Chapter 3

Phase and anti-phase synchronizations of fractional order hyperchaotic systems with uncertainties and external disturbances using nonlinear active control method

3.1 Introduction

Chaos theory has many useful applications in many areas of engineering such as digital communication, secure communication, power electric and power quality, biological systems, chemical reactions analysis and design and information processing. Synchronization and anti-synchronization are interesting area of research in chaos theory (Fujisaka and Yamada (1983), Pecora and Carroll (1990), Chen and Dong (1998), Elabbssy et al. (2006), Lu et alt. (2002), Chen and Lu (2003), Li and Xu (2004)). It is widely used in many fields of physics and engineering (Wang (2003)). Recently, more works have been done in the study of chaos synchronization. Different types of synchronization such as generalized synchronization, complete synchronization, phase synchronization, projective synchronization, lag synchronization function projective, adaptive synchronization etc. (Agrawal et al. (2012a), Chang and Chen (2010), Cai et al. (2012), Srivastava et al. (2014b)) used even in fractional order and also in coupled complex system. Fractional order derivative has become an active field of research to the scientists and engineers since fractional order system response ultimately converges to the integer order system. A wide range of problems in different branches of engineering and biology have already been studied by a number of researchers from different parts of the world to explore the potential of the fractional

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derivative. The usage of first order time derivative with fractional order time derivative is not only applicable for non-Gaussian but also for Non-Markovian systems. Another important point is that fractional order systems have gained popularity in the investigation of dynamical systems since they allow a greater flexibility in the model and also for its nonlocal behaviour which takes into account the fact that the future state not only depends upon the present state but also upon all the history of its previous states.

The word chaos is derived from the Greek words "$\chi\alpha$os" is an active research topic for last few decades to the researchers working in the area of nonlinear dynamical system. Chaotic system is a bounded nonlinear deterministic system having aperiodic long term behavior, which is very much sensitive on initial conditions. Hyperchaotic system is classified as chaotic system with more than one positive Lyapunov exponent. Hyperchaotic system is used to improve the security of chaotic communication system. The hyperchaos in fractional order dynamical system was first studied by Li and Chen (2004b), where dynamics of fractional Rossler system was studied through numerical simulations. Chaos synchronization via scalar transmitted signal can be found in the research contribution of Cafagna and Grassi (2011), where an observer based method is used to synchronize a class of fractional order chaotic systems. In 2012, Cafagna and Grassi (2012a) have used the observer- based projective synchronization to synchronize fractional order hyperchaotic Rossler systems. In 2006, Dong et al. (2006) synchronized the hyperchaotic Rossler system with uncertain parameters using nonlinear active control method. In 2014, Bhalekar (2014) synchronized the fractional order hyperchaotic systems using active control method. But to the best of author's knowledge, this powerful method has not yet been used during the study of synchronization and anti-synchronization of hyperchaotic systems in fractional order cases.

The synchronization between chaotic systems with uncertainties and disturbances are not easy jobs for researchers since there are always possibilities of destroying synchronization under the effects of those parameters especially for fractional order systems. In 2012, Chen et al. (2012) have studied disturbance-observer-based robust synchronization control of uncertain chaotic systems. Jawaadaa et al. (2012) studied robust active sliding mode anti-synchronization of hyperchaotic systems with uncertainties and external disturbances. But phase and anti phase synchronizations of fractional order hyperchaotic systems with uncertainties and external disturbances using nonlinear active control method is first of its kind.

In this chapter the phase and anti-phase synchronizations between non-identical fractional order hyperchaotic systems viz., Lu and 4D integral order hyperchaotic systems are studied using nonlinear active control method in the presence of parametric uncertainties and external disturbances. Numerical simulation results are displayed graphically which clearly exhibit that the nonlinear active control method is effective, easy to implement and reliable for both the phase and anti-phase synchronizations of two nonlinear fractional order uncertain hyperchaotic systems.

3.2 Problem formulation

Consider an uncertain fractional order chaotic system as a master system as

$$D_t^q x = (A_1 + \Delta A_1)x + f_1(x) + d_1(t), \qquad 0 < q < 1 \tag{3.1}$$

and another uncertain fractional order chaotic system as the slave system as

$$D_t^q y = (A_2 + \Delta A_2)y + f_2(y) + d_2(t) + u(t), \tag{3.2}$$

where $x=[x_1,x_2,.......x_n]^T \in R^n$ and $y=[y_1,y_2,.......y_n]^T \in R^n$ are the state vectors, $A_1, A_2 \in R^{n\times n}$ are constant matrices with proper dimensions, $f_1, f_2 : R^n \to R^n$ are the nonlinear functions of the systems, $\Delta A_1, \Delta A_2 \in R^{n\times n}$ are parametric uncertainties of chaotic systems with $|\Delta A_1| \le \delta_1, |\Delta A_2| \le \delta_2$, where δ_1, δ_2 are positive constants and $d_1(t), d_2(t)$ are the external disturbances of uncertain chaotic systems with $|d_1(t)| \le \rho_1, |d_2(t)| \le \rho_2$, where $\rho_1, \rho_2 > 0$ and $u(t) \in R^n$ is the control input vector of the uncertain chaotic system (3.2). Now controller $u(t)$ is to be designed in such a way that the master and slave systems are synchronized or anti-synchronised through the proper definitions of errors.

