existing traditional materials. Costly composite parts are manufactured by utilizing proper design and various manufacturing techniques. There are lot of flexibilities in designing composite materials keeping in view of meeting up specifications, which have led to a considerable development of those materials and their diversity. One of the greatest achievements in the area of composite material research is that metals are gradually replaced by composites in airplane structure. Composite materials are also used in spacecrafts, solar panels, racing car bodies, storage tanks etc, while it is known that even the presence of micro-cracks and its propagation may cause safe structure to fail. Since components of engineering structures are very expensive, therefore idea of replacement is not acceptable. It is always better to predict in which direction failure may occur rather than replacing the components. Again crack propagation in composite materials is complicated than that in homogeneous media due to existence of interfaces in the structure of composite media. The problems become more complicated when crack/cracks occur at the interfaces. In composite materials the mechanical properties of interfaces play an important role in overall scenario during construction of engineering structures and machines. Moreover, from mathematical point of view since the nature of stress singularity for a crack imbedded in a homogeneous medium and for an interface crack are different there is no smooth transition from one solution to the other as the crack distance from the interface goes to zero. Thus this area of research in composite media should be more focused. The problem of interfacial crack bonded between two dissimilar elastic half- planes was solved by Delale and Erdogan (1988). The problem of static stress was solved by Zhang et al. (2008) using finite element method for two parallel cracks in composite materials.

The dynamic stress intensity factor of a crack in non-homogeneous composite materials was identified by Wang et al (2000). Many authors have solved the problem of multiple cracks using classical approach through reducing it into a pair of Integral equations, which are solved numerically. But solutions of these types of Integral equations are complex and have slow convergence. If approximate methods are used then it becomes a convenient tool for the analysis of multiple cracks.

Simulation of solid deformable models has many applications in computer graphics, robotics, and virtual reality. Earlier applications had been restricted to isotropic materials which have the property of equally elastic in all directions. But the anisotropic materials which have vast space in nature are not to be ignored. Nowadays lot of research are going on a particular subset of anisotropic materials named orthotropic materials, which exhibit different stiffness in three orthogonal planes of symmetry. The orthotropic materials used in the formation of composite material in the present article are α-Uranium, Beryllium, Beech-wood, glass-epoxy, graphite-epoxy, epoxy-boron. These materials are useful while dealing with problems of fracture in composite materials, probably due to their serious engineering applications.

In recent years, the advancement of space research demands a deep insight into the character of materials, particularly of the anisotropic materials which are habitually used in missiles, high speed aircraft, aerospace structures, rocket engines, turbine blades etc. This is due to the advent and increased use of high modules, high strength and low weight composite materials in the various fields of modern technology.

The materials such as carbon fibre-reinforced plastic (CFRP), fiber-reinforced metals and numerous composite materials are essentially anisotropic elastic in character. Therefore, when a crack is found in such a composite material, anisotropic theory of elasticity should be used to determine the stresses and other physical quantities. This leads to an increasing interest in the analysis of elasto-static and elasto-dynamic crack problems in anisotropic media. But due to the inbuilt complicacy in the stress-strain relations and in equations of equilibrium or in equations of motion (Lekhnitskii (1963)), the study of the problems of fracture mechanics in such anisotropic media is comparatively difficult. Orthotropy is a particular type of anisotropy. Technologically important materials with orthotropic elastic behavior are of frequent occurrence. Accordingly, study of fracture mechanics problems in orthotropic elastic materials has raised attention of many researchers.

Some significant works in orthotropic materials may be referred to those, which were carried out by Ang and Williams (1961), Dhaliwal (1973), Konishi and Atsumi (1973), Tupholme (1974), Satpathy and Parhi (1978), Kushawaha (1978), Das and Behera (1982), Kassir and Tse (1983), Arcisz and Sih (1984), Danyluk and singh (1984), Gilbert and Lin Wei (1985), Loboda and Tauchert(1985), Viola and Piva (1986), Piva (1987), Georgiadis and Papadopoulos (1987), Piva and Viola (1988), Sweeney (1988), Itou (1989), Viola et al. (1989), Yum and hong (1991), Tan and Gao (1992) and Cripps (2007).

Li et al. (2012) have studied the development of stress intensity factors for deep surface cracks in plates and cylinders, Okada et al. (2012) had calculated the stress intensity factor in plates by using the tetrahedral finite element, Tanaka et al. (2015) had used spline-based wavelet Galerkin method to analyze the dynamics stress concentration problems, Okada et al. (2016) have calculated the stress intensity factors for semi-elliptical cracks with high aspect ratios by using the tetrahedral finite element.

