Chapter 7

Dual function projective synchronization of fractional order complex chaotic systems

7.1 Introduction

The concept of dynamical system is originated from Newtonian mechanics, which has many applications in engineering. The mathematical models of a dynamical system express the evolution of system in terms of equation of motion and initial value. Dynamical systems are exponentially sensitive to initial condition, which is popularly known as the butterfly effect. In 1963, E. N. Lorenz (Lorenz (1963)) found the canonical chaotic attractor first time. The term chaos in highly associated with nonlinear systems, which creates the occurrences of irregular solution while the equation of motion is deterministic. Later it has been detected in a large number of dynamical systems of various physical natures. It has been extended to the fractional order systems by the researchers and from the literature survey it is seen that in past few years. The area of chaotic dynamics in fractional order systems has been growing rapidly (Podlubny (1999), Hilfer (2001)).

The chaos synchronization problems have been studied and applied by the scientists and engineers in many scientific and engineering fields. The results have an important role in chaotic communications. It offers a potential advantage over non-coherent detection in terms of noise performance and data when the basis functions are recovered from noisy distorted received signals (Feng and Qiu (2004), Kolumban et al. (1998), Xu et al.

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(2004)). In the section 6.1, it is already mentioned that the idea of synchronization of chaotic systems was first given by L. M. Pecora and T. L. Carroll. They showed that synchronization between the systems is possible through simple coupling and later the idea was expanded in various fields of science and engineering.

Dual synchronization is a special circumstance in synchronization in which two different pairs of chaotic systems, i.e., two master systems and two slave systems are synchronized. In 1996, Tsimring and Sushchik (1996) were first investigated the idea of multiplexing chaos using synchronization in a small map and an electric circuit model. After this in 2000, the concept of dual synchronization was given by Liu and Davids (2000). They introduced the dual synchronization of 1-D discrete chaotic systems. In 2003, Uchida et al. (2003b) studied the dual synchronization of chaos in one-way coupled microchip lasers. The dual synchronization of chaos was also studied in microchip lasers (Uchida et al. (2003a)) in the same year. The dual synchronization between Lorenz and Rossler systems was investigated by Ning et al. (2007). In 2008, Salarieh and Shahrokhi (2008) studied the dual synchronization of chaotic systems via time-varying gain proportional feedback. In 2013, Xiao et al. (2013) have studied the dual synchronization of fractional-order chaotic systems via a linear controller. These have motivated the author to study on the dual function projective synchronization of fractional order complex chaotic systems. The dual synchronization of chaotic systems is used experimentally in communication applications (Uchida et al. (2003c)). In 2007, the complex Lu system was proposed by Mahmoud et al. (2007b), and later the fractional order complex Lu system was studied by the Jiang et al. (2014). In 2008, Tigan and Opris (2008) proposed a 3D chaotic system called T-system and its dynamical behavior was studied in details by Liu et al. (2014). The several complex dynamical systems of physical interest have been studied and proposed by the many researchers (Ning and Haken (1990), Nian et al. (2010), Roldan et al. (1993), Toronov and Derbov (1997), Luo and Wang (2013)), where chaotic systems were efficiently used in various important fields of science and engineering. The complex chaotic systems are successfully used in laser physics where the atomic polarization and electric field amplitudes in a ring laser system of two-level atoms are complex quantities (Fowler et al. (1983), Rauh et al. (1996)). The application of complex chaotic systems is already described in the section 4.1 of chapter 4.

In this chapter the dual function projective synchronization of fractional order complex chaotic systems using active control method is studied. During the study of complex chaotic systems, fractional order T and Lu systems are taken. The numerical simulation and results of this chapter are displayed graphically which clearly exhibit that the active control method is effective, easy to implement and reliable for the dual function projective synchronization of fractional order complex chaotic systems.

7.2 Problem statement

Let us consider first two fractional order complex chaotic systems as master (drive) system as

Master systems-I:

$$\frac{d^q X_m}{dt^q} = AX_m + f(X_m),$$

where $X_m = X_{m_1} + jX_{m_2}$ is complex state variable with

$$\frac{d^q X_{m_1}}{dt^q} = AX_{m_1} + f_1(X_{m_1})$$

$$\frac{d^q X_{m_2}}{dt^q} = AX_{m_2} + f_2(X_{m_2}). \qquad (7.1)$$

Master system-II:

$$\frac{d^q Y_m}{dt^q} = BY_m + g(Y_m),$$

where $Y_m = Y_{m_1} + jY_{m_2}$ is complex state variable with

$$\frac{d^q Y_{m_1}}{dt^q} = BY_{m_1} + g_1(Y_{m_1})$$

$$\frac{d^q Y_{m_2}}{dt^q} = BY_{m_2} + g_2(Y_{m_2}). \qquad (7.2)$$

The linear combination of the master systems I & II, gives rise to

$$V_m = \sum_{i=1}^{n} a_i X_{m_1 i} + \sum_{i=1}^{n} a_i X_{m_2 i} + \sum_{i=1}^{n} b_i Y_{m_1 i} + \sum_{i=1}^{n} b_i Y_{m_2 i}$$

$$= [a_1, a_2, ...a_n] X_{m_1} + [a_1, a_2, ...a_n] X_{m_2} + [b_1, b_2, ...b_n] Y_{m_1} + [b_1, b_2, ...b_n] Y_{m_2}$$

$$= A_1^T X_{m_1} + A_1^T X_{m_2} + B_1^T Y_{m_1} + B_1^T Y_{m_2} = [A_1^T \;\; A_1^T] \begin{bmatrix} X_{m_1} \\ X_{m_2} \end{bmatrix} + [B_1^T \;\; B_1^T] \begin{bmatrix} Y_{m_1} \\ Y_{m_2} \end{bmatrix}$$

$$= A_{11}^T X_m + B_{11}^T Y_m = E_{11}^T \begin{bmatrix} X_m \\ Y_m \end{bmatrix},$$

where $A_1 = [a_1, a_2, ...a_n]^T$ and $B_1 = [b_1, b_2, ...b_n]^T$ are known and $A_{11} = [A_1^T \;\; A_1^T]^T$, $B_{11} = [B_1^T \;\; B_1^T]^T$ and $E_{11} = [A_{11}^T \;\; B_{11}^T]^T$.

