Chapter 6

Combination synchronization of fractional order n-chaotic systems using active backstepping design

6.1 Introduction

Chaos synchronization is an interesting phenomenon of nonlinear dynamical systems and it may occur when two or more chaotic systems are coupled or one chaotic system drives the other. After introduction of the synchronization method between chaotic systems by L. M. Pecora and T. L. Carroll in the year 1990, it has been intensively studied due to its potential applications in various fields viz., ecological system, physical system, chemical system, secure communications etc. (Blasius et al. (1999), Lakshmanan and Murali (1996), Han et al. (1995), Cuomo and Oppenheim (1993), Murali and Lakshmanan (2003)). In recent years different schemes have been successfully applied to chaos synchronization viz., linear and nonlinear feedback control method, active control method, adaptive control method, sliding mode control method, backstepping method etc. (Chen and Lu (2002a), Huang et al. (2004), Agrawal et al. (2012b), Chen and Lu (2002b), Razminia and Baleanu (2013), Park (2006)).

The method here to use is backstepping design, which has been employed by many researchers for controlling and synchronizing chaotic systems as well as hyperchaotic systems. It consists in a recursive procedure that links the choice of a Lyapunov function with the design of a controller. Backstepping design is recognised as powerful design method for chaos synchronization. The design can guarantee global stability, tracking and transient performance for a broad class of strict-feedback nonlinear

---

The contents of this chapter have been communicated in **Nonlinear Dynamic (Springer)** after minor revision.

systems (Zhang et al. (2005), Kokotovic (1992), Krstic et al. (1995). To stabilize and track chaotic systems, the method had been successfully used by Mascolo and Grassi (1999). In 2006, the backstepping control was used by Bin et al. (2006) to synchronize two coupled chaotic neurons in external electrical stimulation. Wang and Ge (2001) proposed the Adaptive synchronization of uncertain chaotic systems via backstepping design. Backstepping design was successfully applied by Tan et al. (2003) during synchronization of the chaotic systems and also by Yu and Zhang (2003) to control the uncertain behavior of chaotic systems. Recently, Park (2006), Wu et al. (2009) have shown that the back stepping method is very simple, reliable and powerful for controlling the chaotic behavior and synchronization of chaotic systems. But to the best of my knowledge the synchronization of fractional order systems using backstepping control has not yet been studied by any researcher. The theme of the present study is to investigate the synchronization procedure for a number of fractional order chaotic systems using this simple and reliable backstepping method.

Initially the prediction of a system had been confined through finding the analytical solution of the formal modelling of the systems via mathematical modelling with a set of parameters and initial/boundary conditions. But after the advent of modern computers and related software packages, the simulation has become a useful technique of modelling of many streams of science and engineering as well as computational sociology. Nowadays it is used in technology to optimize the performance, safety engineering, also during modelling of natural and human systems. Simulation is described as the limitation of operation of a real world system over time. Thus before performing simulation, it requires to develop a model which will represent key features of the selected physical or abstract systems. Thus simulation basically represents the operation of the system over time. During synchronization of identical or non-identical chaotic systems the simulation is used to find requirement of minimum time after which the states of slave system behave similar to the master system.

The synchronization of three chaotic dynamical systems in integer order are first studied by the Runzi et al. (2011). It is seen from literature survey that the synchronization between three and more chaotic systems are few in numbers. Runzi et al. (2011) had stated the cause of investigation between two drive systems and one response system through a physical application in secure communication as transmitted signals can be splitted into several parts, each part loaded in different drive systems which shows that transmitted signals have stronger anti-attack ability and anti-translated capability than that transmitted by the usual transmission model. This has motivated me to study the generalization of synchronization between chaotic systems, when the systems have memory effects. In this chapter a new type of synchronization scheme known as combination synchronization of fractional order n-chaotic systems is proposed. The backstepping method is applied during synchronization of fractional order chaotic systems using Lyapunov stability theory and a new lemma for Caputo derivative. The combination synchronization of three and four fractional order chaotic systems are shown numerically and graphically to show the effectiveness and feasibility of the proposed scheme and method.

6.2 The scheme of combination synchronization of fractional order n-chaotic systems

In this scheme, (n-1) drive systems and one response system are assumed to be in fractional order system.

The drive systems are considered as

$$D_t^q x_1 = f_1(x_1), \tag{6.1}$$

$$D_t^q x_2 = f_2(x_2), \tag{6.2}$$

………………..

………………..

$$D_t^q x_{n-1} = f_{n-1}(x_{n-1}), \tag{6.3}$$

and the response system is taken as

$$D_t^q x_n = f_n(x_n) + U(x_1, x_2, ..., x_n), \tag{6.4}$$

where $x_1 = (x_1^1, x_2^1, ..., x_n^1)$, $x_2 = (x_1^2, x_2^2, ..., x_n^2)$, ………………, $x_{n-1} = (x_1^{n-1}, x_2^{n-1}, ..., x_n^{n-1})$ and $x_n = (x_1^n, x_2^n, ..., x_n^n)$ with $x_1, x_2, ..., x_{n-1}, x_n \in R^n$ are the state vectors of the n-chaotic systems. $f_1, f_2, ..., f_{n-1}, f_n : R^n \to R^n$ are the n-continuous vector functions and $U(x_1, x_2, ..., x_n) : \underbrace{R^n \times R^n \times .... \times R^n}_{n\ times} \to R^n$ is a controller which will be designed latter.

Definition 6.1 The fractional order (n-1) drive systems and one response system follow combination synchronization among n-chaotic systems if there exists n constant matrixes called scaling matrixes $A_1, A_2, ..., A_n \in R^n$ with $A_n \neq 0$ such that

$\lim_{n\to+\infty} \|A_1 x_1 + A_2 x_2 + ... - A_n x_n\| = 0$, where $\| \, . \, \|$ represents the matrix norm.

