Chapter 5

Chaos control and function projective synchronization of fractional order systems through back stepping method

5.1 Introduction

The chaos control and chaotic dynamics of fractional order systems are important topics to the researchers, especially chaos control and chaos synchronization problems in nonlinear dynamical system. Chaos synchronization is a process wherein two identical or non-identical chaotic systems adjust with a given property of their motion to a common behavior due to a coupling. Thus it appears to be structurally stable. These ideas have motivated to the researchers to construct mathematical model for synchronization of two different fractional order chaotic systems. The advantages of fractional order systems allow greater flexibilities in the models. The fractional order differential operator is non local but integer order differential operator is a local operator in the sense that fractional order differential operator takes into account the fact that the future state not only depends upon the present state but also upon all of the history of its previous states. For this realistic property, fractional calculus which was in earlier stage considered as mathematical curiosity now becomes the object of extensive development of fractional order partial differential equations for the purpose of engineering applications.

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Function projective synchronization is the generalization of projective synchronization which is one of the important synchronization methods that has been widely investigated to obtain faster communication with its proportional feature. In Function projective synchronization the drive and response systems are synchronized with a scaling function. It is obvious that the presence of the scaling function in function projective synchronization additionally improves the security in communication. Many types of function projective synchronizations are focused only on integer-order chaotic systems (Chen and Li (2007), Ojoniyi (2014), Du et al. (2008), Du et al. (2010), Zhang and Li (2012)), whereas there are few results about the function projective synchronization for the fractional order chaotic systems (Zhou and Zhu (2011), Zhou and Cao (2010), Agrawal and Das (2014)).

In the present chapter the dynamical behaviour and chaos control of fractional order T-system have been studied. It is found that the chaotic attractor exists in the fractional-order T- system. Fractional Routh–Hurwitz conditions are used to analyze the stability conditions in the fractional-order T-system and the conditions for linear feedback control have been obtained for controlling chaos in the considered system. The backstepping method is used for function projective synchronization of fractional order T-system and Lorenz system.

5.2 System's description and its stability

5.2.1 Fractional order T-system

The chaotic dynamical T-system is introduced by Gheorghe Tigan (Tigan (2004), Tigan (2005)), which is described by

$$\frac{dx}{dt} = a(y - x)$$

$$\frac{dy}{dt} = (c-a)x - axz \qquad (5.1)$$

$$\frac{dz}{dt} = -bz + xy \, ,$$

where a, b, c are the parameters and x, y, z are state variables of the system. When the values of the parameters are taken as $(a, b, c) = (2.1, 0.6, 30)$, and the maximal Lyapunov exponent of system is 0.37, the T-system exhibits chaos.

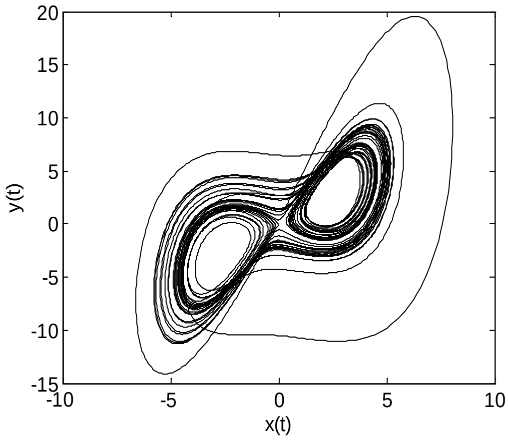
The fractional order T-system is given by

$$\frac{d^q x}{dt^q} = a(y-x)$$

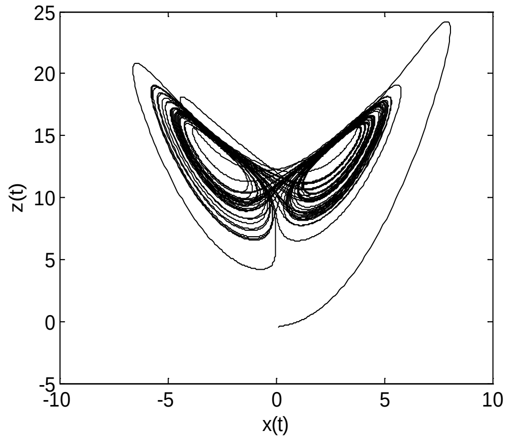
$$\frac{d^q y}{dt^q} = (c-a)x - axz \qquad (5.2)$$