The next two fractional order complex systems is taken as response (slave) system as

Response system-I:

$$\frac{d^q X_s}{dt^q} = CX_s + h(X_s) + u^{(1)}(t).$$

Taking $X_s = X_{s_1} + jX_{s_2}$ as complex state variable, we get

$$\frac{d^q X_{s_1}}{dt^q} = CX_{s_1} + h_1(X_{s_1}) + u_1^{(1)}(t)$$

$$\frac{d^q X_{s_2}}{dt^q} = CX_{s_2} + h_2(X_{s_2}) + u_2^{(1)}(t)\,. \tag{7.3}$$

Response system-II:

$$\frac{d^q Y_s}{dt^q} = DY_s + I(Y_s) + u^{(2)}(t).$$

Taking $Y_s = Y_{s_1} + jY_{s_2}$ as complex state variable, we get

$$\frac{d^q Y_{s_1}}{dt^q} = DY_{s_1} + I_1(Y_{s_1}) + u_1^{(2)}(t)$$

$$\frac{d^q Y_{s_2}}{dt^q} = DY_{s_2} + I_2(Y_{s_2}) + u_2^{(2)}(t)\,, \tag{7.4}$$

where $u^{(1)}(t) = u_1^{(1)}(t) + j\,u_2^{(1)}(t)$, $u^{(2)}(t) = u_1^{(2)}(t) + j\,u_2^{(2)}(t)$ are control functions, $u_1^{(1)}(t) = [u_1^{(1)}, u_3^{(1)}, ..., u_{2n-1}^{(1)}]^T$, $u_2^{(1)}(t) = [u_2^{(1)}, u_4^{(1)}, ..., u_{2n}^{(1)}]^T$ and $u_1^{(2)}(t) = [u_1^{(2)}, u_3^{(2)}, ..., u_{2n-1}^{(2)}]^T$, $u_2^{(2)}(t) = [u_2^{(2)}, u_4^{(2)}, ..., u_{2n}^{(2)}]^T$.

The linear combination gives

$$V_s = \sum_{i=1}^{n} a_i X_{s_1 i} + \sum_{i=1}^{n} a_i X_{s_2 i} + \sum_{i=1}^{n} b_i Y_{s_1 i} + \sum_{i=1}^{n} b_i Y_{s_2 i}$$

$$= [a_1, a_2, ...a_n] X_{s_1} + [a_1, a_2, ...a_n] X_{s_2} + [b_1, b_2, ...b_n] Y_{s_1} + [b_1, b_2, ...b_n] Y_{s_2}$$

$$= A_1^T X_{s_1} + A_1^T X_{s_2} + B_1^T Y_{s_1} + B_1^T Y_{s_2} = [A_1^T \;\; A_1^T] \begin{bmatrix} X_{s_1} \\ X_{s_2} \end{bmatrix} + [B_1^T \;\; B_1^T] \begin{bmatrix} Y_{s_1} \\ Y_{s_2} \end{bmatrix}$$

$$= A_{11}^T X_m + B_{11}^T Y_m = E_{11}^T \begin{bmatrix} X_S \\ Y_s \end{bmatrix}.$$

The goal is to obtain the dual function projective synchronization between master and slave systems.

Now defining the error function between the master and slave systems as

$e = V_s - A'(x)V_m$, where $A'(x) = diag\{K_1(X_1), K_2(X_2), ..., K_n(X_n)\}$ is the function scaling matrix.

Therefore, for dual function projective synchronization the active control method is used for designing the control functions in such a way that the origin becomes asymptotically stable equilibrium point of the error dynamics i.e., $\lim_{t\to\infty}\|X_s - A'(x)X_m\| = 0$ and $\lim_{t\to\infty}\|Y_s - A'(x)Y_m\| = 0$.

The demonstration of active control method is given in section 7.4 and the schematic diagram is described through Fig. 7.1 for dual synchronization of complex systems.