It is noted that if $A_1 \neq 0, A_2 = A_3 = ... = A_{n-1} = 0,\ A_n = I$, then this problem is reduced to the projective synchronization, where I is an $n \times n$ identity matrix. If the scaling matrix A_1 is considered as a function, then synchronization problem is reduced into function projective synchronization problem. Again if $A_1 = A_2 = ... = A_{n-1} = 0$, then the problem becomes a chaos control problem.

6.3 Systems' descriptions

6.3.1 Fractional order Newton-Leipnik systems

The fractional order Newton-Leipnik system (Sheu et al. (2008)) was first studied in the year 2008, which is given by

$$\frac{d^q x_1}{dt^q} = -a_1 x_1 + x_2 + 10 x_2 x_3$$

$$\frac{d^q x_2}{dt^q} = -x_1 - 0.4 x_2 + 5 x_1 x_3 \tag{6.5}$$

$$\frac{d^q x_3}{dt^q} = a_2 x_3 - 5 x_1 x_2,$$

when a_2 takes the values outside of this interval. If a_2 becomes close to zero, the system shows uninteresting dynamic and if $a_2 \geq 0.8$, the given system becomes explosive i.e., the solution of this system diverges to infinity for any initial condition other than the critical points.

For the parameters' values $a_1 = 0.4$, $a_2 = 0.175$ and the initial condition $(0.19,\ 0,\ -0.18)$, the Newton-Leipnik system shows chaotic behaviour at $q = 0.95$ which is depicted through Fig. 6.1(a).

6.3.2 Fractional order Liu system

The Liu system (Liu et al. (2009)) was studied in the year 2009, which was later extended to fractional order Liu system by Gejji and Bhalekar (2010) as

$$\frac{d^q y_1}{dt^q} = -b_1 y_1 - b_4 y_2^2$$

$$\frac{d^q y_2}{dt^q} = b_2 y_1 - b_5 y_1 y_3 \tag{6.6}$$

$$\frac{d^q y_3}{dt^q} = -b_3 y_3 + b_6 y_1 y_2 .$$

The phase portrait of the system is described through Fig. 6.1(b), which shows that the system exhibits chaos at the lowest fractional order $q = 0.92$ for the values of parameters $b_1 = 1,\ b_2 = 2.5,\ b_3 = 5,\ b_4 = 1,\ b_5 = 4,\ b_6 = 4$ and initial condition $(0.2,\ 0,\ 0.5)$.

6.3.3 Fractional order Lotka-Voltra system

The fractional order Lotka-Voltra system (Petras (2011)) is given as

$$\frac{d^q z_1}{dt^q} = c_1 z_1 - c_2 z_1 z_2 + c_5 z_1^2 - c_6 z_3 z_1^2$$

$$\frac{d^q z_2}{dt^q} = -c_3 z_2 + c_4 z_1 z_2 \tag{6.7}$$

$$\frac{d^q z_3}{dt^q} = -c_7 z_3 + c_6 z_3 z_1^2 .$$

The chaotic attractor of the system is described through Fig. 6.1(c) at $q = 0.95$ for the values of the parameters $c_1 = c_2 = c_3 = c_4 = 1,\ c_5 = 2,\ c_6 = 2.7,\ c_7 = 3$ and initial conditions $(1,\ 1.4,\ 1)$.

6.3.4 Fractional order Chen system

The fractional order Chen system (Li and Chen (2004a)) is considered as

$$\frac{d^q w_1}{dt^q} = d_1 (w_2 - w_1)$$

$$\frac{d^q w_2}{dt^q} = (d_3 - d_1) w_1 - w_1 w_3 + d_3 w_2 \tag{6.8}$$

$$\frac{d^q w_3}{dt^q} = w_1 w_2 - d_2 w_3 .$$

Fig. 6.1(d) shows the chaotic attractors of the system at the fractional order $q = 0.95$ for the parameters' values $d_1 = 35,\ d_2 = 3,\ d_3 = 28$ and the initial condition $(1, 1.4, 1)$.