$$\frac{d^q z}{dt^q} = -bz + xy \, ,$$

where $0 < q < 1$. Figs. 5.1(a)-(d) depicts that the T-system shows the regular chaotic behaviour at fractional order $q = 0.95$ for the values of parameters $(a, b, c) = (2.1, 0.6, 30)$ and the initial condition $(0.1, 1.2, -0.5)$. For those parameters' values and initial condition, the trajectories of the T-system are depicted through Figs. 5.2(a)-(d) to show the stable nature of the trajectories at $q = 0.94$.



(a)



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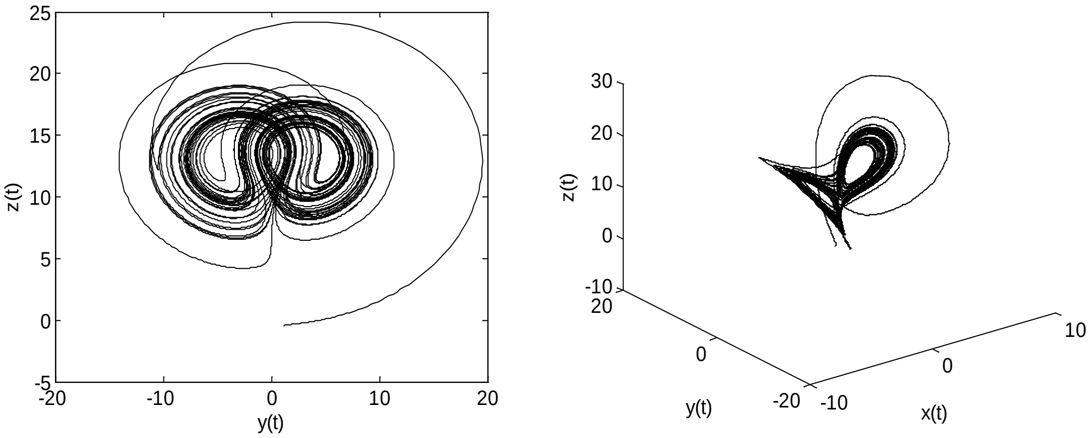
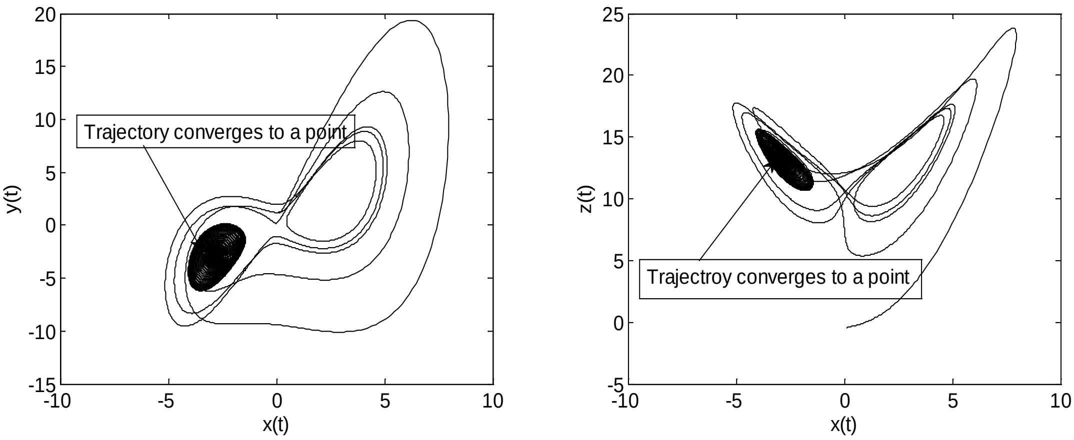
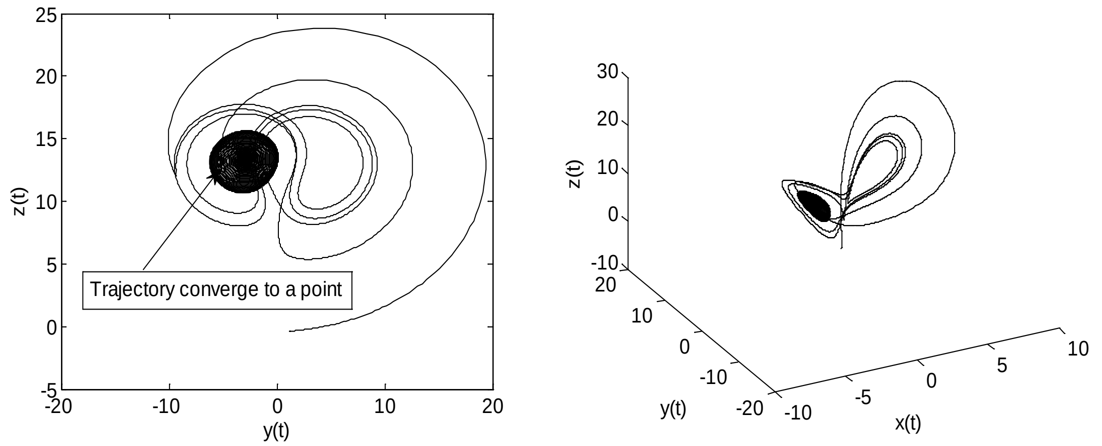