The geometry of two and more collinear cracks has practical importance in fracture mechanics since experimental studies have shown that minute cracks are developed near the ends of the main crack for fracturing. Tranter (1961) has solved the elastostatic problem of determining the distribution of stresses when two coplanar cracks are opened by a constant pressure in an infinite elastic medium. Lowengrub and Srivastava

(1968) have considered the problem of an infinitely long strip containing two coplanar Griffith cracks. Dhaliwal and Singh (1981) have solved the same problem analytically. Plane strain problem of determination of stress distribution in the vicinity of three collinear cracks embedded in an infinite isotropic elastic medium has considered by Parihar and Sowdamini (1985).

As regards the dynamic crack problem in anisotropic media, research has been restricted mainly to the case of a single crack because of severe mathematical complexities encountered in finding solution of problems involving two or more cracks. De and Patra (1993) have solved the problem of propagation of two collinear Griffith cracks in an orthotropic strip by using integral transform technique.

In material structures, pre-existing cracks subjected to a given set of external loads become extremely important for the purpose of design and safe-life prediction of material structures. Another part of crack interaction is the interaction of crack with a field of micro-cracks. Crack propagation in many brittle materials accompanies micro-cracking (Chudnovsky and Kachanov (1983)) which may substantially affect the overall mechanics of fracture propagation. A number of authors have suggested that an interaction produces stress shielding i.e., reduction of stress intensity factors at the main crack tip. A number of solution (both analytical and numerical) have been considered involving one or several micro-cracks in two dimension by Rubinstein (1985, 1986), Rose (1986), Horii and Nasser (1987), Kachanov and Montagut (1989).

The improved stress intensity factor for the selected configurations in cracked plates is given by R. Evans (2014). An approximate analytical solution for plasticity-corrected stress intensity factor is developed by Dai et al. (2014) on the basis of Eshelby equivalent inclusion theory and transformation toughening theory both for Mode-I and

Mode-II cracks. In recent years, multiphase materials such as modern composites, diffusion bonded materials and thermal barrier coatings, have a great use in technological applications. Particularly, mechanical failure initiating largely at the interfacial regions in many of these materials has led to extensive studies for the purpose of understanding the interaction between the flaws that may exist in these region and the applied loads and other environmental factors. From a view point of mechanical strength, the optimal design of the interfacial regions usually involves trade-offs between strength and toughness. However, since joints of such high interfacial strength generally have poor impact resistance, one may need an interfacial region which is either highly flexible or has a relatively low fracture resistance. In such situations, the solutions appear to be the introduction of a relatively ductile organic interlayer between two materials. It is therefore clear that the mechanical properties and relative dimension as well as the strength parameters of the interfacial region would play a major role in the failure of processes initiating at an interface. In fracture oriented failures such as subcritical crack growth and static or dynamic fracture, the crack driving force would heavily depend on the thickness and constitutive properties of the interfacial region, the size, location and orientation of the crack and on mechanical properties of the adjacent materials as well as on the magnitude of the external loads and geometry of the media.

During last few decades many researchers were studied to extend the linear fracture mechanics in homogenous material to bimaterial interface crack problems and made important contributions to the bimaterial interface fracture mechanics. However, the analytical solutions of interfacial cracks given by the previous researches became useless during applications to the real engineering problems. Recently a number of numerical methods are developed to analyze the interfacial crack problems.

In bonding of two materials with different mechanical elastic properties, very often it is not possible to obtain a homogeneous perfect bond due to existence of entrapped imperfections, e.g., in the joints involving ceramics and metals used in manufacturing electronic devices and a variety of reinforced composites, the cavity and other imperfections with weak bonds strength existing on the interface usually have very sharp corners. For the purpose of analysis, these imperfections may all be classified as singular surface across with displacement or stress vector suffers a discontinuity. These imperfections generally form the nucleus of the fracture initiation and propagation in the medium. Thus the study of stress distribution near the edge of the crack is of great importance in fracture of composites. From the mathematical point of view since the nature of the stress singularities for a crack embedded in homogeneous medium and for an interface crack is different, there is no smooth transition from one solution to the other as the crack distance from the interface goes to zero. For very small values of this distance, one also encounters convergence problems in the numerical analysis. This is primarily due to the fact that as the crack interface distance goes to zero, the Fredholm kernals in the system become unbounded. For the interface crack, if one separates these singular parts of the kernals, the system of singular integral equations become first or second kind, the solutions of which require different numerical techniques.