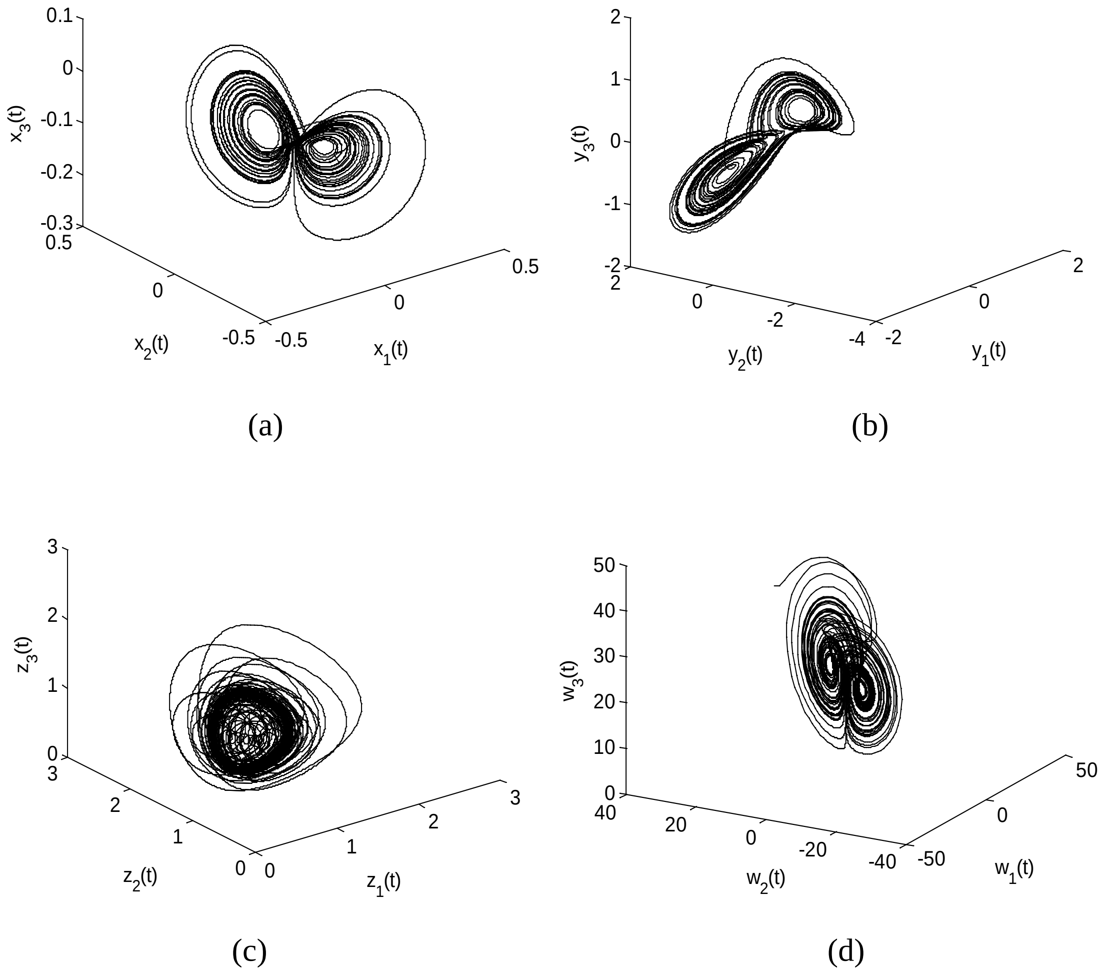


Fig. 6.1 Phase portraits of fractional order (a) Newton-Leipnik system; (b) Liu system; (c) Lotka-Voltra system; (d) Chen system for the order of derivative $q = 0.95$.

6.4 Synchronization of fractional order Newton-Leipnik, Lotka-Voltra and Liu systems

For the study of synchronization between three fractional order chaotic systems, two systems Newton-Leipnik (6.5) and Lotka-Voltra (6.7) are considered as drive system-I and drive system-II and third system Liu system is considered as response system. The response system with the control functions u_1, u_2, u_3 is defined as

$$\frac{d^q y_1}{dt^q} = -b_1 y_1 - b_4 y_2^2 + u_1$$

$$\frac{d^q y_2}{dt^q} = b_2 y_1 - b_5 y_1 y_3 + u_2 \tag{6.9}$$

$$\frac{d^q y_3}{dt^q} = -b_3 y_3 + b_6 y_1 y_2 + u_3 .$$

Defining the error functions as $e_1 = y_1 - z_1 - x_1$, $e_2 = y_2 - z_2 - x_2$, $e_3 = y_3 - z_3 - x_3$, we obtain the error system as

$$\frac{d^q e_1}{dt^q} = -b_1 e_1 - b_4 e_2 + \phi_1 + u_1$$

$$\frac{d^q e_2}{dt^q} = b_2 e_2 - b_5 e_1 e_3 - b_5 e_3 (z_1 + x_1) - b_5 e_1 (z_3 + x_3) + \phi_2 + u_2 \tag{6.10}$$

$$\frac{d^q e_3}{dt^q} = -b_3 e_3 + b_6 e_1 e_2 + b_6 e_2 (z_1 + x_1) + b_6 e_1 (z_2 + x_2) + \phi_3 + u_3 ,$$

where

$$\phi_1 = -b_1 (z_1 + x_1) - b_4 (z_2 + x_2) - c_1 z_1 + c_2 z_1 z_2 - c_5 z_1^2 + c_6 z_3 z_1^2 + a_1 x_1 - x_2 - 10 x_2 x_3$$

$$\phi_2 = -b_5 (z_1 + x_1)(z_3 + x_3) + b_2 (z_2 + x_2) + c_3 z_2 - c_4 z_1 z_2 + x_1 + 0.4 x_2 - 5 x_1 x_3$$

$$\phi_3 = b_6 (z_1 + x_1)(z_2 + x_2) - b_3 (z_3 + x_3) + c_7 z_3 - c_6 z_3 z_1^2 - a_2 x_3 + 5 x_1 x_2 .$$

Now the control functions would be designed using backstepping approach for combination synchronization of three fractional order chaotic systems.

Theorem 6.1 If the control functions are chosen as

$$u_1 = -\phi_1$$

$$u_2 = b_5 v_1 (z_3 + x_3) - b_2 v_2 + b_4 v_1 - v_2 - \phi_2$$

$$u_3 = -b_6 v_1 (z_2 + x_2) + (b_5 - b_6) v_2 (z_1 + x_1) + (b_5 - b_6) v_1 v_2 - \phi_3 ,$$

where $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$, then the drive systems I & II will be combination synchronized with response system.

Proof: To achieve the results, let us use the active backstepping procedure through following three steps.

Step-I: Defining $v_1 = e_1$, we get

$$\frac{d^q v_1}{dt^q} = \frac{d^q e_1}{dt^q} = -b_1 v_1 - b_4 e_2 + \phi_1 + u_1, \tag{6.11}$$

where $e_2 = \alpha_1(v_1)$ is regarded as an virtual controller. For designing $\alpha_1(v_1)$ to stabilize v_1 - subsystem, choosing the Lyapunov function V_1 as

$$V_1 = \frac{1}{2} v_1^2 .$$

The q -th order fractional derivative of V_1 w. r. to t is

$$\frac{d^q V_1}{dt^q} = \frac{1}{2} \frac{d^q v_1^2}{dt}$$

$$\leq v_1 \frac{d^q v_1}{dt^q} \text{ (using Lemma-1.1)}$$

$$i.e., \quad \leq v_1[-b_1 v_1 - b_4 \alpha_1(v_1) + \phi_1 + u_1].$$

If we take $\alpha_1(v_1) = 0$ and $u_1 = -\phi_1$, then $\frac{d^q V_1}{dt^q} \leq -b_1 v_1^2 < 0$, which implies that sub-system (6.11) is asymptotically stable. Since virtual control function $\alpha_1(v_1)$ is an estimate function, defining the error variable v_2 between e_2 and $\alpha_1(v_1)$ as

$$v_2 = e_2 - \alpha_1(v_1) .$$

We obtain the following (v_1, v_2) -subsystem as

$$\frac{d^q v_1}{dt^q} = -b_1 v_1 - b_4 v_2$$

$$\frac{d^q v_2}{dt^q} = b_2 v_2 - b_5 v_1 e_3 - b_5 e_3 (z_1 + x_1) - b_5 v_1 (z_3 + x_3) + \phi_2 + u_2, \tag{6.12}$$

where $e_3 = \alpha_2(v_1, v_2)$ is regarded as an virtual controller.