(c) (d)

Fig. 5.1 Phase portraits of fractional order T-system for fractional order $q = 0.95$.



(a) (b)



(c) (d)

Fig. 5.2 Phase portraits of fractional order T-system for fractional order $q = 0.94$.

5.2.2 Equilibrium points and stability

To find equilibrium points of the system (5.2), we have

$$a(y-x)=0$$

$$(c-a)x-axz=0 \tag{5.3}$$

$$-bz+xy=0,$$

and the equilibrium points are obtained as

$E_1=(0,\,0,\,0)$, $E_2=(2.8234,\,2.8234,\,13.2857)$ and $E_3=(-2.8234,\,-2.8234,13.2857)$.

The Jacobian matrix of the system (5.2) at the equilibrium point $\overline{E}(\overline{x},\,\overline{y},\,\overline{z})$ is given as

$$J(\overline{E})=\begin{bmatrix} -a & a & 0 \\ c-a-a\overline{z} & 0 & -a\overline{x} \\ \overline{y} & \overline{x} & -b \end{bmatrix}. \tag{5.4}$$

The characteristic polynomial of above Jacobian matrix is

$$P(\lambda)=\lambda^3+2.7\lambda^2+(2.1\overline{x}^2+4.41\overline{z}-57.33)\lambda+4.41\overline{x}^2+4.41\overline{x}\,\overline{y}+2.646\overline{z}-35.154. \tag{5.5}$$

At the equilibrium point $E_1=(0,\,0,\,0)$, the equation (5.5) becomes

$$P(\lambda)=\lambda^3+2.7\lambda^2-57.33\lambda-35.154. \tag{5.6}$$

The eigenvalues of the equation (5.6) are $\lambda_1=-8.7761,\,\lambda_2=6.6761,\,\lambda_3=-0.60$. It is seen that the equilibrium points E_1 is a saddle point of index 1 and from definition 1.3 it is unstable for $0<q<1$.

At the equilibrium point $E_2=(2.8234,\,2.8234,\,13.2857)$, the equation (5.5) becomes

$$P(\lambda)=\lambda^3+2.7\lambda^2+18.0002\lambda+70.3093. \tag{5.7}$$

The eigenvalues of equation (5.7) are $\lambda_1=-3.4294,\,\lambda_{2,3}=0.3647\pm4.5132i$ the equilibrium point E_2 is the saddle points of index 2 the (definition 1.4). So E_2 is stable for $q < 0.949$. Similarly the equilibrium point $E_3 = (-2.8234, -2.8234, 13.2857)$ is stable for $q < 0.949$.

