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# PREFACE

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Chapter 1 deals with the basics of dynamical systems, classification of dynamical systems, history of fractional calculus, Definitions of Riemann-Liouville and Caputo derivatives, Chaos theory, Synchronization, types of synchronization, Definition of Lyapunov exponent, Stability theory of fractional order systems and three synchronization methods viz., Active control method, Nonlinear control method and Backstepping method which have used in this thesis. Each topic plays an important role during preparation of chapters of this thesis.

In chapter 2, the stability analysis and chaos control of the fractional order Vallis and El-Nino systems have been studied. The chaos control of these systems is studied using nonlinear control method with the help of a new lemma for Caputo derivative and Lyapunov stability theory. The synchronization between the systems for different fractional order cases and numerical simulation through graphical plots for different particular cases clearly exhibit that the method is easy to implement and reliable for synchronization of fractional order chaotic systems. The comparison of time of synchronization when the systems pair approaches from standard order to fractional order is the key feature of this chapter.

In chapter 3, the phase and anti-phase synchronizations between fractional order hyperchaotic Lu and 4D integral order systems with parametric uncertainties and disturbances are studied using nonlinear active control method. A new lemma is used to design the controller. Numerical simulations are presented to demonstrate the effectiveness of the method to synchronize and anti-synchronize the fractional order hyperchaotic systems. The striking feature of this chapter is the comparison of time of



synchronization and anti-synchronization with and without the presence of uncertainties and external disturbances through graphical presentations for different particular cases.

The chapter 4 addresses the complex projective synchronization between fractional order Lorenz and Lu complex systems using nonlinear control method. During complex projective synchronization, controllers are designed on the basis of Lyapunov stability theory using a new lemma. The numerical and graphical results show that the nonlinear control method is easy to implement and reliable to achieve complex projective synchronization of fractional order complex chaotic systems.

The chapter 5 is related to the chaos control and the function projective synchronization of fractional order T-system and Lorenz chaotic system using backstepping method. Based on the stability theory, the condition for local stability of nonlinear three-dimensional commensurate fractional order system has been discussed. Feedback control method is used to control the chaos in the considered fractional order T-system. Numerical simulations of function projective synchronization between fractional order T system and Lorenz system is carried out using MATLAB and the results are depicted through graphs.

In chapter 6, a scheme using active backstepping design method to achieve combination synchronization of fractional order n-chaotic systems is proposed. In this proposed scheme the controllers are designed with the help of a new lemma and Lyapunav function in a systematic way. Synchronization between fractional order three and four systems have been shown as examples of synchronization of n-chaotic systems. Numerical simulation and graphical results clearly exhibit that the method of this new procedure is easy to implement and reliable for synchronization of fractional order chaotic systems.

In chapter 7, the dual function projective synchronization among various fractional order complex chaotic systems is investigated using active control technique. Two master systems and two response (salve) systems with control functions are taken for dual function projective synchronization. The control functions are designed with the help of active control technique. The fractional order complex T and complex Lu systems are taken to illustrate synchronization process. Numerical simulation and graphical results are presented to demonstrate the effectiveness of the method.