1.7 Mathematical methods and techniques

1.7.1 Laplace Transform

Let $f(x)$ is a piecewise continuous on every finite interval on the semi-axis $x \geq 0$ and there exist some constants M and c_0 s.t. $|f(x)| < Me^{c_0 x}$, for all $x \geq 0$, then Laplace transform £(f) exist for all $p > c_0$.

Then Laplace transform of *f(t)* and the corresponding inverse formula are defined by

$$£(f) = \int_0^\infty e^{-px} f(x)dx = F(p), \quad p > c_0, \tag{1.6}$$

and

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt} F(p)\, dp, \; c > c_0. \tag{1.7}$$

If the Laplace transform of a given function exists, it is uniquely determined. Conversely, the inverse of a given function is essentially unique.

1.7.2 Fourier Transform

Let $f(x)$ is absolutly integrable on the *x*-axis and piecewise continuous on every finite interval, then Fourier Transform $\hat{f}(w)$ of $f(x)$, is given by

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx, \qquad \text{exists.} \tag{1.8}$$

Dirichlet conditions for inverse Fourier transform are given as follows:

Let $f : R \to R$ suppose that

(i) $\int_{-\infty}^{\infty} |f| dt$ converses

(ii) In any finite interval, f, f' are piecewise continuous with at most finitely many maxima/minima/discontinuities. Then inverse Fourier transform is given as

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w) e^{iwx}\, dw. \tag{1.9}$$

1.7.3 Dirac Delta Function

In many crack problems the boundary conditions are concerned with point loading conditions with Dirac Delta function, which is usually defined through the functional relation

$$\int_{-\infty}^{\infty} f(\xi)\,\delta(\xi - x)\,d\xi = f(x)\,. \tag{1.10}$$

Laplace transformation of $\delta(x)$ gives

$$\int_{0}^{\infty} \delta(x)\,e^{-t\,x}\,d\,x = H(t)\,, \tag{1.11}$$

where $H(t)$ is the Heaviside unit step function and it is used to solve the problems related to the impact loading.

1.7.4 The Holder's Condition

A function $\phi(t)$ will be said to satisfy a Holder condition on the arc L, if for any two points t_1 and t_2 of L,

$$\left|\phi(t_1) - \phi(t_2)\right| \le A\left|t_1 - t_2\right|^{\mu}, \tag{1.12}$$

where A and μ are positive constants. A is called Holder's constant and μ Holder's index. A function which satisfies a Holder's condition will be said to obey H condition or, when it is necessary to specify the index μ, the $H(\mu)$ condition. A function satisfying the Holder's condition on the arc L is clearly continuous.

1.7.5 Singular Integral Equation

Generelly the singular integral equations in the field of applied physics and engineering occur due to the formulation of the mixed boundary value problems. Such singular integral equations play an important role for solving crack problem in fracture mechanics.

$$A(x)\,\phi(x)+\frac{1}{\pi}\int_a^b B(t)\,\phi(t)\frac{dt}{t-x}+\int_a^b K(x,t)\,\phi(t)\,dt = f(x), \qquad a<x<b, \tag{1.13}$$

where $\phi = \phi_i, (i = 1,2,........N)$ are the unknown functions and $A = (a_{ij}),\ B = (b_{ij}),\ (i, j = 1,\ 2,..........N)$ are known functions with $A \mp B$ non-singular in $a < x < b$, $K = (k_{ij})$ is Fredholm type kernal and vectors $f = f_i$ consist of the known functions satisfying the Holder condition in $[a, b]$. The system of singular integral equation (1.13) may be obtained by considering the plane or axis symmetric elastostatic problem for layered materials containing $N/2$ non-coplanar cracks, or at the interfaces. If there are no interfacial cracks then the matrix A becomes null matrix and singular integral equation (1.13) is reduced into system of singular integral equation of first kind otherwise (1.13) is of second kind (Erdogan and Gupta (1971), Karpenko (1967)).

