Chapter 4

Complex projective synchronization of fractional order complex dynamical systems using nonlinear control method

4.1 Introduction

The dynamical systems have been widely applied to describe a variety of physical phenomena, such as amplitudes of electromagnetic fields, detuned laser systems and thermal connection of liquid flow (Ning and Haken (1990), Roldan et al. (1993), Toronov and Derbov (1997), Luo and Wang (2013)) etc. The several complex dynamical systems of physical interest have been studied and proposed by the many researchers (Ning and Haken (1990), Mahmoud and Mahmoud (2010), Liu and Liu (2011), Liu and Liu (2010), Nian et al. (2010). The complex dynamical systems are used in various important fields of physics and engineering. One of the examples is laser physics where the atomic polarization and electric field amplitudes in a ring laser system of two-level atoms are complex quantities (Fowler et al. (1983), Rauh et al. (1996)). The complex chaotic systems are efficiently used in communications, and also in security of the transmitted information (Mahmoud et al. (2007a), Mahmoud et al. (2008)). In recent years, the results of many researchers have proposed about the properties of dynamics in real space and complex space (Mahmoud and Mahmoud (2010), Liu and Wang (2007), Wu et al. (2012), Liu and Zheng (2009), Liu et al. 2011)). So there are plenty of the scopes for researchers to explore the dynamical behavior in fractional order complex nonlinear systems.

The contents of this chapter have been communicated in **ASME Journal of Computational and Nonlinear Dynamics.**

In 1982, complex Lorenz system was first introduced by Fowler et al. (1982), and its dynamical properties studied by Mahmoud et al. (2007a). But the factional order complex Lorenz system was proposed and studied by Luo and Wang (2013). In 2007, the complex Lu system was proposed by Mahmoud et al. (2007b), and fractional order complex Lu system was studied by the Jiang et al. (2014).

In 2012, Wu et al. (2012) have studied complex projective synchronization in coupled chaotic complex dynamical systems. In 2013, Mahmud and Mahmud (2013) have studied complex modified projective synchronization of two chaotic complex nonlinear systems. In 2014, Liu (2014) has done complex modified hybrid projective synchronization of different dimensional fractional order complex chaos and real hyperchaos. But in fractional order systems, the complex projective synchronizations has not been studied.

In this chapter, the complex projective synchronization is studied in fractional order complex dynamical systems using nonlinear control method. The effectiveness of the method for synchronization is shown through error analysis and results are displayed graphically.

4.2 Nonlinear control method for projective synchronization of fractional order complex nonlinear systems

Let us consider two fractional order complex nonlinear systems of which first one is master (drive) system as

$$\frac{d^q x_m}{dt^q} = Ax_m + f(x_m),$$

where $x_m = x_{m_1} + jx_{m_2}$ is complex state variable with

$$\frac{d^q x_{m_1}}{dt^q} = Ax_{m_1} + f_1(x_{m_1})$$

$$\frac{d^q x_{m_2}}{dt^q} = Ax_{m_2} + f_2(x_{m_2}) \qquad (4.1)$$

and the second one is response (slave) system as

$$\frac{d^q y_s}{dt^q} = By_s + g(y_s) + u(t),$$

where $y_s = y_{s_1} + jy_{s_2}$ is complex state variable, and $u(t) = u_1^r(t) + ju_2^i(t)$ is control function with

$$\frac{d^q y_{s_1}}{dt^q} = By_{s_1} + g_1(y_{s_1}) + u_1^r(t)$$

$$\frac{d^q y_{s_2}}{dt^q} = By_{s_2} + g_2(y_{s_2}) + u_2^i(t)\,, \qquad (4.2)$$

where $u_1^r(t) = (u_1, u_3, \ldots\ldots\ldots., u_{2n-1})^T$ and $u_2^i(t) = (u_2, u_4, \ldots\ldots\ldots., u_{2n})^T$. Now we define complex error function e_r between drive and response systems for complex projective synchronization as

$$e_r = e_{r_1} + je_{r_2} = y_s - Mx_m,$$

where $M = M_1 + jM_2 = diag(\phi_1, \phi_2, \ldots\ldots\ldots\ldots, \phi_n)$, $\phi_l = m_l^r + jm_l^i$, $l = 1, 2, \ldots\ldots\ldots\ldots, n$. Then the error function will be

$$e_{r_1} = y_{s_1} - M_1 x_{m_1} + M_2 x_{m_2}$$

$$e_{r_2} = y_{s_2} - M_1 x_{m_2} + M_2 x_{m_1}$$

Case-I: If $\lim_{t\to\infty} e_{r_1} = \lim_{t\to\infty} \left\| y_{s_1} - M_1 x_{m_1} + M_2 x_{m_2} \right\| = 0$ and $\lim_{t\to\infty} e_{r_2} = \lim_{t\to\infty} \left\| y_{s_2} - M_1 x_{m_2} - M_2 x_{m_1} \right\| = 0,$ $e_{r_1} = (e_1, e_3, \ldots\ldots\ldots\ldots, e_{2n-1})^T$ and

$e_{r_2} = (e_2, e_4, ..., e_{2n})^T$, then we obtain complex projective synchronization between the systems (4.1) and (4.2).