Step II: To stabilize (v_1, v_2) - subsystem (6.12), taking Lyapunov function as

$$V_2 = V_1 + \frac{1}{2}v_2^2 = \frac{1}{2}v_1^2 + \frac{1}{2}v_2^2.$$

The $q-th$ order fractional derivative of V_2 w. r. to t is

$$\frac{d^q V_2}{dt^q} = \frac{1}{2}\frac{d^q v_1^2}{dt^q} + \frac{1}{2}\frac{d^q v_2^2}{dt^q}$$

$$\le v_1 \frac{d^q v_1}{dt^q} + v_2 \frac{d^q v_2}{dt^q}, \text{ (from Lemma 1.1)}$$

i.e., $\le -b_1 v_1^2 - b_4 v_1 v_2 + v_2 [b_2 v_2 - b_5 v_1 \alpha_2(v_1, v_2) - b_5 \alpha_2(v_1, v_2)(z_1 + x_1) - b_5 v_1 (z_3 + x_3) + \phi_2 + u_2]$

If $\alpha_2(v_1, v_2) = 0$ and $u_2 = b_5 v_1 (z_3 + x_3) - b_2 v_2 + b_4 v_1 - v_2 - \phi_2$, then

$\frac{d^q V_2}{dt^q} \le -b_1 v_1^2 - v_2^2 < 0$, which implies that $(v_1,\ v_2)$-subsystem (6.12) is asymptotically stable.

Again defining the error variable as

$$v_3 = e_3 - \alpha_2(v_1, v_2),$$

the (v_1, v_2, v_3) - subsystem becomes

$$\frac{d^q v_1}{dt^q} = -b_1 v_1 - b_4 v_2$$

$$\frac{d^q v_2}{dt^q} = -v_2 + b_4 v_1 - b_5 v_1 v_3 - b_5 v_3 (z_1 + x_1) \tag{6.13}$$

$$\frac{d^q v_3}{dt^q} = -b_3 v_3 + b_6 v_1 v_2 + b_6 v_2 (z_1 + x_1) + b_6 v_1 (z_2 + x_2) + \phi_3 + u_3.$$

Step III: To stabilize the (v_1, v_2, v_3) - subsystem (6.13), choosing the following Lyapunov function V_3 as

$$V_3 = V_2 + \frac{1}{2}v_3^2 = \frac{1}{2}v_1^2 + \frac{1}{2}v_2^2 + \frac{1}{2}v_3^2.$$

The fractional derivative of V_3 is

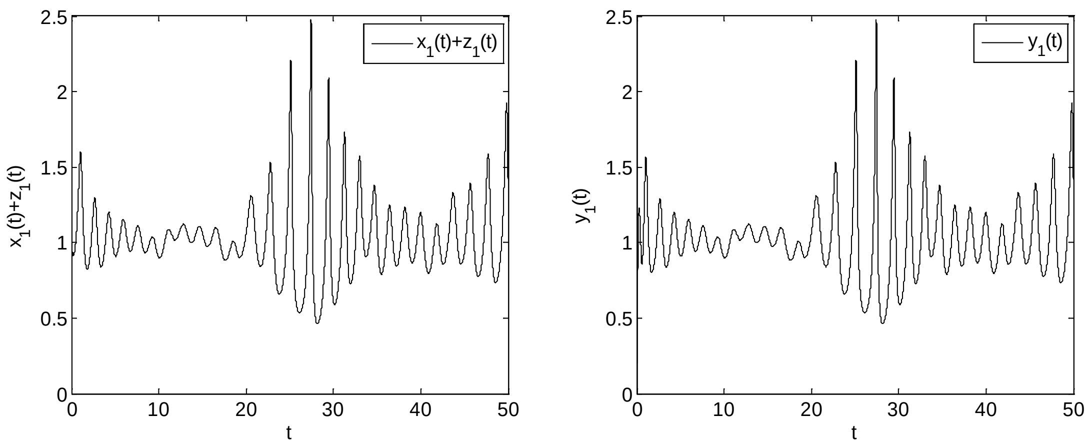
$$\frac{d^q V_3}{dt^q} = \frac{1}{2}\frac{d^q v_1^2}{dt^q} + \frac{1}{2}\frac{d^q v_2^2}{dt^q} + \frac{1}{2}\frac{d^q v_3^2}{dt^q}$$

$$\leq v_1 \frac{d^q v_1}{dt^q} + v_2 \frac{d^q v_2}{dt^q} + v_3 \frac{d^q v_3}{dt^q}, \text{ (from Lemma1.1)}$$