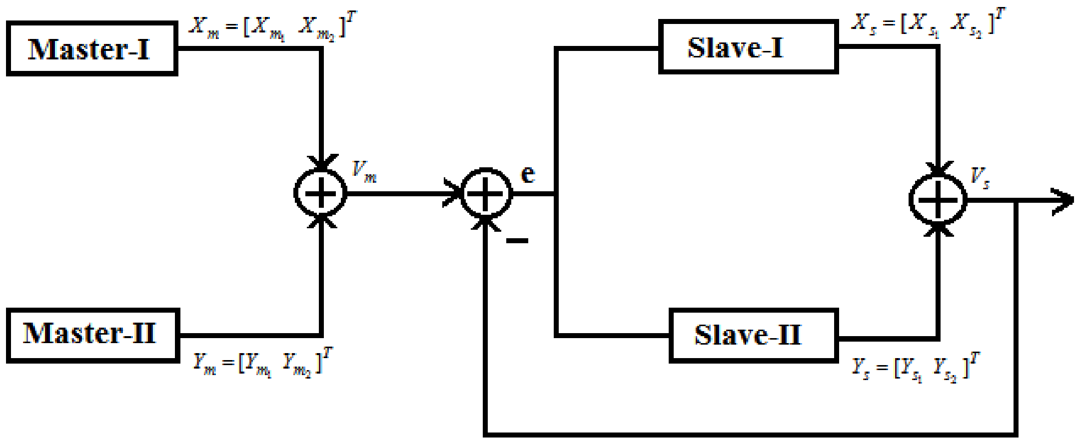


Fig. 7.1 Dual synchronization scheme of complex chaotic systems

Definition 7.1 The master systems (7.1), (7.2) and response systems (7.3), (7.4) are said to be dual function projective synchronized, if $\lim_{t\to+\infty}\|e\| = 0$, where $\|\cdot\|$ denotes matrix norm.

7.3 Systems' descriptions

7.3.1 The fractional order complex T system

The fractional order complex T system (Xiao et al. (2013)) is described as

$$\frac{d^q x_1}{dt^q} = a_1(x_2 - x_1)$$

$$\frac{d^q x_2}{dt^q} = (a_2 - a_1)x_1 - a_1 x_1 x_3 \qquad (7.5)$$

$$\frac{d^q x_3}{dt^q} = \frac{1}{2}(\bar{x}_1 x_2 + x_1 \bar{x}_2) - a_3 x_3,$$

where $x = (x_1, x_2, x_3)^T$ is the state variable vector of the system, $x_1 = x_{11} + jx_{12}$ and $x_2 = x_{13} + jx_{14}$ are complex variables, $x_3 = x_{15}$ is real variable and a_1, a_2, a_3 are parameters with $a_1 \neq 0$. This system possesses chaotic attractors which are shown in Fig. 7.2, when the parameters are taken as $a_1 = 2.1, a_2 = 30, a_3 = 0.6$ and initial condition $x(0) = [-1 + 9j,\ 8 - 5j,\ -1]$ at $q = 0.94$.