Case-II: When $\phi_1 = \phi_2 = ... = \phi_n = j$, then the complex complete synchronization is obtained between master and response systems.

Case-III: If we take $\phi_1 = \phi_2 = ... = \phi_n =$ a real number, then the projective synchronization is obtained between complex systems (4.1) and (4.2).

The error system reduced to

$$\frac{d^q e_r}{dt^q} = By_s + g(y_s) - M(Ax_m + f(x_m)) + u(t),$$

which can be separated into real and imaginary parts as

$$\frac{d^q e_{r_1}}{dt^q} = Be_{r_1} + (B-A)M_1 x_{m_1} - (B-A)M_2 x_{m_2} + g_1(y_{s_1}) - M_1 f_1(x_{m_1}) + M_2 f_2(x_{m_2}) + u_1^r(t)$$

$$\frac{d^q e_{r_2}}{dt^q} = Be_{r_2} + (B-A)M_1 x_{m_2} + (B-A)M_2 x_{m_1} + g_2(y_{s_2}) - M_1 f_2(x_{m_2}) - M_2 f_1(x_{m_1}) + u_2^i(t). \quad (4.3)$$

Now defining the Lyapunov function as

$$V = \frac{1}{2}(e_{r_1}^T e_{r_1} + e_{r_2}^T e_{r_2})$$

and taking $q-th$ order derivative of V, we get

$$\frac{d^q V}{dt^q} = \frac{1}{2}\left(\frac{d^q}{dt^q}(e_{r_1}^T e_{r_1}) + \frac{d^q}{dt^q}(e_{r_2}^T e_{r_2})\right)$$

$$= \frac{1}{2}\left(\frac{d^q e_{r_1}^2}{dt^q} + \frac{d^q e_{r_1}^2}{dt^q}\right)$$

$$\leq e_{r_1}\frac{d^q e_{r_1}}{dt^q} + e_{r_2}\frac{d^q e_{r_2}}{dt^q}. \text{ (using lemma (1.1))}$$

Substituting the values of $\frac{d^q e_{r_1}}{dt^q}$, $\frac{d^q e_{r_2}}{dt^q}$ from equation (4.3) and choosing the appropriate control functions $u_1^r(t)$, $u_2^i(t)$ to make $q-th$ order derivative of Lyapunov function negative definite. i.e., $\frac{d^q V}{dt^q} < 0$, which helps to get the synchronization between the systems (4.1) and (4.2).

4.3 Systems' descriptions

4.3.1 Fractional order complex Lorenz system

The fractional order complex Lorenz system (Luo and Wang (2013)) is given as

$$\frac{d^q x_1}{dt^q} = a_1(x_2 - x_1)$$

$$\frac{d^q x_2}{dt^q} = a_2 x_1 - x_2 - x_1 x_3 \tag{4.4}$$

$$\frac{d^q x_3}{dt^q} = \frac{1}{2}(\bar{x}_1 x_2 + x_1 \bar{x}_2) - a_3 x_3,$$

where $0 < q < 1$ is the fractional order derivative and a_1, a_2, a_3 are system parameters, $x_1 = x_1' + jx_2'$ and $x_2 = x_3' + jx_4'$, $j = \sqrt{-1}$ are the complex state variables and $x_3 = x_5'$ is a real state variable. Now we separate into real and imaginary part of system (4.4) in the following form

$$\frac{d^q x_1'}{dt^q} = a_1(x_3' - x_1')$$

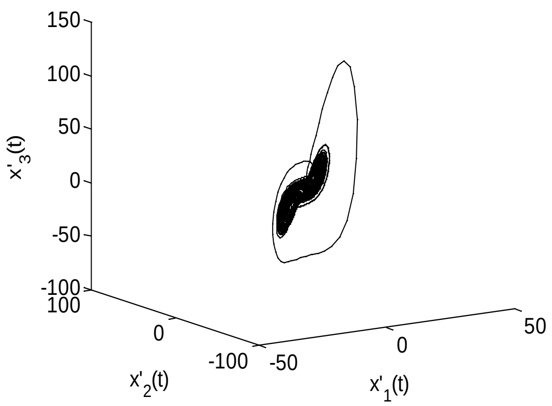
$$\frac{d^q x_2'}{dt^q} = a_1(x_4' - x_2')$$