(i) Fredholm integral equation of first kind

In the above equation (1.13), let A be a null matrix and B be non-singular. Thus, multiplying equation (1.13) by B^{-1}, the dominant part of the system may easily be uncoupled. Without any loss of generality, it will be assumed that the interval (a, b) in which the functions ψ_i and f_i are defined is normalized to be $(-1, 1)$. Hence the integral equation has to be considered as

$$\frac{b}{\pi}\int_{-1}^{1}\frac{\phi(t)}{t-x}\,dt+\int_{-1}^{1}K(x,t)\phi(t)\,dt = f(x), \qquad -1<x<1, \tag{1.14}$$

where $\phi(x)$ is unknown function, and $f(x)$ and $K(x,t)$ are known functions which are H-continuous in the closed interval $[-1, 1]$.

Since $\phi(x)$ and $K(x,t)$ both are H-continuous functions, the second term of equation (1.14) is bounded function of *x*. Therefore the singular behavior of $\phi(x)$ may be obtained through the dominat part of the equation (1.14) as

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\phi(t)}{t-x}dt = F(x), \qquad -1 < x < 1, \tag{1.15}$$

where $F(x)$ contains the input function $f(x)$ and the term coming from the part of the integral equation with the Fredholm Kernal.

Let,

$$\Phi(z) = \frac{1}{2\pi i}\int_{-1}^{1}\frac{\phi(z)}{t-z}dz. \tag{1.16}$$

The boundary values of the sectionally holomorphic function $\Phi(z)$ are related with the following Plemelj formulas:

$$\Phi^{+}(z) - \Phi^{-}(z) = \phi(x),$$

$$\Phi^{+}(z) + \Phi^{-}(z) = \frac{1}{\pi i}\int_{-1}^{1}\frac{\phi(t)\,dt}{t-x}. \tag{1.17}$$

The following Riemann-Hilbert Problem to determine $\Phi(z)$ using equation (1.15) and (1.17) is obtained as

$$\Phi^{+}(z) + \Phi^{-}(z) = -iF(x), \quad -1 < x < 1. \tag{1.18}$$

The solution of equation (1.18) may be expressed as

$$\Phi(z) = -\frac{X(z)}{2\pi}\int_{-1}^{1}\frac{F(t)\,dt}{(t-z)\,X^{+}(t)} + C\,X(z), \tag{1.19}$$

where C is a constant quantity and $X(z)$ is the fundamental solution of the problem satisfying the following homogeneous boundary conditions

$$X^{+}(z) + X^{-}(z) = 0, \quad -1 < x < 1,$$

$$X^{+}(z) + X^{-}(z) = 0, \quad -1 < x < 1. \tag{1.20}$$

The general solution of the equation (1.20) is given by

$$X(Z) = (Z-1)^{\frac{1}{2}+N}(Z+1)^{\frac{1}{2}+M}, \qquad (1.21)$$

where $X(Z)$ will be taken as the branch for which $Z^k X(Z) \to 1$ as $Z \to \infty$ and the equation (1.20) will be satisfied provided N and M are integers.

Equation (1.17) and (1.19) yield the solution of the integral equation (1.15) as

$$\phi(x) = \Phi^+(z) - \Phi^-(z) = 2C\, X^+(x) - \frac{X^+(z)}{\pi}\int_{-1}^{1} \frac{F(t)\, dt}{(t-z)\, X^+(t)}, \quad -1 < x < 1. \qquad (1.22)$$

Equation (1.18) implies the following restrictions on N and M as

$$-1 < \frac{1}{2} + N < 1,\ -1 < \frac{1}{2} + M < 1. \qquad (1.23)$$

This Implies that $\phi(x)$ is either a potential type quantity or flux type quantity. For determining the index of the problem for arbitrary integers N and M as

$$\kappa = -(M+N), \qquad (1.24)$$

the physical nature of the singularity of the function $\phi(x)$ at the end points -1 and +1 has to be considered.

The real function $w(x)$ is given by

$$w(x) = i(-1)^N X^+(x) = (1-x)^{\frac{1}{2}+N}(x+1)^{\frac{1}{2}+M}, \qquad -1 < x < 1. \qquad (1.25)$$

Case I: When $\kappa = 1$. If the function $\phi(x)$ has inferable singularities at both ends, then $N = -1,\ M = 0$. Therefore $X(z)$ and $w(x)$ is given as

$$X(z) = (z^2-1)^{-\frac{1}{2}},\ w(x) = (1-x^2)^{-\frac{1}{2}}. \qquad (1.26)$$

In this case one extra condition is needed to determined the constant C which appears in

(1.19) and (1.22), which gives unique solution. $\phi(x)$ must satisfy an additional condition with aid of equation (1.15) which may be expressed as

$$\int_{-1}^{1} \phi(x)\, dx = A, \tag{1.27}$$

where A is a known constant.