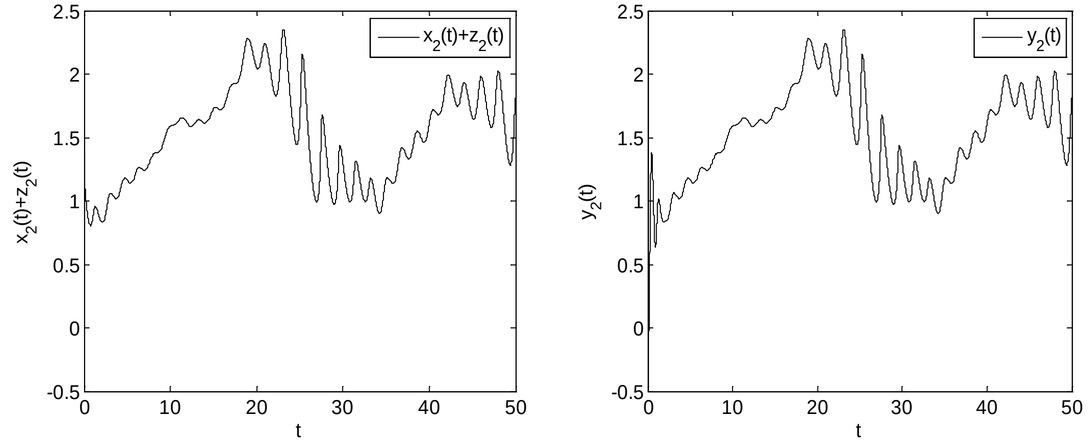
i.e., $\leq -b_1 v_1^2 - v_2^2 - b_5 v_1 v_2 v_3 - b_5 v_2 v_3 (z_1 + x_1) + v_3[-b_3 v_3 + b_6 v_1 v_2 + b_6 v_2 (z_1 + x_1)$

$+ b_6 v_1 (z_2 + x_2) + \phi_3 + u_3].$

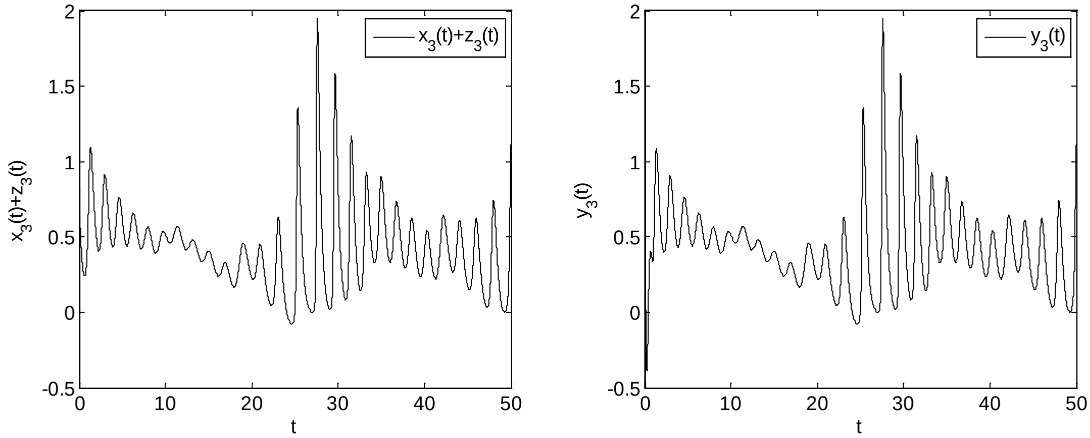
Taking $u_3 = -b_6 v_1 (z_2 + x_2) + (b_5 - b_6) v_2 (z_1 + x_1) + (b_5 - b_6) v_1 v_2 - \phi_3$, we obtain $\frac{d^q V_3}{dt^q} \leq -b_1 v_1^2 - v_2^2 - b_3 v_3^2 < 0$. Thus the system is asymptotically stable. Thus for $v_1 = e_1, v_2 = e_2 - \alpha_1(v_1) = e_2$ and $v_3 = e_3 - \alpha_2(v_1, v_2) = e_3$, the error systems $e_i \to 0,\ i = 1, 2, 3,$ which helps to obtain combination synchronization among the three considered fractional order systems.



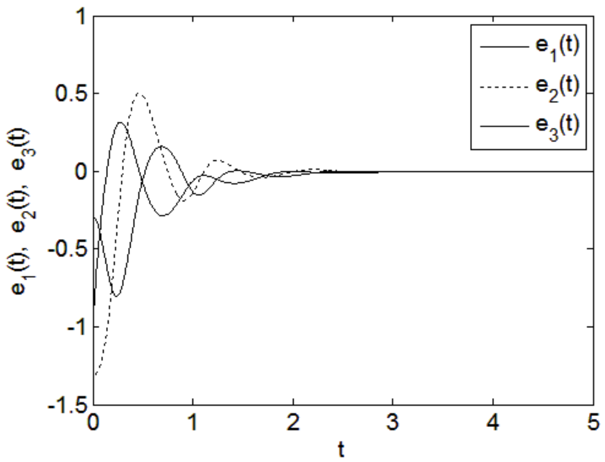
(a)



(b)



(c)



(d)

Fig. 6.2 Combination synchronization among three fractional order chaotic systems (6.5), (6.6) and (6.7) for fractional order $q = 0.95$: (a) between $x_1(t) + z_1(t)$ and $y_1(t)$; (b) between $x_2(t) + z_2(t)$ and $y_2(t)$; (c) between $x_3(t) + z_3(t)$ and $y_3(t)$; (d) the evaluation of error functions $e_1(t), e_2(t)$ and $e_3(t)$.

6.5 Synchronization of fractional order Newton-Leipnik, Liu, Lotka-Voltra systems and Chen system

In this section to synchronize four fractional order chaotic systems, we consider fractional order Newton-Leipnik system (6.5), fractional order Liu system (6.6) and fractional order Lotka-Voltra system (6.7) as the drive systems I, II and III respectively. The fractional order Chen system (6.8) is taken as response system with control function u_1', u_2', u_3' as

$$\frac{d^q w_1}{dt^q} = d_1(w_2 - w_1) + u_1'$$

$$\frac{d^q w_2}{dt^q} = (d_3 - d_1)w_1 - w_1 w_3 + d_3 w_2 + u_2' \tag{6.14}$$

$$\frac{d^q w_3}{dt^q} = w_1 w_2 - d_2 w_3 + u_3'.$$

Defining error functions as $e_1 = w_1 - z_1 - y_1 - x_1$, $e_2 = w_2 - z_2 - y_2 - x_2$, $e_3 = w_3 - z_3 - y_3 - x_3$, we obtain the error system as