$$\frac{d^q x_3'}{dt^q} = a_2 x_1' - x_3' - x_1' x_5' \tag{4.5}$$

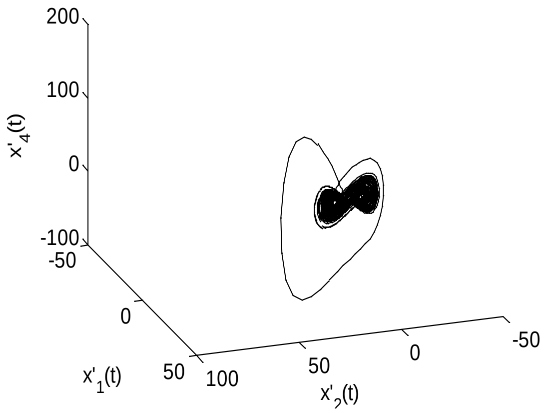
$$\frac{d^q x_4'}{dt^q} = a_2 x_2' - x_4' - x_2' x_5'$$

$$\frac{d^q x_5'}{dt^q} = x_1' x_3' + x_2' x_4' - a_3 x_5'$$

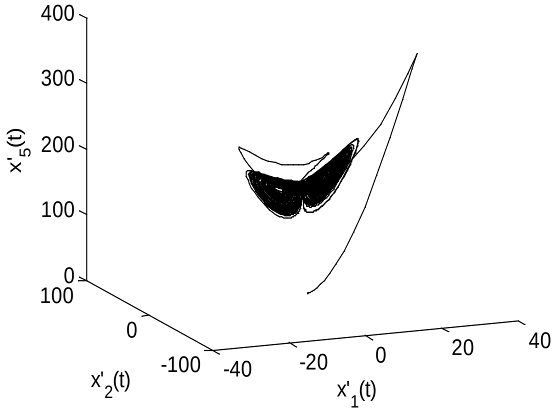
when the values of the parameters are taken as $a_1 = 10,\ a_2 = 180,$ and $a_3 = 1$, the system is chaotic and the phase portraits of the system (4.5) at $q = 0.95$ in (a) $x_1' - x_2' - x_3'$ space, (b) $x_1' - x_2' - x_4'$ space, (c) $x_1' - x_2' - x_5'$ space, (d) $x_2' - x_3' - x_4'$ space, (e) $x_2' - x_3' - x_5'$ space, (f) $x_3' - x_4' - x_5'$ space are shown through Fig. 4.1.



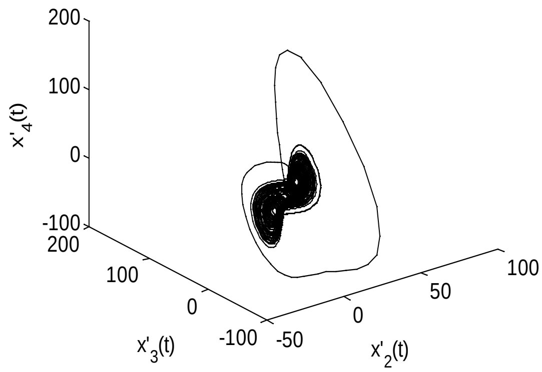
(a)



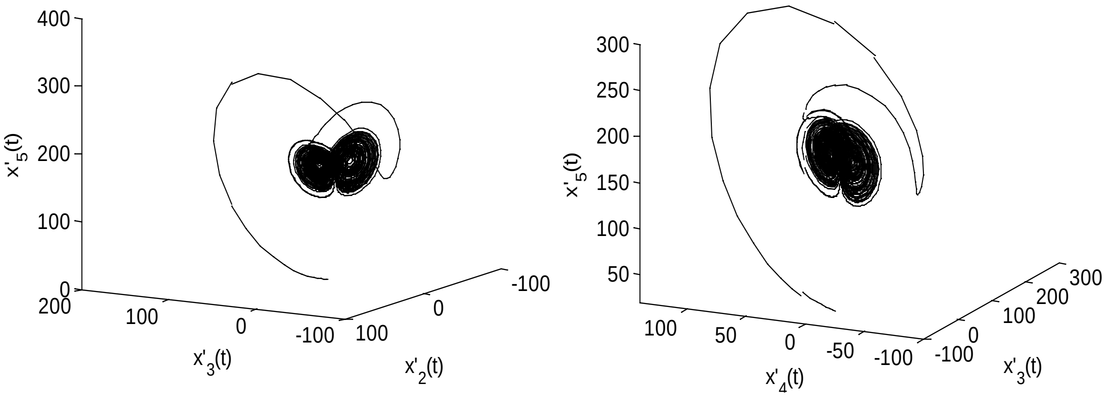
(b)



(c)



(d)



(e) (f)

Fig. 4.1 Phase portraits of the complex Lorenz system for the order of derivative $q = 0.95$.