Case II: When $\kappa = 0$. If $\phi(x)$ is bounded at one end and has integrable singularity at the other, in that case either

$$N = -1,\ M = 0,\ X(z) = (z-1)^{\frac{1}{2}}(z+1)^{-\frac{1}{2}},\ w(x) = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$$

or $N = -1,\ M = 1,\ X(z) = (z-1)^{-\frac{1}{2}}(z+1)^{\frac{1}{2}},\ w(x) = (1-x)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}}$.

In this case, the condition at infinity requires that in equation (1.19) the constant C be zero and with this the equation (1.14) gives the unique solution.

Case III: When $\kappa = -1$. If the function $\phi(x)$ is bounded at both the ends then

$$N = 0,\ M = 1 \ \text{ and } \ X(z) = (z^2-1)^{\frac{1}{2}},\ w(x) = (1-x^2)^{\frac{1}{2}}.$$

In this case the condition at infinity requires that, in addition to $C = 0$, the solution must satisfy the following consistency condition

$$\int_{-1}^{1} \frac{F(t)\, dt}{w(t)} = 0. \tag{1.28}$$

The general solution of the unknown function $\phi(x)$ is given by

$$\phi(x) = g(x)\, w(x), \qquad -1 < x < 1, \tag{1.29}$$

where $g(x)$ is a bounded continuous function in the closed interval [-1, 1]. A numerical method is used for the determination of $g(x)$ as

$$-1 < \alpha, \beta < 1, \quad \alpha + \beta = -\kappa = (0, \mp 1),$$

$$w(x)=(1-x)^{\alpha}(1+x)^{\beta} \text{ and } g(x)=\sum_{0}^{\infty}B_j P_j^{(\alpha,\beta)}(x)\,. \tag{1.30}$$

Equations (1.14) and (1.30) yield

$$\sum_{j=0}^{\infty}B_j\left(-\frac{2^{-\kappa}}{\sin\pi\alpha}P_{j-\kappa}^{(-\alpha,-\beta)}(x)+p_j(x)\right)=f(x),\quad -1<x<1\,, \tag{1.31}$$

where

$$p_j(x)=\int_{-1}^{1}k(x,t)P_j^{(\alpha,\beta)}(t)\,w(t)\,dt.$$

The orthogonality relation of Jacobi polynomial is given by

$$\int_{-1}^{1}p_n^{(\alpha,\beta)}(t)P_k^{(\alpha,\beta)}(t)\;w(t)\,dt=\begin{cases}0, & n\neq k\\ \theta_k^{(\alpha,\beta)}, & n=k\end{cases} \tag{1.32}$$

where

$$\theta_k^{(\alpha,\beta)}=\frac{2^{\alpha+\beta+1}}{2k+\alpha+\beta+1}\frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{k!\,\Gamma(k+\alpha+\beta+1)},\quad (k=0,1,....)\,.$$

Therefore the equation (1.31) is reduced to an infinite system of algebraic equations in unknown coefficients B_j. Multiplying both sides of the equation (1.31) by $w(-\alpha,-\beta,t)P_k^{(-\alpha,-\beta)}(t)$, $(k=0,1,2...)$ and integrating from -1 to 1, and using the orthogonality relation of Jacobi polynomial, we get

$$-\frac{2^{-\kappa}b}{\sin\pi\alpha}\theta_k^{(-\alpha,-\beta)}B_{k+\kappa}+\sum_{j=0}^{\infty}b_{kj}B_j=c_k,\quad (k=0,1,....)\,, \tag{1.33}$$

where

$$b_{kj}=\int_{-1}^{1}P_k^{(-\alpha,-\beta)}(x)\,p_j(x)(1-x)^{-\alpha}(1+x)^{-\beta}\,dx\,,$$

$$F_k=\int_{-1}^{1}P_k^{(-\alpha,-\beta)}(x)(1-x)^{-\alpha}(1+x)^{-\beta}f(x)dx\,.$$

Case I: When $\kappa = -1$. In this case $B_{-1} = 0$ and the consistency condition (1.28) is automatically satisfied.