Equation (7.5) can be written in the form

$$\frac{d^q x_{11}}{dt^q} = a_1(x_{13} - x_{11})$$

$$\frac{d^q x_{12}}{dt^q} = a_1(x_{14} - x_{12})$$

$$\frac{d^q x_{13}}{dt^q} = (a_2 - a_1)x_{11} - a_1 x_{11} x_{15} \qquad (7.6)$$

$$\frac{d^q x_{14}}{dt^q} = (a_2 - a_1)x_{12} - a_1 x_{12} x_{15}$$

$$\frac{d^q x_{15}}{dt^q} = x_{11}x_{13} + x_{12}x_{14} - a_3 x_{15}.$$

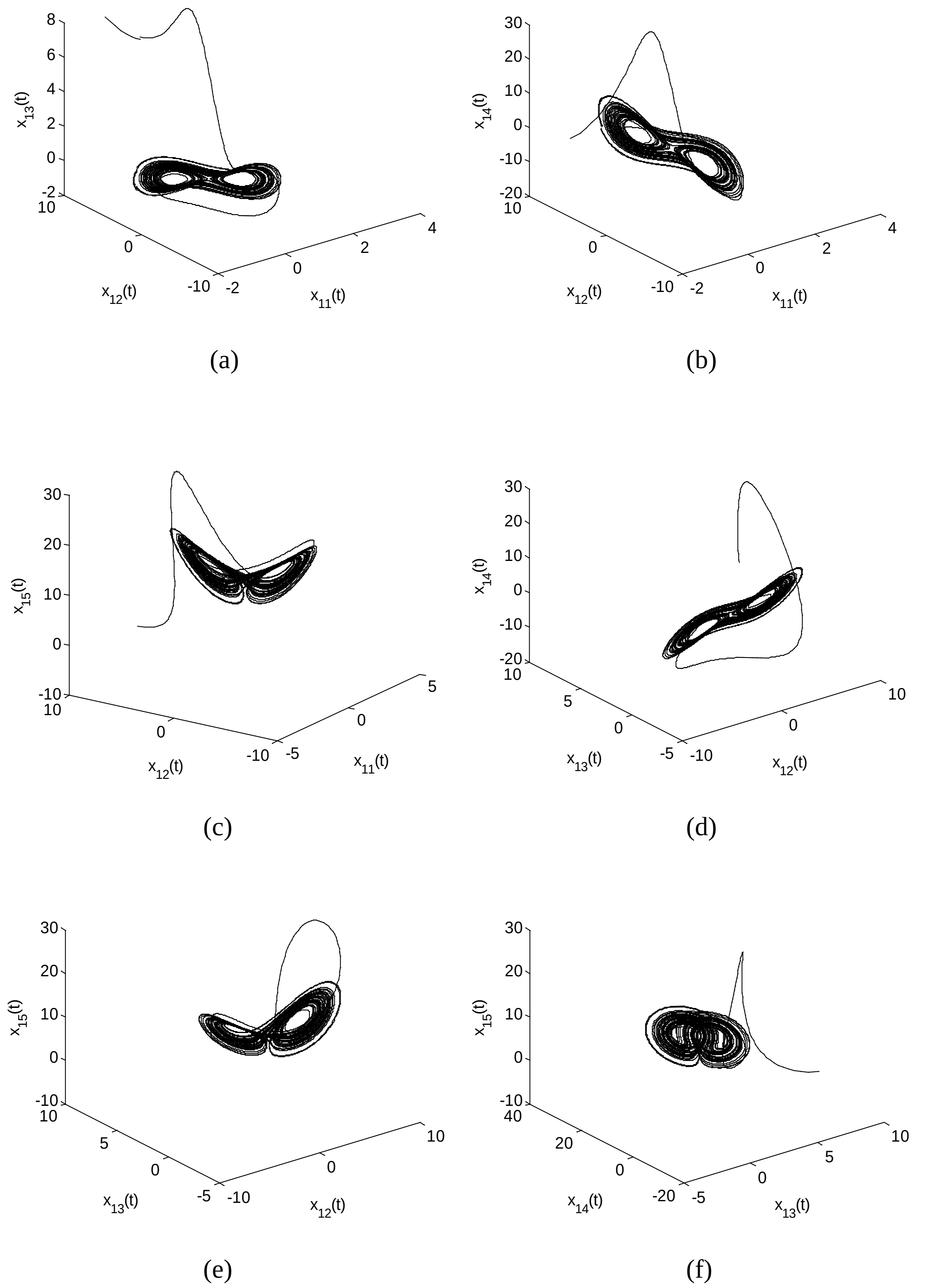


Fig. 7.2 Phase portraits of fractional order complex T system for fractional order $q = 0.94$.

7.3.2 The fractional order complex Lu system

The fractional order complex Lu system is already described through equation (4.6) under the section 4.3.2 in chapter 4.

Let the state variables of the system is written in the form $y_1 = y_{11} + jy_{12}$ and $y_2 = y_{13} + jy_{14}$ are complex variables while $y_3 = y_{15}$ is real variable.

Separating into real and imaginary parts, we get

$$\frac{d^q y_{11}}{dt^q} = b_1(y_{13} - y_{11})$$

$$\frac{d^q y_{12}}{dt^q} = b_1(y_{14} - y_{12})$$

$$\frac{d^q y_{13}}{dt^q} = b_2 y_{13} - y_{11} y_{15} \tag{7.7}$$

$$\frac{d^q y_{14}}{dt^q} = b_2 y_{14} - y_{12} y_{15}$$

$$\frac{d^q y_{15}}{dt^q} = y_{11} y_{13} + y_{12} y_{14} - b_3 y_{15}.$$

7.4 Dual function projective synchronization of fractional order complex T system and Lu system using active control method

The fractional order complex T system is considered as master systems-I and fractional order complex Lu system is taken as master system-II, which are already defined in equations (7.6) and (7.7) respectively.

The response systems-I and response system-II with control functions are defined as

Response system-I:

$$\frac{d^q x_{21}}{dt^q} = a_1(x_{23} - x_{21}) + u_1^{(1)}$$

$$\frac{d^q x_{22}}{dt^q} = a_1(x_{24} - x_{22}) + u_2^{(1)}$$

$$\frac{d^q x_{23}}{dt^q} = (a_2 - a_1)x_{21} - a_1 x_{21} x_{25} + u_3^{(1)} \tag{7.8}$$

$$\frac{d^q x_{24}}{dt^q} = (a_2 - a_1)x_{22} - a_1 x_{22} x_{25} + u_4^{(1)}$$

$$\frac{d^q x_{25}}{dt^q} = x_{21}x_{23} + x_{22}x_{24} - a_3 x_{25} + u_5^{(1)}.$$

Response system-II:

$$\frac{d^q y_{21}}{dt^q} = b_1(y_{23} - y_{21}) + u_1^{(2)}$$

$$\frac{d^q y_{22}}{dt^q} = b_1(y_{24} - y_{22}) + u_2^{(2)}$$

$$\frac{d^q y_{23}}{dt^q} = b_2 y_{23} - y_{21} y_{25} + u_3^{(2)} \tag{7.9}$$

$$\frac{d^q y_{24}}{dt^q} = b_2 y_{24} - y_{22} y_{25} + u_4^{(2)}$$

$$\frac{d^q y_{25}}{dt^q} = y_{21}y_{23} + y_{22}y_{24} - b_3 y_{25} + u_5^{(2)}.$$

Now error functions are defined as $e_{1i} = x_{2i} - k_{1i}x_{1i}$, where $i = 1, 2, ..., 5$ and $e_{2i} = y_{2i} - k_{2i}y_{1i}$, where $i = 1, 2, ..., 5,$ and k_{1i}, k_{2i} are the scaling functions.

The error systems become

$$\frac{d^q e_{11}}{dt^q} = a_1(e_{13} - e_{11}) + a_1(k_{13} - k_{11})x_{13} + u_1^{(1)}(t)$$

$$\frac{d^q e_{12}}{dt^q} = a_1(e_{14} - e_{12}) + a_1(k_{14} - k_{12})x_{14} + u_2^{(1)}(t)$$

$$\frac{d^q e_{13}}{dt^q} = (a_2 - a_1)e_{11} + (a_2 - a_1)(k_{11} - k_{13})x_{11} - a_1 x_{21} x_{25} + k_{13} a_1 x_{11} x_{15} + u_3^{(1)}(t)$$

$$\frac{d^q e_{14}}{dt^q} = (a_2 - a_1)e_{12} + (a_2 - a_1)(k_{12} - k_{14})x_{12} - a_1 x_{22} x_{25} + k_{14} a_1 x_{12} x_{15} + u_4^{(1)}(t)$$

$$\frac{d^q e_{15}}{dt^q} = -a_3 e_{15} + x_{21} x_{23} + x_{22} x_{24} - k_{15}(x_{11} x_{13} + x_{12} x_{14}) + u_5^{(1)}(t)$$