4.3.2 Fractional order complex Lu system

The fractional order complex Lu system (Jiang et al. (2014)) is described as

$$\begin{aligned} \frac{d^q y_1}{dt^q} &= b_1(y_2 - y_1) \\ \frac{d^q y_2}{dt^q} &= -y_1 y_3 + b_2 y_2 \\ \frac{d^q y_3}{dt^q} &= \frac{1}{2}(\bar{y}_1 y_2 + y_1 \bar{y}_2) - b_3 y_3, \end{aligned} \tag{4.6}$$

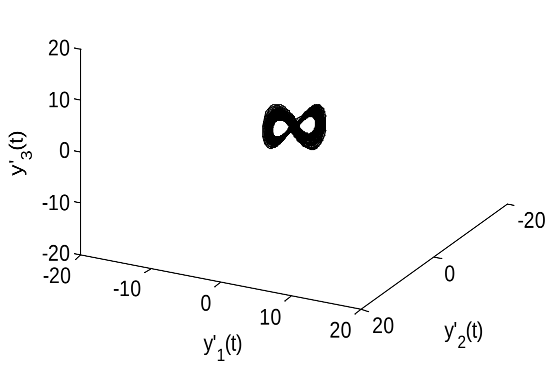
where $y_1 = y_1' + jy_2'$ and $y_2 = y_3' + jy_4'$, $j = \sqrt{-1}$ are the complex state variables and $y_3 = y_5'$ is a real state variable. System (4.6) separate into real and imaginary part as follows

$$\begin{aligned} \frac{d^q y_1'}{dt^q} &= b_1(y_3' - y_1') \\ \frac{d^q y_2'}{dt^q} &= b_1(y_4' - y_2') \\ \frac{d^q y_3'}{dt^q} &= -y_1' y_5' + b_2 y_3' \end{aligned} \tag{4.7}$$

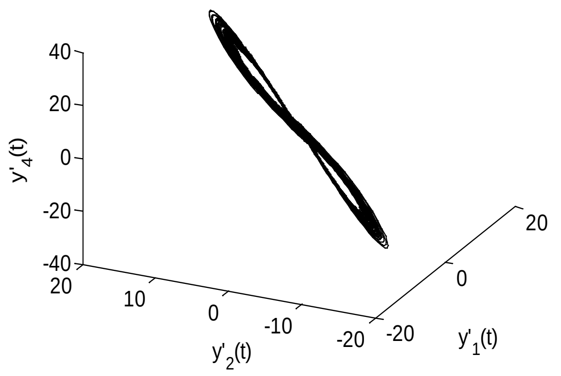
$$\frac{d^q y_4'}{dt^q} = -y_2' y_5' + b_2 y_4'$$

$$\frac{d^q y_5'}{dt^q} = y_1' y_3' + y_2' y_4' - b_3 y_5' .$$

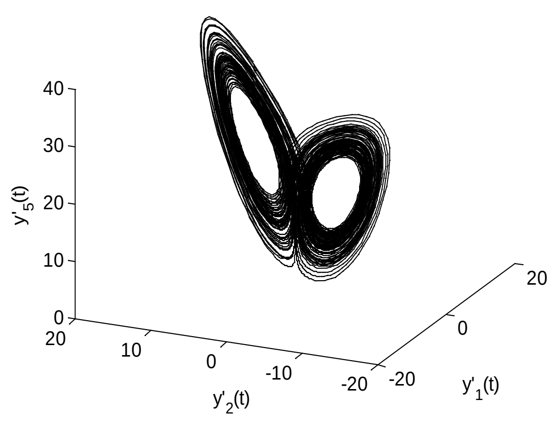
The phase portraits of (4.7) are depicted through Fig. 4.2 in (a) $y_1' - y_2' - y_3'$ space, (b) $y_1' - y_2' - y_4'$ space, (c) $y_1' - y_2' - y_5'$ space, (d) $y_2' - y_3' - y_4'$ space, (e) $y_2' - y_3' - y_5'$ space, (f) $y_3' - y_4' - y_5'$ space at $q = 0.95$ for the parameters' values $b_1 = 40,\ b_2 = 22$ and $b_3 = 5$.



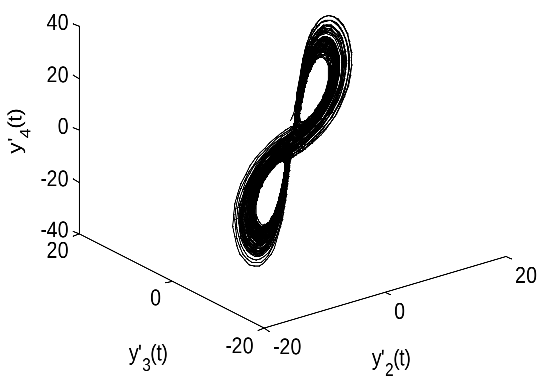
(a)