Case II: When $\kappa = 0$. In this case equation (1.33) yields the system of $n \times n$ algebraic equations which give unique solution.

Case II: When $\kappa = 1$. In this case equation (1.33) contains one more unknown as compared to the number of equations , and the equilibrium or compatibility condition (1.27) provides the necessary additional equation. From equations (1.27) and (1.29), we get

$$B_0 = A / \theta_0^{(\alpha,\beta)} \tag{1.34}$$

and remaining constants $B_1, \ldots.. B_n$ are determined from equations (1.33).

(ii) Fredholm integral equation of second kind

In general system of the singular integral equation (1.13) if one diagonalizes the matrices A and B simultaneously, the dominant part of the system can be uncoupled. Therefore, any numerical technique developed for a solution of a single equation may be generalized for a system of equations. So it is enough to consider the solution of the following singular integral equation of the form

$$a\,\phi(x) + \frac{b}{\pi}\int_{-1}^{1} \phi(t)\frac{dt}{t-x} + \int_{-1}^{1} K(x,t)\phi(t)\,dt = f(x), \qquad -1 < x < 1 . \tag{1.35}$$

The interval is normalized to (-1, 1), without loss of generality it is assumed that a and b are some constants (real or complex), and $f(x)$ and $K(x,t)$ are H-continuous functions may be real or complex.

The fundamental function of the above equation (1.35) is given by

$$a\;\phi(x) + \frac{b}{\pi}\int_{-1}^{1} \phi(t)\frac{dt}{t-x}\,dt = F(x), \;\; |x| < 1 , \tag{1.36}$$

where F contains the input function f and the bounded term containing the Fredholm kernel *K(x,t)*. Defining sectionally holomorphic function as

$$\Phi(z) = \frac{1}{2\pi i}\int_{-1}^{1}\frac{\phi(z)}{t-z}\,dz\,, \tag{1.37}$$

and using the Plemej formulas, equation (1.36) gives

$$(a+ib)\Phi^{+}(z) - (a-ib)\Phi^{-}(z) = F(x)\,, \qquad |x|<1\,. \tag{1.38}$$

The fundamental solution of the equation (1.38) may be obtained as

$$X(Z) = (Z-1)^{\alpha}(Z+1)^{\beta}\,, \tag{1.39}$$

with $\alpha = \frac{1}{2\pi i}\ln\left(\frac{a-ib}{a+ib}\right) + N\,,\ \beta = \frac{1}{2\pi i}\ln\left(\frac{a-ib}{a+ib}\right) + M\,,$

$-1 < \mathrm{Re}\,\alpha, \mathrm{Re}\,\beta < 1\,,\ \kappa = -(\alpha+\beta) = -(N+M)\,,$

where $X(Z)$ is the branch for which $Z^{k}X(Z) \to 1$ as $Z \to \infty$, N and M are integers. It is assumed that $\phi(x)$ is either potential or flux to apply the restrictions on α and β. Here the index $\kappa = 0, \mp 1$ is determined by the physics of the problem.

Fundamental function of integral equation is obtained from equation (1.39) and given as

$$w(x) = (-1)^{-\alpha}X^{+}(x) = (1-x)^{\alpha}(x+1)^{\beta} \tag{1.40}$$

and hence the solution of equation (1.35) is given as

$$\phi(t) = g(t)\,w(t), \qquad |t|<1, \tag{1.41}$$

where $g(t)$ is a bounded and continuous function represented by an infinite series. From equation (1.40), it can be observed that $w(t)$ is the weight function associated with Jacobi polynomial $P_n^{(\alpha,\beta)}(t),\ (n = 0,1,2,3,.....)$.

Now $\phi(t) = \sum_{n=0}^{\infty} c_n\, w(t)\, P_n^{(\alpha,\beta)}(t),$ (1.42) where $P_n^{(\alpha,\beta)}(t)$ is Jacobi polynomial and c_n $(n=0,1,2..........)$ are undetermined constants.