$$\frac{d^q e_{21}}{dt^q} = b_1(e_{23} - e_{21}) + b_1(k_{23} - k_{21}) y_{13} + u_1^{(2)}(t) \qquad (7.10)$$

$$\frac{d^q e_{22}}{dt^q} = b_1(e_{24} - e_{22}) + b_1(k_{24} - k_{22}) y_{14} + u_2^{(2)}(t)$$

$$\frac{d^q e_{23}}{dt^q} = b_2 e_{23} - y_{21} y_{25} + k_{23} y_{11} y_{15} + u_3^{(2)}(t)$$

$$\frac{d^q e_{24}}{dt^q} = b_2 e_{24} - y_{22} y_{25} + k_{24} y_{12} y_{15} + u_4^{(2)}(t)$$

$$\frac{d^q e_{25}}{dt^q} = -b_3 e_{25} + y_{21} y_{23} + y_{22} y_{24} - k_{25}(y_{11} y_{13} + y_{12} y_{14}) + u_5^{(2)}(t)\,.$$

Here the goal is to design the control functions $u_i^j(t),\ j = 1, 2,\ i = 1, 2, ..., 5.$ (Agrawal et al. (2012b)) as

$$u_1^{(1)}(t) = -a_1(k_{13} - k_{11}) x_{13} + v_1^{(1)}(t)$$

$$u_2^{(1)}(t) = -a_1(k_{14} - k_{12}) x_{14} + v_2^{(1)}(t)$$

$$u_3^{(1)}(t) = -(a_2 - a_1)(k_{11} - k_{13}) x_{11} + a_1 x_{21} x_{25} - k_{13} a_1 x_{11} x_{15} + v_3^{(1)}(t)$$

$$u_4^{(1)}(t) = -(a_2 - a_1)(k_{12} - k_{14}) x_{12} + a_1 x_{22} x_{25} - k_{14} a_1 x_{12} x_{15} + v_4^{(1)}(t)$$

$$u_5^{(1)}(t) = -x_{21} x_{23} - x_{22} x_{24} + k_{15}(x_{11} x_{13} + x_{12} x_{14}) + v_5^{(1)}(t)$$

$$u_1^{(2)}(t) = -b_1(k_{23} - k_{21}) y_{13} + v_1^{(2)}(t)$$

$$u_2^{(2)}(t) = -b_1(k_{24} - k_{22}) y_{14} + v_2^{(2)}(t)$$

$$u_3^{(2)}(t) = y_{21} y_{25} - k_{23} y_{11} y_{15} + v_3^{(2)}(t)$$

$$u_4^{(2)}(t) = y_{22} y_{25} - k_{24} y_{12} y_{15} + v_4^{(2)}(t)$$

$u_5^{(2)}(t) = -y_{21}y_{23} - y_{22}y_{24} + k_{25}(y_{11}y_{13} + y_{12}y_{14}) + v_5^{(2)}(t)$,

which leads to the following error systems as

$$\frac{d^q e_{11}}{dt^q} = a_1(e_{13} - e_{11}) + v_1^{(1)}(t)$$

$$\frac{d^q e_{12}}{dt^q} = a_1(e_{14} - e_{12}) + v_2^{(1)}(t)$$

$$\frac{d^q e_{13}}{dt^q} = (a_2 - a_1)e_{11} + v_3^{(1)}(t)$$

$$\frac{d^q e_{14}}{dt^q} = (a_2 - a_1)e_{12} + v_4^{(1)}(t)$$

$$\frac{d^q e_{15}}{dt^q} = -a_3 e_{15} + v_5^{(1)}(t)$$

$$\frac{d^q e_{21}}{dt^q} = b_1(e_{23} - e_{21}) + v_1^{(2)}(t) \tag{7.11}$$

$$\frac{d^q e_{22}}{dt^q} = b_1(e_{24} - e_{22}) + v_2^{(2)}(t)$$

$$\frac{d^q e_{23}}{dt^q} = b_2 e_{23} + v_3^{(2)}(t)$$

$$\frac{d^q e_{24}}{dt^q} = b_2 e_{24} + v_4^{(2)}(t)$$

$$\frac{d^q e_{25}}{dt^q} = -b_3 e_{25} + v_5^{(2)}(t).$$

The error system (7.11) is considered as a control problem, which is a linear system with control inputs $v_i^{(j)}(t),\ j = 1, 2,\ i = 1, 2, .., 5.$ as the functions of $e_{ji}(t),\ j = 1, 2,\ i = 1, 2, .., 5.$ Now to design control inputs the above system is stabilized so that $e_{ji}(t),\ j = 1, 2,\ i = 1, 2, .., 5$ converge to zero as time *t* approaches to infinity which implies that fractional order complex T and complex Lu systems are synchronized.