5.2.3 Control of chaos

The fractional order T-system with controller is given by

$$\frac{d^q x}{dt^q} = a(y - x) - k_1(x - \overline{x})$$

$$\frac{d^q y}{dt^q} = (c - a)x - axz - k_2(y - \overline{y}) \tag{5.8}$$

$$\frac{d^q z}{dt^q} = -bz + xy - k_3(z - \overline{z}),$$

where k_1, k_2, k_3 are control parameters and $(\overline{x}, \overline{y}, \overline{z})$ is the equilibrium points. Jacobian matrix of the system (5.8) at equilibrium point $\overline{E}(\overline{x}, \overline{y}, \overline{z})$ is given as

$$J(\overline{E}) = \begin{bmatrix} -a - k_1 & a & 0 \\ c - a - a\overline{z} & -k_2 & -a\overline{x} \\ \overline{y} & \overline{x} & -b - k_3 \end{bmatrix}.$$

At $a = 2.1, b = 0.6, c = 30$, we get the corresponding characteristic polynomial as

$$\begin{aligned} P(\lambda) = \lambda^3 &+ (k_1 + k_2 + k_3 + 2.7)\lambda^2 + [2.1\overline{x}^2 + 4.41\overline{z} + (k_3 + 3/5)(k_1 + k_2 + 2.1) \\ &+ k_2(k_1 + 2.1) - 58.59]\lambda + [4.41\overline{x}\,\overline{y} + 2.1\overline{x}^2(k_1 + k_2 + 2.1) - 2.1k_2\overline{x}^2 \\ &+ (k_3 + 3/5)(4.41\overline{z} + k_2(k_1 + 2.1) - 58.59)]. \end{aligned} \tag{5.9}$$

In view of fractional order Routh-Hurwitz conditions, we get

$$a_1 = k_1 + k_2 + k_3 + 2.7$$

$$a_2 = 2.1\overline{x}^2 + 4.41\overline{z} + (k_3 + 3/5)(k_1 + k_2 + 2.1) + k_2(k_1 + 2.1) - 58.59 \tag{5.10}$$

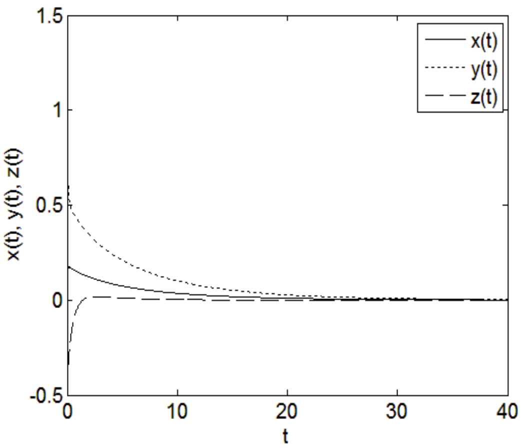
$$a_3 = 4.41\overline{x}\,\overline{y} + 2.1\overline{x}^2(k_1 + k_2 + 2.1) - 2.1k_2\overline{x}^2 + (k_3 + 3/5)(4.41\overline{z} + k_2(k_1 + 2.1) - 58.59).$$