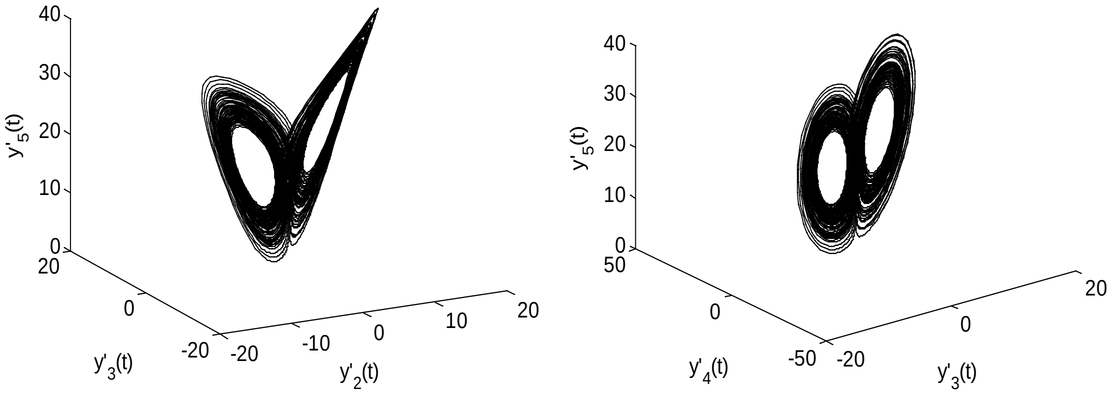
(b)



(c)



(d)



(e) (f)

Fig. 4.2 Phase portraits of the complex Lu system for the order of derivative $q = 0.95$.

4.4 Complex projective synchronization between fractional order complex Lorenz and Lu systems

Fractional order complex Lorenz system is taken as master system as

$$\begin{aligned}
\frac{d^q x_1'}{dt^q} &= a_1(x_3' - x_1') \\
\frac{d^q x_2'}{dt^q} &= a_1(x_4' - x_2') \\
\frac{d^q x_3'}{dt^q} &= a_2 x_1' - x_3' - x_1' x_5' \\
\frac{d^q x_4'}{dt^q} &= a_2 x_2' - x_4' - x_2' x_5' \\
\frac{d^q x_5'}{dt^q} &= x_1' x_3' + x_2' x_4' - a_3 x_5'
\end{aligned} \tag{4.8}$$

and fractional order complex Lu systems is taken as response system as

$$\frac{d^q y_1}{dt^q} = b_1(y_2 - y_1) + u_1(t)$$

$$\frac{d^q y_2}{dt^q} = -y_1 y_3 + b_2 y_2 + u_2(t) \tag{4.9}$$

$$\frac{d^q y_3}{dt^q} = \frac{1}{2}(\bar{y}_1 y_2 + y_1 \bar{y}_2) - b_3 y_3 + u_3(t),$$

where $u_1(t) = u_1'(t) + ju_2'(t)$, $u_2(t) = u_3'(t) + ju_4'(t)$ and $u_3(t) = u_5'(t)$ are control functions.

The equation (4.9) reduces to

$$\frac{d^q y_1'}{dt^q} = b_1(y_3' - y_1') + u_1'(t)$$

$$\frac{d^q y_2'}{dt^q} = b_1(y_4' - y_2') + u_2'(t)$$

$$\frac{d^q y_3'}{dt^q} = -y_1' y_5' + b_2 y_3' + u_3'(t) \tag{4.10}$$

$$\frac{d^q y_4'}{dt^q} = -y_2' y_5' + b_2 y_4' + u_4'(t)$$

$$\frac{d^q y_5'}{dt^q} = y_1' y_3' + y_2' y_4' - b_3 y_5' + u_5'(t) .$$

We define the error function between drive and response systems as $e_1 = e_1' + je_2' = y_1 - M_1 x_1$, $e_2 = e_3' + je_4' = y_2 - M_2 x_2$ and $e_3 = e_5' = y_3 - M_3 x_3$, where M is scaling matrix and taken as $M = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} = \begin{bmatrix} m_1 + jm_2 & 0 & 0 \\ 0 & m_3 + jm_4 & 0 \\ 0 & 0 & m_5 \end{bmatrix}$.

Separating real and imaginary parts of above error function as

$$e_1' = y_1' - m_1 x_1' + m_2 x_2'$$

$$e_2' = y_2' - m_1 x_2' - m_2 x_1'$$

$$e_3' = y_3' - m_3 x_3' + m_4 x_4' \tag{4.11}$$

$$e_4' = y_4' - m_3 x_4' - m_4 x_3'$$

$$e_5' = y_5' - m_5 x_5',$$

we obtain the error dynamical system as

$$\frac{d^q e_1'}{dt^q} = b_1(e_3' - e_1') - m_1[a_1 x_3' - (a_1 - b_1)x_1'] + m_2[a_1 x_4' - (a_1 - b_1)x_2'] + m_3 b_1 x_3 - m_4 b_1 x_4 + u_1'(t)$$