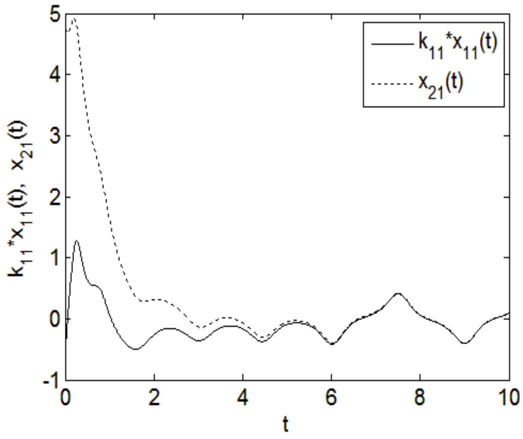
There are many choices for control inputs. Let us choose $[v_i^{(j)}(t)] = A[e_{ji}]$, $j = 1, 2,\ i = 1, 2, .., 5$, where A is the 10×10 matrix. In order to make the closed loop system stable, the matrix A should be selected in such a way that the feedback system will have the eigenvalues $\lambda_i,\ i = 1,\ 2, ..., 10$ with negative real parts. There is no unique choice for matrix A, but a good choice can be as follows

$$A = \begin{bmatrix} a_1 - 1 & 0 & -a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 - 1 & 0 & -a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(a_2 - a_1) & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(a_2 - a_1) & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_3 - 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_1 - 1 & 0 & -b_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_1 - 1 & 0 & -b_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b_2 - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b_2 - 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_3 - 1 \end{bmatrix}$$

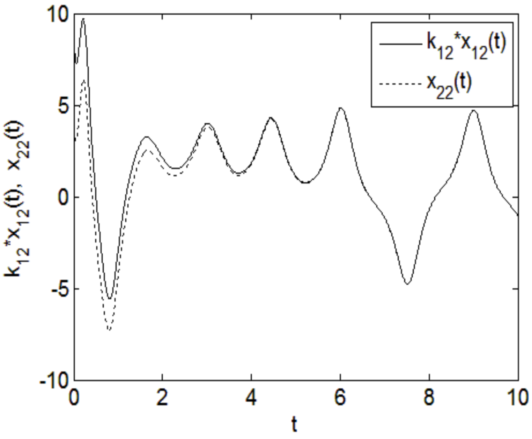
Then the error system (7.11) is reduced to

$$\frac{d^q e_{ij}}{dt^q} = -e_{ij}, \quad i = 1,\ 2;\ j = 1, 2, 3, 4, 5. \tag{7.12}$$

All the eigenvalues of the error systems (7.12) are negative and hence the condition $|\arg(\lambda_i)| > (q\pi/2)$ for $0 < q \le 1$ is satisfied. Therefore the systems are stable and required dual function projective synchronization is obtained which are shown graphically through Figs. 7.3(a)-(j).


$k_{11}*x_{11}(t)$
$x_{21}(t)$
$k_{11}*x_{11}(t)$, $x_{21}(t)$
t

(a)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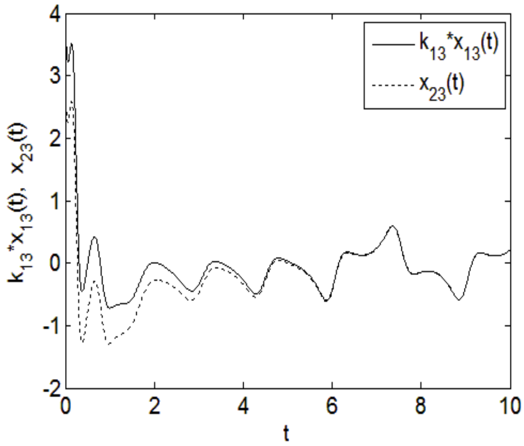
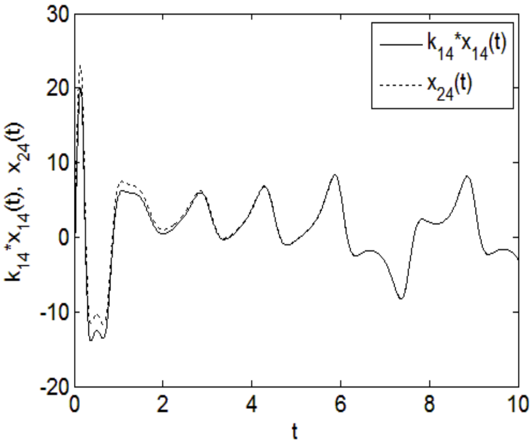
$x_{22}(t)$
$k_{12}*x_{12}(t)$, $x_{22}(t)$
t

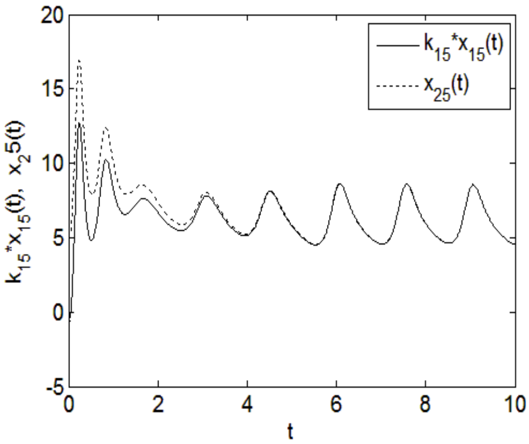
(b)


$k_{13}*x_{13}(t)$
$x_{23}(t)$
$k_{13}*x_{13}(t)$, $x_{23}(t)$
t

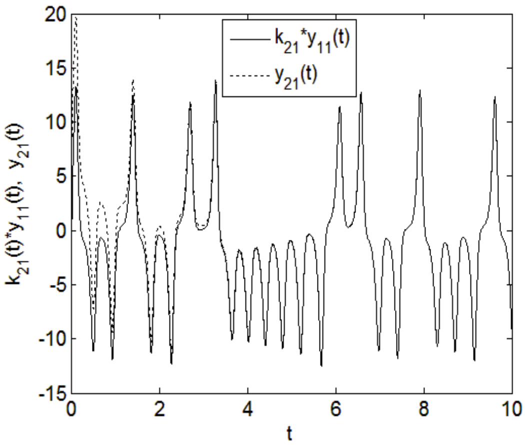
(c)