$$\frac{d^q e_1}{dt^q} = d_1(e_2 - e_1) + \psi_1 + u_1'$$

$$\frac{d^q e_2}{dt^q} = (d_3 - d_1)e_1 - e_1 e_3 - e_3(z_1 + y_1 + x_1) - e_1(z_3 + y_3 + x_3) + d_3 e_2 + \psi_2 + u_2' \tag{6.15}$$

$$\frac{d^q e_3}{dt^q} = e_1 e_2 + e_2(z_1 + y_1 + x_1) + e_1(z_2 + y_2 + x_2) - d_2 e_3 + \psi_3 + u_3',$$

where

$$\psi_1 = d_1 z_2 + (d_1 + b_4 y_2)y_2 + (d_1 - 1)x_2 - (d_1 + c_1)z_1 - (d_1 - b_1)y_1 - (d_1 - a_1)x_1 + c_2 z_1 z_2 - c_5 z_1^2$$
$$+ c_6 z_3 z_1^2 - 10 x_2 x_3$$

$$\psi_2 = (d_3 - d_1)(z_1 + y_1 + x_1) - (z_1 + y_1 + x_1)(z_3 + y_3 + x_3) + d_3(z_2 + y_2 + x_2) + c_3 z_2 - c_4 z_1 z_2$$

$$- b_2 y_2 + b_5 y_1 y_3 + x_1 + 0.4 x_2 - 5 x_1 x_3$$

$$\psi_3 = (z_1 + y_1 + x_1)(z_2 + y_2 + x_2) - d_2(z_3 + y_3 + x_3) + c_7 z_3 - c_6 z_3 z_1^2 + b_3 y_3 - b_6 y_1 y_2 - a_2 x_3$$

$$+ 5 x_1 x_2 .$$

Next control functions u_1', u_2' and u_3' would be designed using backstepping approach to achieve combination synchronization among four fractional order chaotic systems.

Theorem 6.2 If the control functions are chosen as

$$u_1' = -\psi_1$$

$$u_2' = -d_3 v_2 + v_1(z_3 + y_3 + x_3) - d_2 v_1 - v_2 - \psi_2$$

$$u_3' = -v_1(z_2 + y_2 + x_2) - \psi_3 ,$$

where $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$, then the drive systems (6.5), (6.6) and (6.7) will be combination synchronized with response system (6.14).

Proof: For synchronization, backstepping procedure is used through following steps.

Step-I: Considering $v_1 = e_1$,

$$\frac{d^q v_1}{dt^q} = \frac{d^q e_1}{dt^q} = d_1(e_2 - e_1) + \psi_1 + u_1' , \tag{6.16}$$

where $e_2 = \alpha_1(v_1)$ is regarded as an virtual controller. To stabilize v_1-subsystem, let us define the Lyapunov function V_1 as

$$V_1 = \frac{1}{2} v_1^2 ,$$

whose fractional derivative is

$$\frac{d^q V_1}{dt^q} = \frac{1}{2}\frac{d^q v_1^2}{dt^q} \le v_1 \frac{d^q v_1}{dt^q}$$

i.e., $\leq v_1[d_1(\alpha_1(v_1)-v_1)+\psi_1+u_1']$.

Taking $\alpha_1(v_1)=0$ and $u_1'=-\psi_1$, we get $\frac{d^qV_1}{dt^q}\leq -d_1v_1^2<0$, which implies that v_1-subsystem (6.16) is asymptotically stable. For the virtual control function $\alpha_1(v_1)$, a variable v_2 between e_2 and $\alpha_1(v_1)$ is defined as

$$v_2=e_2-\alpha_1(v_1).$$

Then (v_1,v_2)-subsystem is obtained as

$$\frac{d^qv_1}{dt^q}=d_1(v_2-v_1)$$

$$\frac{d^qv_2}{dt^q}=(d_3-d_1)v_1-v_1e_3-e_3(z_1+y_1+x_1)-v_1(z_3+y_3+x_3)+d_3v_2+\psi_2+u_2'. \tag{6.17}$$

Let us consider $v_3=\alpha_2(v_1,v_2)$ is a virtual controller.

Step II: In this step to stabilize (v_1,v_2)-subsystem (6.17), let us define the Lyapunov function V_2 as

$$V_2=V_1+\frac{1}{2}v_2^2=\frac{1}{2}v_1^2+\frac{1}{2}v_2^2.$$

Now

$$\frac{d^qV_3}{dt^q}=\frac{1}{2}\frac{d^qv_1^2}{dt^q}+\frac{1}{2}\frac{d^qv_2^2}{dt^q}$$

$$\leq v_1\frac{d^qv_1}{dt^q}+v_2\frac{d^qv_2}{dt^q},$$

i.e., $\leq d_1v_1v_2-d_1v_1^2+v_2[(d_3-d_1)v_1-v_1\alpha_2(v_1,v_2)-\alpha_2(v_1,v_2)(z_1+y_1+x_1)$

$-v_1(z_3+y_3+x_3)+d_3v_2+\psi_2+u_2']$.

Taking $\alpha_2(v_1, v_2) = 0$ and $u_2' = -d_3 v_2 + v_1(z_3 + y_3 + x_3) - d_2 v_1 - v_2 - \psi_2$, we get

$\frac{d^q V_2}{dt^q} \le -d_1 v_1^2 - v_2^2 < 0$, which makes subsystem (6.17) asymptotically stable.