5.2.4 Stabilizing the point E_1

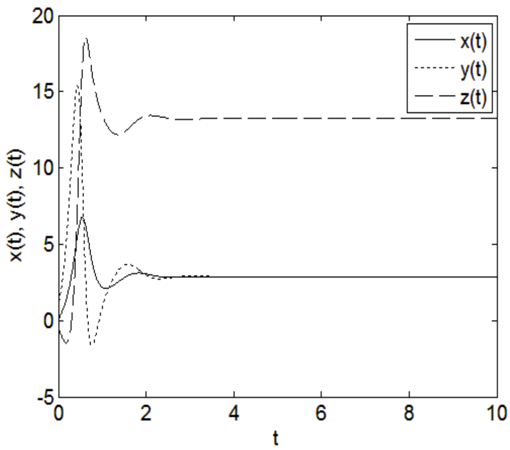
Putting the value of E_1 in equation (5.10) and taking $k_1 = 4, k_2 = 10$ and $k_3 = 1$, we have $D(P) > 0$, $a_1 > 0, a_3 > 0, a_1 a_2 - a_3 > 0$. All eigenvalues of the equation (5.9) are real and negative. So the system (5.8) is locally asymptotically stable for $0 < q < 1$, which is shown through Fig. 5.3(a).

5.2.5 Stabilizing the points E_2 and E_3

Substituting the value of E_2 in equation (5.10) and considering $k_1 = 1, k_2 = 5$ and $k_3 = -3/5$, we have $D(P) > 0$, $a_1 > 0, a_3 > 0, a_1 a_2 - a_3 > 0$. Hence all the eigenvalues of equation (5.9) are real and negative. So the system (5.8) is locally asymptotically stable for $0 < q < 1$ (Fig. 5.3(b)). Similarly for $k_1 = 1, k_2 = 6$ and $k_3 = -2$, the system (5.8) is locally asymptotically stable for $0 < q < 1$ at the equilibrium point E_3 (Fig. 5.3(c)).



(a)



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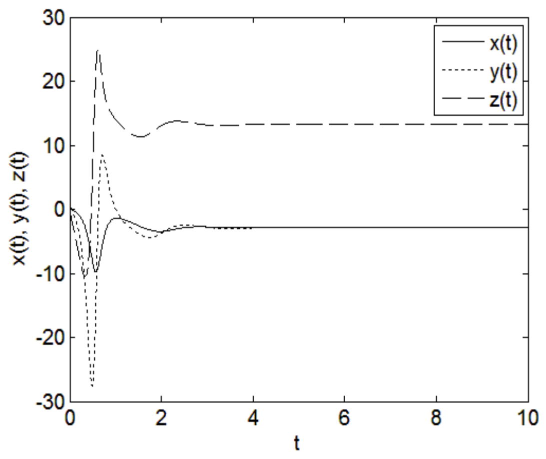

(c)

Fig. 5.3 Plots of $x(t), y(t), z(t)$ of the controlled system (5.8): (a) at equilibrium point E_1; (b) at the equilibrium point E_2; (c) at the equilibrium point E_3.

5.3 Function projective synchronization between fractional order non identical T-system and Lorenz system

The fractional order Lorenz system (Wu and Shen (2009), Grigorenko (2003)) is given by

$$\frac{d^q x}{dt^q} = \alpha(y - x)$$

$$\frac{d^q y}{dt^q} = x(\gamma - z) - y \tag{5.11}$$

$$\frac{d^q z}{dt^q} = xy - \beta z\,,$$

where α is the Prandtl number, γ is the Rayleigh number and β is the size of the region approximated by the system. The phase portraits of Lorenz system is shown through Fig. 5.4 for the parameters' values $\alpha = 10, \beta = 8/3$, $\gamma = 28$ and initial condition = (0.1, 0.1, 0.1) at $q = 0.993$.

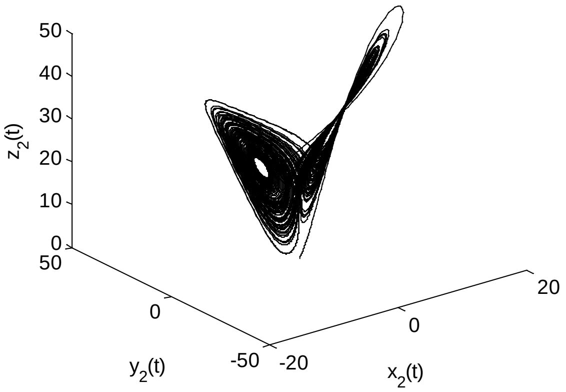

Fig. 5.4 Phase portrait of the Lorenz system for the order of derivative $q = 0.993$.