$k_{14}*x_{14}(t)$
$x_{24}(t)$
$k_{14}*x_{14}(t)$, $x_{24}(t)$
t

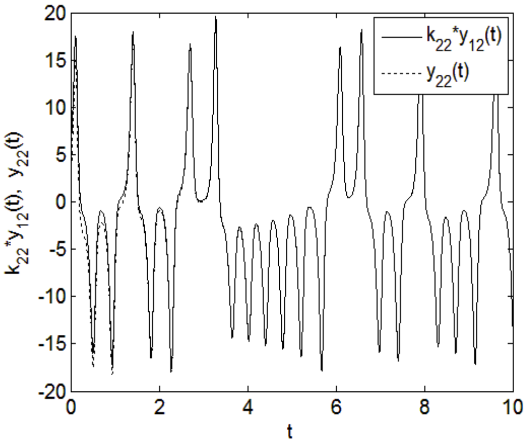
(d)


$k_{15}*x_{15}(t)$
$x_{25}(t)$
$k_{15}*x_{15}(t)$, $x_2 5(t)$
t

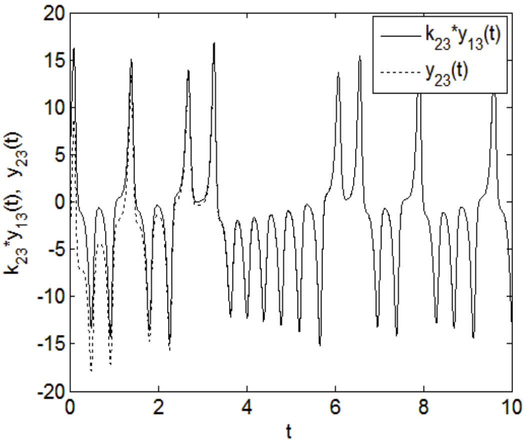
(e)


$k_{21}*y_{11}(t)$
$y_{21}(t)$
$k_{21}(t)*y_{11}(t)$, $y_{21}(t)$
t

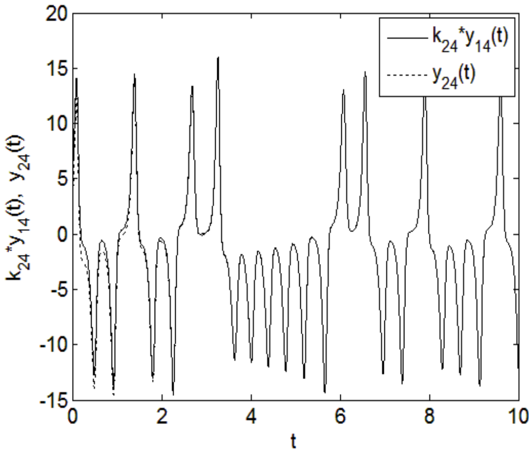
(f)


k22*y12(t)
y22(t)
k22*y12(t), y22(t)
t

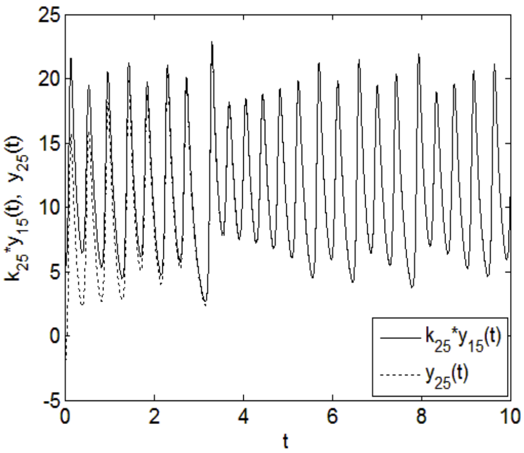
(g)


k23*y13(t)
y23(t)
k23*y13(t), y23(t)
t

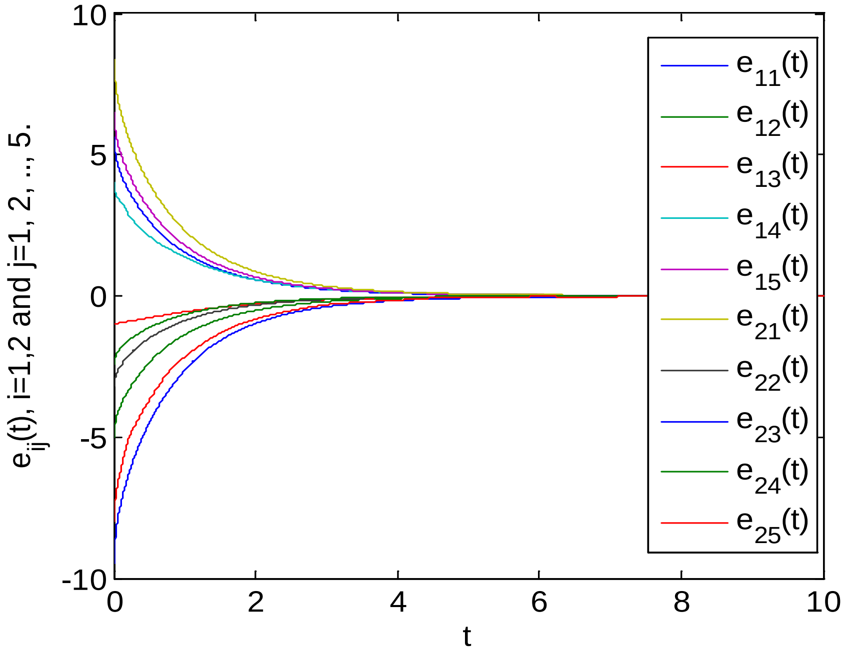
(h)


k24*y14(t)
y24(t)
k24*y14(t), y24(t)
t

(i)


k25*y15(t)
y25(t)
k25*y15(t), y25(t)
t

(j)