Considering $v_3 = e_3 - \alpha_2(v_1, v_2)$, the (v_1, v_2, v_3) -subsystem is obtained as

$$\frac{d^q v_1}{dt^q} = d_1(v_2 - v_1)$$

$$\frac{d^q v_2}{dt^q} = -v_2 - d_1 v_1 - v_1 v_3 - v_3(z_1 + y_1 + x_1) \tag{6.18}$$

$$\frac{d^q v_3}{dt^q} = v_1 v_2 + v_2(z_1 + y_1 + x_1) + v_1(z_2 + y_2 + x_2) - d_2 v_3 + \psi_3 + u_3'.$$

Step III: In order to stabilize (v_1, v_2, v_3)- subsystem (6.18), choosing the Lyapunov function as

$$V_3 = V_2 + \frac{1}{2}v_3^2 = \frac{1}{2}v_1^2 + \frac{1}{2}v_2^2 + \frac{1}{2}v_3^2,$$

we get

$$\frac{d^q V_3}{dt^q} = \frac{1}{2}\frac{d^q v_1^2}{dt^q} + \frac{1}{2}\frac{d^q v_2^2}{dt^q} + \frac{1}{2}\frac{d^q v_3^2}{dt^q}$$

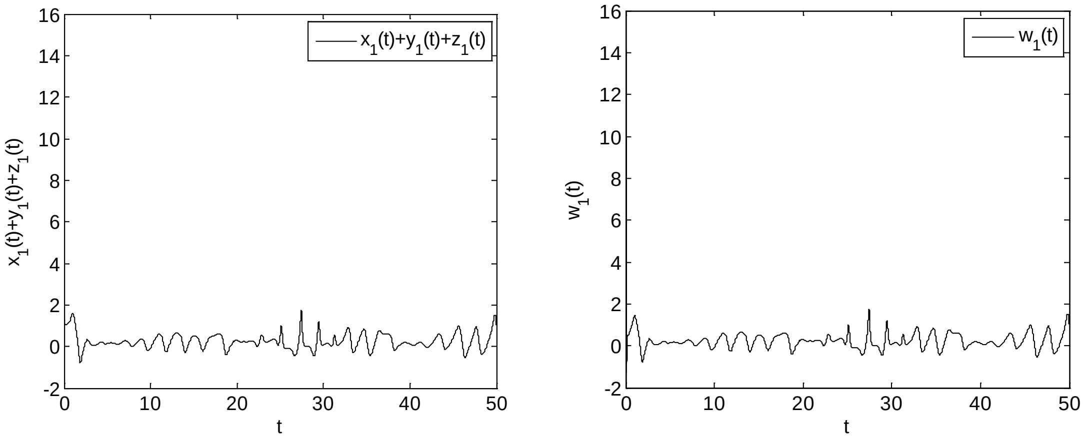
$$\le v_1 \frac{d^q v_1}{dt^q} + v_2 \frac{d^q v_2}{dt^q} + v_3 \frac{d^q v_3}{dt^q},$$

i.e., $\le v_1[d_1(v_2 - v_1)] + v_2[-v_2 - d_1 v_1 - v_1 v_3 - v_3(z_1 + y_1 + x_1)] + v_3[v_1 v_2 + v_2(z_1 + y_1 + x_1)$

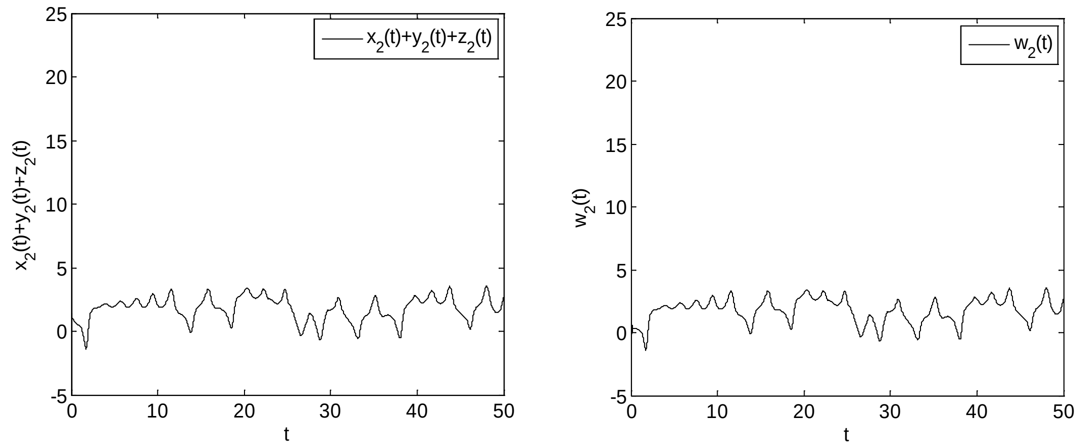
$$+ v_1(z_2 + y_2 + x_2) - d_2 v_3 + \psi_3 + u_3']$$

$$= -d_1 v_1^2 - v_2^2 + v_3[v_1(z_2 + y_2 + x_2) - d_2 v_3 + \psi_3 + u_3'] \ .$$

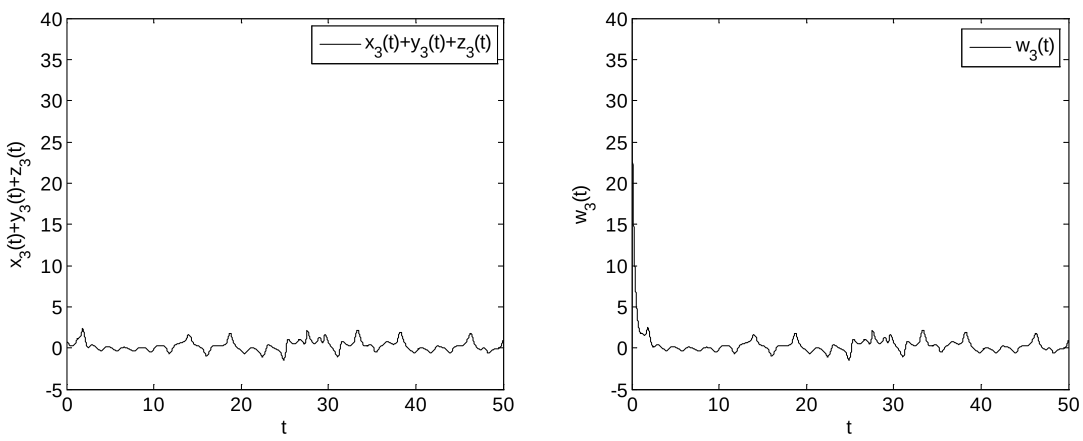
If $u_3' = -v_1(z_2 + y_2 + x_2) - \psi_3$, $\frac{d^q V_3}{dt^q} \le -d_1 v_1^2 - v_2^2 - d_2 v_3^2 < 0$ negative definite. In view of $v_1 = e_1$, $v_2 = e_2 - \alpha_1(v_1) = e_2$, $v_3 = e_2 - \alpha_2(v_1, v_2) = e_3$, the error states $e_i \to 0,\ i = 1, 2, 3$ will converge to zero after a finite period of time, and thus the combination synchronization among four fractional order chaotic systems will be achieved.