In this section in order to do function projective synchronization between fractional order T-system and Lorenz system,

we consider fractional order T-system as the master system as

$$\begin{aligned} \frac{d^q x_1}{dt^q} &= a(y_1 - x_1) \\ \frac{d^q y_1}{dt^q} &= (c-a)x_1 - a x_1 z_1 \\ \frac{d^q z_1}{dt^q} &= -b z_1 + x_1 y_1 \end{aligned} \tag{5.12}$$

and the fractional Lorenz system is taken as slave system as

$$\begin{aligned} \frac{d^q x_2}{dt^q} &= \alpha(y_2 - x_2) + u_1(t) \\ \frac{d^q y_2}{dt^q} &= x_2(\gamma - z_2) - y_2 + u_2(t) \\ \frac{d^q z_2}{dt^q} &= x_2 y_2 - \beta z_2 + u_3(t). \end{aligned} \tag{5.13}$$

Taking the error states as $e_1 = x_2 - m_1 x_1$, $e_2 = y_2 - m_2 y_1$ and $e_3 = z_2 - m_3 z_1$, where m_1, m_2 and m_3 are scaling functions, the error dynamical system becomes

$$\frac{d^q e_1}{dt^q} = \alpha(e_2 - e_1) + \psi_1 + u_1(t)$$

$$\frac{d^q e_2}{dt^q} = e_1(\gamma - e_3) - e_1 m_3 z_1 - e_3 m_1 x_1 - e_2 + \psi_2 + u_2(t) \qquad (5.14)$$

$$\frac{d^q e_3}{dt^q} = e_1 e_2 + e_1 m_2 y_1 + e_2 m_1 x_1 - \beta e_3 + \psi_3 + u_3(t),$$

where $\psi_1 = \alpha m_2 y_1 - \alpha m_1 x_1 - m_1 a(y_1 - x_1)$

$$\psi_2 = m_1 \gamma x_1 - m_1 m_3 z_1 x_1 - m_2[y_1 + (c-a)x_1 - a x_1 z_1]$$

$$\psi_3 = m_1 m_2 x_1 y_1 - m_3[(\beta - b) z_1 + x_1 y_1]$$

Step I: Let us consider $w_1 = e_1$, then fractional derivative of w_1 is

$$\frac{d^q w_1}{dt^q} = \frac{d^q e_1}{dt^q} = \alpha(e_2 - w_1) + \psi_1 + u_1(t), \qquad (5.15)$$

where $e_2 = \alpha_1(w_1)$ is regarded as virtual controller. To stabilize w_1 - subsystem, we define the Lyapunov function V_1 as

$$V_1 = \frac{1}{2} w_1^2 .$$

whose q-th time derivative w.r.to t is

$$\frac{d^q V_1}{dt^q} = \frac{1}{2}\frac{d^q w_1^2}{dt^q} \le w_1 \frac{d^q w_1}{dt^q} \text{ (using Lemma 1.1)}$$

i.e., $\le w_1[\alpha(\alpha_1(w_1) - w_1) + \psi_1 + u_1(t)]$.

Taking $\alpha_1(w_1) = w_1 - \dfrac{w_1}{\alpha}$ and $u_1(t) = -\psi_1$, we get $\dfrac{d^q V_1}{dt^q} \le -w_1^2 < 0$ is negative definite, which implies that w_1-subsystem (5.15) is asymptotically stable. For the virtual control function $\alpha_1(w_1)$, we define a variable w_2 between e_2 and $\alpha_1(w_1)$ as

$w_2 = e_2 - \alpha_1(w_1)$.

Then, (w_1, w_2)-subsystem is obtained as

$$\frac{d^q w_1}{dt^q} = \alpha w_2 - w_1,$$

$$\frac{d^q w_2}{dt^q} = -\alpha w_2 + w_1(\gamma - e_3) - w_1 m_3 z_1 - e_3 m_1 x_1 + \psi_2 + u_2(t), \tag{5.16}$$

where $e_3 = \alpha_2(w_1, w_2)$ may be considered as virtual controller.