(k)

Fig. 7.3. State trajectories of the master systems (7.6) and (7.7) and response systems (7.8) and (7.9) for order $q = 0.96$: (a) synchronization between $k_{11} * x_{11}(t)$ and $x_{21}(t)$; (b) synchronization between $k_{12} * x_{12}(t)$ and $x_{22}(t)$; (c) synchronization between $k_{13} * x_{13}(t)$ and $x_{23}(t)$; (d) synchronization between $k_{14} * x_{14}(t)$ and $x_{24}(t)$; (e) synchronization between $k_{15} * x_{15}(t)$ and $x_{25}(t)$; (f) synchronization between $k_{21} * y_{11}(t)$ and $y_{21}(t)$; (g) synchronization between $k_{22} * y_{12}(t)$ and $y_{22}(t)$; (h) synchronization between $k_{23} * y_{13}(t)$ and $y_{23}(t)$; (i) synchronization between $k_{24} * y_{14}(t)$ and $y_{24}(t)$; (j) synchronization between $k_{25} * y_{15}(t)$ and $y_{25}(t)$; (k) The evolution of the error functions $e_{ij}(t),\ i = 1,\ 2$ and $j = 1, 2, \ldots, 5$.

7.5 Numerical simulation and results

In this section, earlier considered values of parameters of the fractional order complex T and Lu systems are taken during dual function projective synchronization. The initial conditions of master systems I & II and response systems I & II are taken as $(x_{11}(0),\ x_{12}(0),\ x_{13}(0),\ x_{14}(0),\ x_{15}(0)) = (-1,\ 9,\ 8,\ 5,\ -1)$, $(y_{11}(0),\ y_{12}(0),\ y_{13}(0),\ y_{14}(0),$ $y_{15}(0)) = (-8,\ 6,\ 5,\ 6,\ 10)$ and $(x_{21}(0),\ x_{22}(0),\ x_{23}(0),\ x_{24}(0),\ x_{25}(0)) = (5,\ 4,\ 3,\ -1,\ 6)$, $(y_{21}(0),\ y_{22}(0),\ y_{23}(0),\ y_{24}(0), y_{25}(0)) = (2,\ 4,\ -5,\ 3,\ -2)$ respectively, hence the initial condition of error systems according to definition of error functions will be $(e_{11}(0), e_{12}(0), e_{13}(0), e_{14}(0), e_{15}(0), e_{21}(0), e_{22}(0), e_{23}(0), e_{24}(0), e_{25}(0)) = (5.6, -4.9991, -0.9992, 3.9975, 6.5, 8.3992, -3.1976, -9.5, -2.4, -7.9860)$. We are taking the scaling functions as $k_{11} = a_{11}\cos(a_{12}x_{11}) + a_{13}$, $k_{12} = a_{21}\cos(a_{22}x_{12}) + a_{23}$, $k_{13} = a_{31}\cos(a_{32}x_{13}) + a_{33}$, $k_{14} = a_{41}\cos(a_{42}x_{14}) + a_{43}$, $k_{15} = a_{51}\cos(a_{52}x_{15}) + a_{53}$, $k_{21} = b_{11}\cos(b_{12}x_{21}) + b_{13}$, $k_{22} = b_{21}\cos(b_{22}x_{22}) + b_{23}$, $k_{23} = b_{31}\cos(b_{32}x_{23}) + b_{33}$, $k_{24} = b_{41}\cos(b_{42}x_{24}) + b_{43}$, $k_{25} = b_{51}\cos(b_{52}x_{25}) + b_{53}$, where values of parameters are $a_{11} = 0.5$, $a_{12} = 0.2$, $a_{13} = 0.1$, $a_{21} = 0.8$, $a_{22} = 0.1$, $a_{23} = 0.2$, $a_{31} = 0.1$, $a_{32} = 0.3$, $a_{33} = 0.4$, $a_{41} = 0.4$, $a_{42} = 0.6$, $a_{43} = 0.6$, $a_{51} = 0.2$, $a_{52} = 0.2$, $a_{53} = 0.3$, $b_{11} = 0.3$, $b_{12} = 0.3$, $b_{13} = 0.5$, $b_{21} = 0.9$, $b_{22} = 0.4$, $b_{23} = 0.3$, $b_{31} = 0.7$, $b_{32} = 0.2$, $b_{33} = 0.2$, $b_{41} = 0.1$, $b_{42} = 0.1$, $b_{43} = 0.8$, $b_{51} = 0.4$, $b_{52} = 0.8$, $b_{53} = 0.2$. It is seen from the Fig. 7.3(k) that error functions asymptotically converge to zero as time approaches to infinity for the order of derivative $q = 0.96$ which shows that the master systems I & II are synchronized with the response systems I & II.

7.6 Conclusion

This chapter has successfully demonstrated the dual function projective synchronization among various fractional order complex chaotic systems using active control method. For validation the dual function projective synchronization of fractional order complex T and Lu systems are done. The graphical presentations of numerical results with error functions approach to zero as time becomes large through proper choices of control functions clearly exhibit that our applied method is very much effective and convenient to achieve global dual function projective synchronization of fractional order complex chaotic systems.