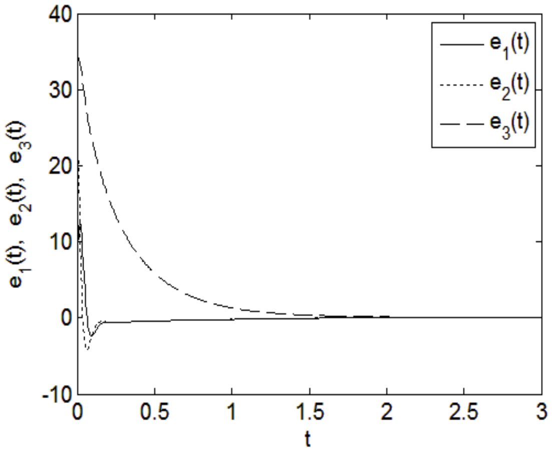
(a)



(b)

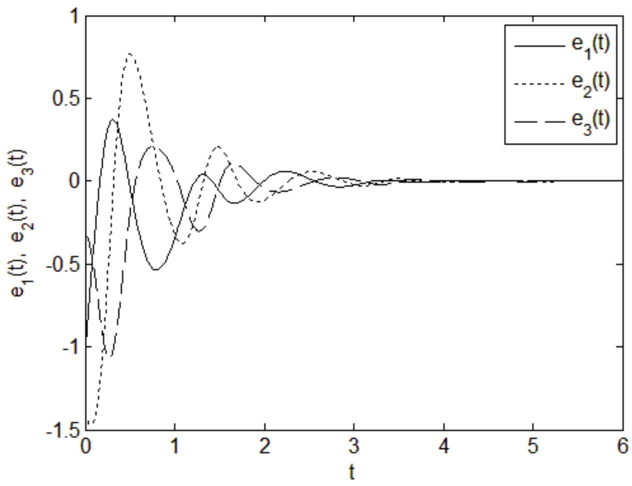


(c)

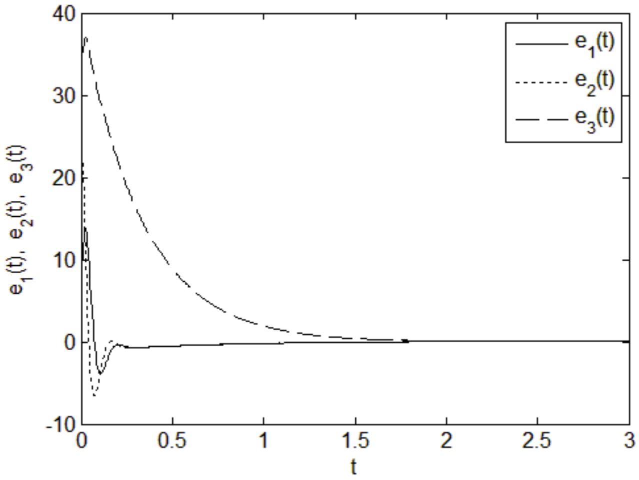


(d)

Fig. 6.3 Combination synchronization among four fractional order chaotic systems (6.5), (6.6), (6.7) and (6.8) for fractional order $q = 0.95$: (a) between $x_1(t) + y_1(t) + z_1(t)$ and $w_1(t)$; (b) between $x_2(t) + y_2(t) + z_2(t)$ and $w_2(t)$; (c) between $x_3(t) + y_3(t) + z_3(t)$ and $w_3(t)$; (d) the evaluation of error functions $e_1(t), e_2(t)$ and $e_3(t)$.



(a)



(b)

Fig. 6.4. The evaluation of error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$ at $q=1$: (a) for three systems ; (b) for four systems.

6.6 Numerical simulation and results

During synchronization the earlier values of the parameters and initial conditions of systems are considered. The time step size is taken as 0.005. Fig. 6.2 shows the synchronization among three fractional order systems are achieved through active backstepping approach at $q=0.95$. Figs. 6.2(a), 6.2(b) and 6.2(c) depict the time response of the state trajectories $x_i(t)+z_i(t)$ and $y_i(t)$, where $i=1, 2, 3$ represent the drive systems (6.5), (6.7) and response system (6.9) respectively. The error states are displayed through Fig. 6.2(d). The synchronization among four fractional order systems are achieved through Fig. 6.3 using the same method at $q=0.95$. Figs. 6.3(a), 6.3(b) and 6.3(c) show the time response of the states $x_i(t)+y_i(t)+z_i(t)$ and $w_i(t)$, where $i=1, 2, 3$ represent the drive systems (6.5), (6.6), (6.7) and the response system (6.8). The error states for this case are described through Fig 6.3(d). It is noticed that it takes less time for synchronization among four systems (Fig. 6.3(d)) compared to that of three systems (Fig. 6.2(d)) for our considered systems in both fractional order as well as integer order case (Fig. 6.4).

6.7 Conclusion

In the present study, the combination synchronization among a number of fractional order drive and response systems is successfully demonstrated using backstepping method. For validation, the combination synchronization of three and four systems are considered taking two systems and three systems as drive system respectively, while one system as response system, which clearly exhibit that the applied method is effective and convenient to achieve global synchronization of a number of non-identical fractional order chaotic systems. It is worth mentioning that this scientific contribution of combination synchronization among the fractional order chaotic systems will be significant towards the further study of nonlinear dynamics among the research community involved in the area of modelling of fractional order dynamical systems.