$$\frac{d^q e_2'}{dt^q} = b_1(e_4' - e_2') - m_1[a_1 x_4' - (a_1 - b_1)x_2'] - m_2[a_1 x_3' - (a_1 - b_1)x_1'] + m_3 b_1 x_4 + m_4 b_1 x_3 + u_2'(t)$$

$$\frac{d^q e_3'}{dt^q} = b_2 e_3' - m_3[a_2 x_1' - (1 + b_2)x_3' - x_1' x_5'] + m_4[a_2 x_2' - (1 + b_2)x_4' - x_2' x_5'] - y_1' y_5' + u_3'(t) \tag{4.12}$$

$$\frac{d^q e_4'}{dt^q} = b_2 e_4' - m_3[a_2 x_2' - (1 + b_2)x_4' - x_2' x_5'] - m_4[a_2 x_1' - (1 + b_2)x_3' - x_1' x_5'] - y_2' y_5' + u_4'(t)$$

$$\frac{d^q e_5'}{dt^q} = -b_3 e_5' - m_5[x_1' x_3' + x_2' x_4' - (a_3 - b_3)x_5'] + y_1' y_3' + y_2' y_4' + u_5'(t).$$

In order to determine the controller, let us the Lyapunov function $V(e)$ as

$$V(e) = \frac{1}{2}(e_1'^2 + e_2'^2 + e_3'^2 + e_4'^2 + e_5'^2), \tag{4.13}$$

whose $q-th$ order derivative w. r. to t is

$$\frac{d^q V(e)}{dt^q} = \frac{1}{2}\left(\frac{d^q e_1'^2}{dt^q} + \frac{d^q e_2'^2}{dt^q} + \frac{d^q e_3'^2}{dt^q} + \frac{d^q e_4'^2}{dt^q} + \frac{d^q e_5'^2}{dt^q}\right),$$

$$\leq \left(e_1' \frac{d^q e_1'}{dt^q} + e_2' \frac{d^q e_2'}{dt^q} + e_3' \frac{d^q e_3'}{dt^q} + e_4' \frac{d^q e_4'}{dt^q} + e_5' \frac{d^q e_5'}{dt^q}\right). \text{ (using lemma 1.1)} \tag{4.14}$$

After putting the values of $\frac{d^q e_1'}{dt^q}$, $\frac{d^q e_2'}{dt^q}$, $\frac{d^q e_3'}{dt^q}$, $\frac{d^q e_4'}{dt^q}$ and $\frac{d^q e_5'}{dt^q}$ from equation (4.12) in equation (4.14) and

Considering $M = \begin{bmatrix} m_1 + jm_2 & 0 & 0 \\ 0 & m_3 + jm_4 & 0 \\ 0 & 0 & m_5 \end{bmatrix} = \begin{bmatrix} 1+j3 & 0 & 0 \\ 0 & 2+j5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$,

so that controllers are

$$u_1'(t) = -b_1 e_3' + [a_1 x_3' - (a_1 - b_1) x_1'] - 3[a_1 x_4' - (a_1 - b_1) x_2'] - m_3 b_1 x_3 + m_4 b_1 x_4$$

$$u_2'(t) = -b_1 e_4' + [a_1 x_4' - (a_1 - b_1) x_2'] + 3[a_1 x_3' - (a_1 - b_1) x_1'] - m_3 b_1 x_4 - m_4 b_1 x_3$$

$$u_3'(t) = -e_3' - b_2 e_3' + 2[a_2 x_1' - (1+b_2) x_3' - x_1' x_5'] - 5[a_2 x_2' - (1+b_2) x_4' - x_2' x_5'] + y_1' y_5' \quad (4.15)$$

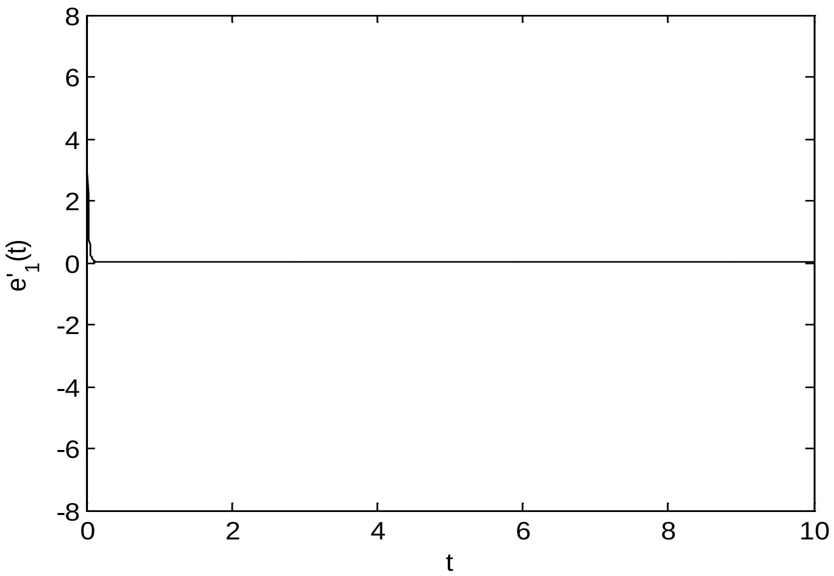
$$u_4'(t) = -e_4' - b_2 e_4' + 2[a_2 x_2' - (1+b_2) x_4' - x_2' x_5'] + 5[a_2 x_1' - (1+b_2) x_3' - x_1' x_5'] + y_2' y_5'$$