$$\frac{1}{\pi}\int_{-1}^{1} w(t)\,P_n^{(\alpha,\beta)}(t)\frac{dt}{t-x}=\cot(\pi\,\alpha)w(x)\,P_n^{(\alpha,\beta)}(x)$$

$$-\frac{2^{\alpha+\beta}\,\Gamma\alpha\;\Gamma n+\beta+1}{\pi\,\Gamma n+\alpha+\beta+1}F(n+1;-n-\alpha-\beta;1-\alpha;\frac{1-x}{2}),$$

$$\left(-1<x<1,\ \mathrm{Re}(\alpha)>-1, \mathrm{Re}(\beta)>-1, \alpha+\beta=-\kappa, \mathrm{Re}\neq(0,1,...)\right),$$

where $\cot\pi\alpha=\cot\pi\left(\frac{1}{2\pi i}\log\left(\frac{a-ib}{a+ib}\right)+N\right)=-\frac{a}{b}$ (1.43)

and $P_{n-\kappa}^{(-\alpha,-\beta)}(x)=\frac{\Gamma n-\kappa-\alpha+1}{\Gamma n-\kappa+1\;\;\Gamma 1-\alpha}F\left(n+1;-n+\kappa+1;1-\alpha;\frac{1-x}{2}\right).$ (1.44)

Combining equation (1.43) and (1.44) yields

$$\frac{1}{\pi}\int_{-1}^{1} w(t)\,P_n^{(\alpha,\beta)}(t)\frac{dt}{t-x}=-\frac{a}{b}w(x)P_n^{(\alpha,\beta)}(x)-\frac{2^{\alpha+\beta}\,\Gamma\alpha\;\Gamma(1-\alpha)}{\pi}P_{n-\kappa}^{(-\alpha,-\beta)}(x),\quad (|x|<1) \tag{1.45}$$

$$\sum_{n=0}^{\infty}c_n\left[-\frac{2^{-\kappa}b}{\sin\pi\alpha}P_{n-\kappa}^{(-\alpha,-\beta)}(x)+h_n(x)\right]=f(x), \tag{1.46}$$

$$h_n(x)=\int_{-1}^{1}w(t)P_n^{(\alpha,\beta)}(t)\,k(x,t)\,dt,\qquad |x|<1. \tag{1.47}$$

The functional equation (1.46) can be reduced to an infinite system of algebraic equations in c_n by expanding both sides into series of Jacobi polynomials $P_k^{(-\alpha,-\beta)}(x)$, $(k=0,1,2,...)$ and comparing the respective coefficients. Using the orthogonality relations of Jacobi polynomials (1.32), we get

$$-\frac{2^{-\kappa}b}{\sin\pi\alpha}\theta_k(-\alpha,-\beta)\,c_{k+\kappa}+\sum_{k=0}^{N}d_{nk}c_n=F_k,\quad (k=0,1,.....N), \tag{1.48}$$

where

$$d_{nk} = \int_{-1}^{1} P_k^{(-\alpha,-\beta)}(x) w(-\alpha,-\beta,x) h_n(x)\, dx\,,$$

$$F_k = \int_{-1}^{1} P_k^{(-\alpha,-\beta)}(x) w(-\alpha,-\beta,x) f(x) dx\,,$$

$$w(-\alpha,-\beta,x) = (1-x)^{-\alpha}(1+x)^{-\beta} = w^{-1}(x)\,.$$

Case I: When $\kappa = -1$, the first term in the series equation (1.46) becomes $c_0 P_1^{(-\alpha,-\beta)}(x)$. Hence, during solving equation (1.32) it can be formally assumed that $c_{-1} = 0$. It is seen that, since $P_0^{(-\alpha,-\beta)}(x) = 1$, the first equation obtained from equation (1.46) for $k = -1$ is seen to be equivalent to the consistency condition

$$\int_{-1}^{1} \left(f(x) - \int_{-1}^{1} k(x,t)\phi(t) dt \right) \frac{dt}{w(t)} = 0. \tag{1.49}$$

Thus equation (1.46) Provides (N+1) linear algebraic equations for the unknown constants $c_0, \ldots\ldots c_N$.

Case II: When $\kappa = 0$, there will be no additional arbitrary constants or conditions, and equation (1.46) gives a unique solution.

Case III: When $\kappa = 1$, the (N+1) equations given by equation (1.46) contains (N+2) unknown constants $c_0, \ldots\ldots c_{N+1}$.

The additional equation for a unique solution is provided by equilibrium or compatibility condition given by

$$\int_{-1}^{1} \phi(t)\, dt = A \tag{1.50}$$

with aid of which that equation (1.42) gives

$$c_0 \theta_0(\alpha, \beta) = A\,. \tag{1.51}$$

This described method is valid for the most general singular integral equation given by equation (1.35). The application of this method can be found for the layered materials with interfacial cracks.