Step II: In this step to stabilize (w_1, w_2)-subsystem (5.16), let us define the Lyapunov function V_2 as

$$V_2 = V_1 + \frac{1}{2}w_2^2 = \frac{1}{2}w_1^2 + \frac{1}{2}w_2^2.$$

Now

$$\frac{d^q V_2}{dt^q} = \frac{1}{2}\frac{d^q w_1^2}{dt^q} + \frac{1}{2}\frac{d^q w_2^2}{dt^q}$$

$$\le w_1 \frac{d^q w_1}{dt^q} + w_2 \frac{d^q w_2}{dt^q},$$

i.e., $\le -w_1^2 - \alpha w_2^2 + w_2[\alpha w_1 + w_1(\gamma - \alpha_2(w_1, w_2)) - w_1 m_3 z_1 - \alpha_2(w_1, w_2) m_1 x_1 + \psi_2 + u_2(t)]$

If $\alpha_2(w_1, w_2) = 0$ and $u_2(t) = -\psi_2 + w_1 m_3 z_1 - w_1\gamma - \alpha w_1$, then $\dfrac{d^q V_2}{dt^q} \le -w_1^2 - \alpha w_2^2 < 0$ makes subsystem (5.16) asymptotically stable.

Considering $w_3 = e_3 - \alpha_2(w_1, w_2)$, we get the following (w_1, w_2, w_3)-subsystem as

$$\frac{d^q w_1}{dt^q} = \alpha w_2 - w_1$$

$$\frac{d^q w_2}{dt^q} = -\alpha(w_1 + w_2) - w_1 w_3 - w_3 m_1 x_1 \qquad (5.17)$$

$$\frac{d^q w_3}{dt^q} = w_1(w_2 + w_1 - \frac{w_1}{\alpha}) + w_1 m_2 y_1 + (w_2 + w_1 - \frac{w_1}{\alpha}) m_1 x_1 - \beta w_3 + \psi_3 + u_3(t).$$

Step III: In order to stabilize (w_1, w_2, w_3)- subsystem (5.17), choosing the Lyapunov function as

$$V_3 = V_2 + \frac{1}{2} w_3^2 = \frac{1}{2} w_1^2 + \frac{1}{2} w_2^2 + \frac{1}{2} w_3^2,$$

we get

$$\frac{d^q V_3}{dt^q} = \frac{1}{2}\frac{d^q w_1^2}{dt^q} + \frac{1}{2}\frac{d^q w_2^2}{dt^q} + \frac{1}{2}\frac{d^q w_3^2}{dt^q}$$

$$\le w_1 \frac{d^q w_1}{dt^q} + w_2 \frac{d^q w_2}{dt^q} + w_3 \frac{d^q w_3}{dt^q},$$

i.e., $\le -w_1^2 - \alpha w_2^2 - \beta w_3^2 + w_3[-w_1 - w_3 m_1 x_1 + w_1(w_2 + w_1 - \frac{w_1}{\alpha}) + w_1 m_2 y_1 + (w_2 + w_1 - \frac{w_1}{\alpha}) m_1 x_1 + \psi_3 + u_3(t)].$

If $u_3(t) = -\psi_3 - (w_2 + w_1 - \frac{w_1}{\alpha}) m_1 x_1 + w_3 m_1 x_1 - w_1(w_2 + w_1 - \frac{w_1}{\alpha}) - w_1 m_2 y_1 + w_1,$

$\frac{d^q V_3}{dt^q} \le -w_1^2 - \alpha w_2^2 - \beta w_3^2 < 0$ negative definite. In view of $w_1 = e_1$, $w_2 = e_2 - \alpha_1(w_1) = e_2 - e_1 + \frac{e_1}{a}$, $w_3 = e_2 - \alpha_2(w_1, w_2) = e_3$, the errors e_1, e_2 and e_3 will converge to zero after a finite period of time, and thus the function projective synchronization between fractional order T-system and Lorenz system is achieved.