$$u_5'(t) = 3[x_1' x_3' + x_2' x_4' - (a_3 - b_3) x_5'] - y_1' y_3' - y_2' y_4',$$

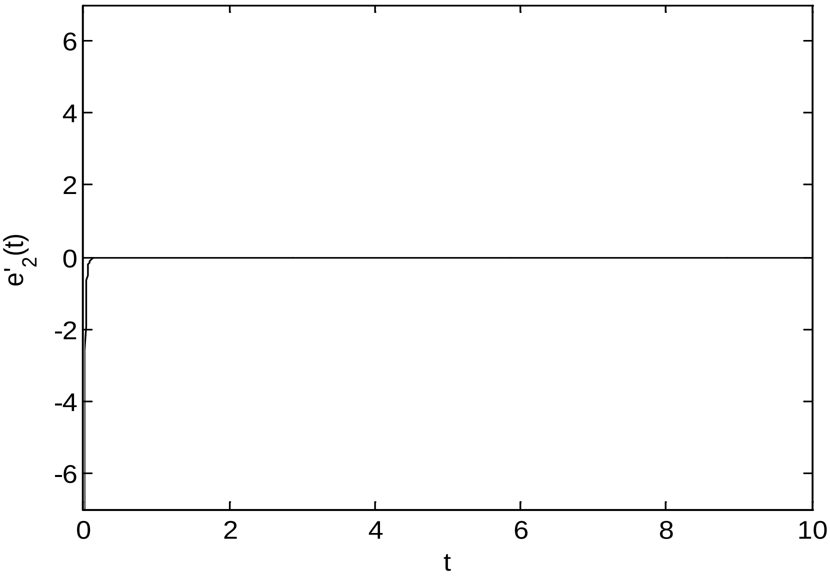
we get the $q-th$ order derivative of the Lyapunov function $V(e)$ as

$\dfrac{d^q V(e)}{dt^q} \le -b_1 e_1'^2 - b_1 e_2'^2 - e_3'^2 - e_4'^2 - b_3 e_5'^2 < 0$, i.e., negative definite.

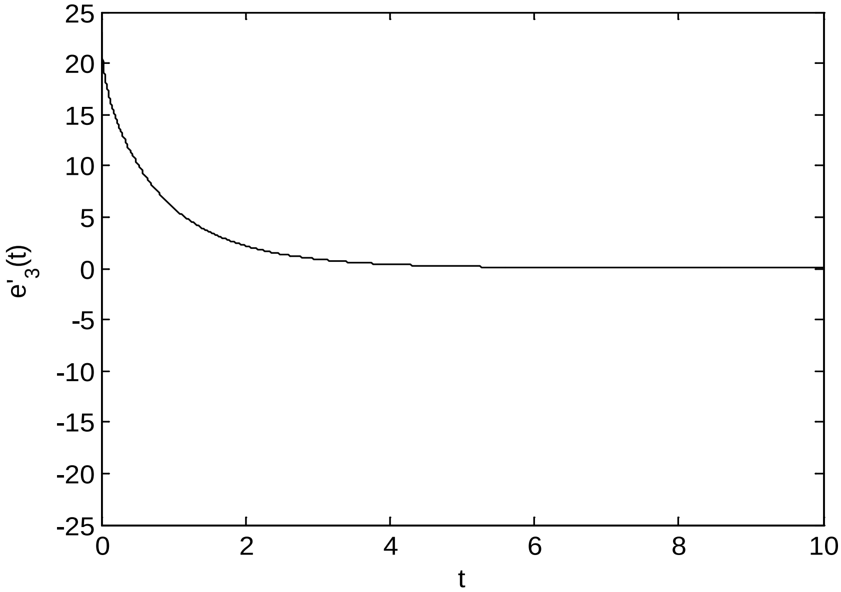
Hence $\lim_{t\to\infty} \|e(t)\| = 0$ and thus the complex projective synchronization between fractional order complex Lorenz and Lu systems is achieved.



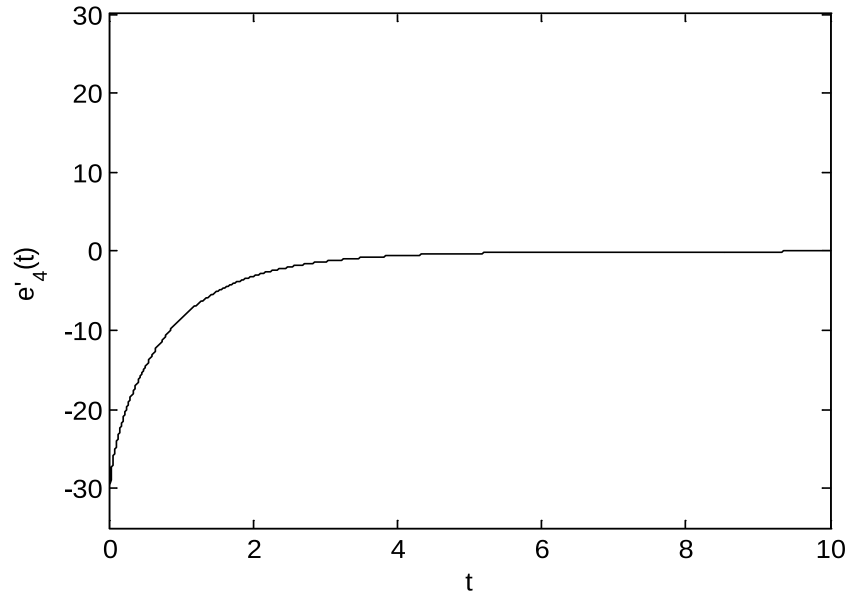
(a)



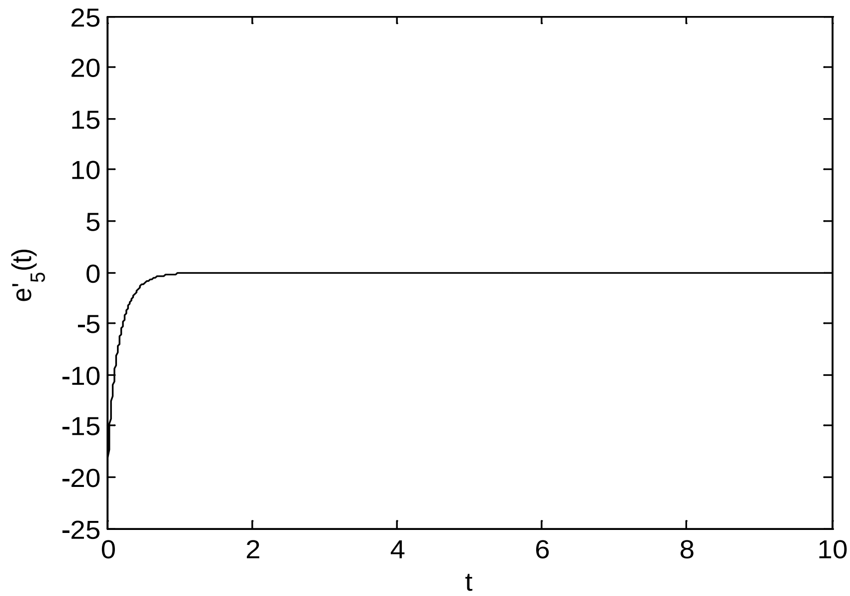
(b)



(c)



(d)



(e)

Fig. 4.3 The evolution of the error functions at $q = 0.95$: (a) evaluation of $e_1'(t)$; (b) evaluation of $e_2'(t)$; (c) evaluation of $e_3'(t)$; (d) evaluation of $e_4'(t)$; (e) evaluation of $e_5'(t)$.

4.5 Results and discussion

In this section, the initial conditions of fractional order complex Lorenz system is taken as $x_1(0) = 2 + j3,\ x_2(0) = 5 + j6$ and $x_3(0) = 9$. The initial condition of fractional order complex Lu system is taken as $y_1(0) = 1 + j2,\ y_2(0) = 3 + j4$ and $y_3(0) = 5$. Choosing the scaling matrix M as $M = diag(1 + j3,\ 2 + j5,\ 3)$, we get the initial condition of error function as $e_1 = 8 - j7,\ e_2 = 23 - j33$ and $e_3 = -22$. For these values of parameters the complex projective synchronization are shown through Fig. 4.3 at $q = 0.95$, where the error functions converge to zero as time becomes large. Thus it can be concluded that the applied method is very much effective to synchronize the complex chaotic systems even for fractional order case.

4.6 Conclusion

The present chapter investigates the complex projective synchronization between two fractional order complex systems viz., Lorenz and Lu systems. Based on Lyapunov stability theory, the synchronization of the systems is done with proper design of control functions. The graphical representation of the numerical results with error states tending to zero as time becomes large clearly exhibits that the nonlinear control method is very much reliable and effective even for synchronization of fractional order complex systems.