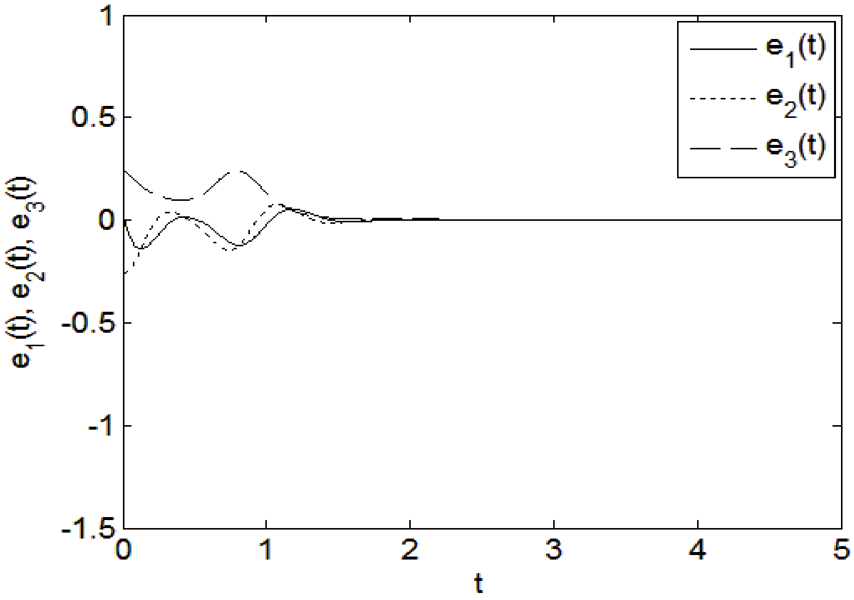

Fig. 5.5 State trajectories of error functions $e_1(t), e_2(t)$ and $e_3(t)$ of master system (5.12) and slave system (5.13) for fractional order $q = 0.993$.

5.4 Numerical simulation and results

In numerical simulation the initial values of master system and slave system are taken as $(x_1(0), y_1(0), z_1(0)) = (0.1, 1.2, -0.5)$ and $(x_2(0), y_2(0), z_2(0)) = (0.1, 0.1, 0.1)$ respectively. Thus the initial value of error systems will be $(e_1(0), e_2(0), e_3(0)) = (0, -1.1, 0.6)$. The time step is taken as 0.005.

Now we are taking the scaling function as periodic function as

$m_1 = a_{11}\cos(a_{12}x_1) + a_{13}$

$m_2 = a_{21}\cos(a_{22}y_1) + a_{23}$

$m_3 = a_{31}\cos(a_{32}z_1) + a_{33}$.

For the values of parameters $a_{11} = 0.5$, $a_{12} = 0.2$, $a_{13} = 0.1$, $a_{21} = 0.1$, $a_{22} = 0.3$, $a_{23} = 0.2$, $a_{31} = 0.2$, $a_{32} = 0.1$, $a_{33} = 0.3$, it is seen from Fig. 5.5 that the error functions asymptotically converge to zero as time become large for the order of derivatives $q = 0.993$, which shows that the master system (5.12) is synchronized with the slave system (5.13).

5.5 Conclusion

Three important goals have been achieved in this chapter. First one, the local stability of the T-system with fractional order time derivative is analyzed. Second one is employing the control function of fractional order T-system at various equilibrium points. The stability of the equilibrium points using the fractional Routh–Hurwitz criterion and the sufficient conditions for control of the fractional order T-system by linear feedback control have been studied. It is observed that the fractional order T-system can be controlled to its equilibrium points. The stability theorems of fractional-order systems guarantee that the chaos control occurs if the necessary conditions are satisfied. Simulation results show that the feedback control is easy to implement even for controlling the fractional order chaotic systems. Third one is the successful implementation of the backstepping method to achieve function projective synchronization between fractional order T-system and Lorenz system.