If the synchronization error is defined by $e = y - x$, then the corresponding error dynamics can be obtained as

$$\begin{aligned} D_t^q e &= (A_2+\Delta A_2)y + f_2(y) + d_2(t) - (A_1+\Delta A_1)x - f_1(x) - d_1(t) + u(t) \\ &= (A_2+\Delta A_2+\Delta A_1)e + d_2(t) + F_1(x,y) - d_1(t) + u(t), \end{aligned} \tag{3.3}$$

where $F_1(x,y) = f_2(y) - f_1(x) + ((A_2+\Delta A_2) - A_1)x - \Delta A_1 y.$

If the anti-synchronization error is defined by $e = y + x$, then the corresponding error dynamics can be obtained as

$$\begin{aligned} D_t^q e &= (A_2+\Delta A_2)y + f_2(y) + d_2(t) + (A_1+\Delta A_1)x + f_1(x) + d_1(t) + u(t) \\ &= (A_2+\Delta A_2+\Delta A_1)e + d_2(t) + F_2(x,y) + d_1(t) + u(t), \end{aligned} \tag{3.4}$$

where $F_2(x,y) = f_2(y) + f_1(x) + ((A_1 - (A_2+\Delta A_2))x - \Delta A_1 y.$

3.3 Nonlinear active control method to design the controller

Let us define the Lyapunov function of error system (3.3) as

$V(e) = \frac{1}{2} e^T e$.

Now $q-th$ order fractional derivative w. r. to t is

$$\frac{d^q V(e)}{dt^q} = \frac{1}{2} \frac{d^q (e^T e)}{dt^q} = \frac{1}{2} \frac{d^q}{dt^q} (e_1^2 + e_2^2 + ... + e_n^2)$$

$$\leq (e_1 \frac{d^q e_1}{dt^q} + e_2 \frac{d^q e_2}{dt^q} + ... + e_n \frac{d^q e_n}{dt^q}). \text{ (using Lemma 1.1)} \qquad (3.5)$$

Putting the values of $\frac{d^q e_1}{dt^q}$, $\frac{d^q e_2}{dt^q}$, ….. $\frac{d^q e_n}{dt^q}$, and choosing the controller $u(t)$ in such a way that the $q-th$ order derivative of the Lyapunov function $V(e)$ becomes negative definite i.e., $\frac{d^q V(e)}{dt^q} < 0$, which implies that the systems (3.1) and (3.2) are synchronized according to definition of error systems.

If there is any eigen value of the error system is equal to zero, then another type of synchronization phenomenon called phase synchronization occurs, in which the difference between various states of synchronized systems may not necessarily converge to zero, but is less than or equal to a constant. The same procedure may be used for anti-phase synchronization process, in which the state vectors have the same absolute values but opposite in sign.

3.4 Systems' descriptions

3.4.1 The fractional order hyperchaotic Lu system

The fractional order Lu hyperchaotic system (Pan et al. (2011)) is given as

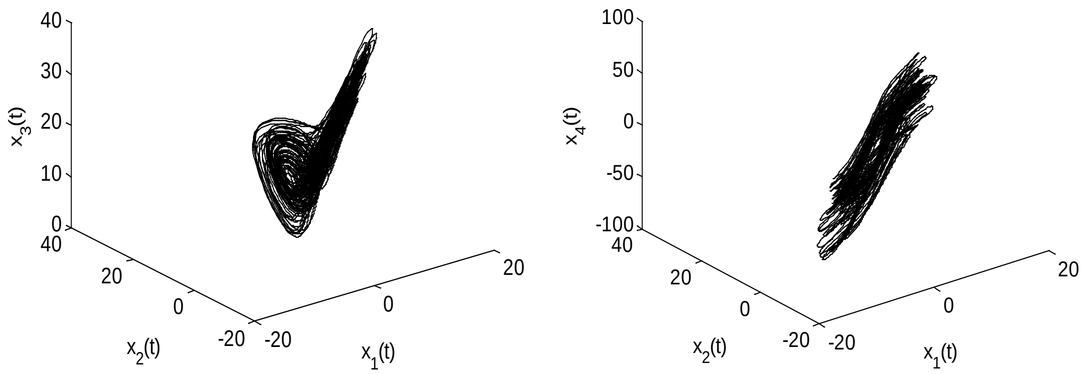
$$\frac{d^q x_1}{dt^q} = a_1 (x_2 - x_1) + x_4$$

$$\frac{d^q x_2}{dt^q} = -x_1 x_3 + a_3 x_2$$

$$\frac{d^q x_3}{dt^q} = x_1 x_2 - a_2 x_3 \tag{3.6}$$

$$\frac{d^q x_4}{dt^q} = x_1 x_2 + a_4 x_4,$$

where x_1, x_2, x_3 and x_4 are states variables and a_1, a_2, a_3 and a_4 are constant parameters. The phase portraits of the system (3.6) in $x_1 - x_2 - x_3$, $x_1 - x_2 - x_4$ spaces are depicted through Fig. 3.1 for $q = 0.95$ at $a_1 = 36, a_2 = 3, a_3 = 20$ and $a_4 = -1$. The lowest order of the systems to the hyperchaotic is 3.8 (at $q = 0.95$), which is the sum of orders of all fractional derivatives in the systems. Thus the range of q is $0.95 \le q \le 1$ for which Lu system is hyperchaotic.



(a) (b)

Fig. 3.1 Phase portraits of Lu hyperchaotic system for $q = 0.95$: (a) in $x_1 - x_2 - x_3$ space; (b) in $x_1 - x_2 - x_4$ space.

3.4.2 The fractional order hyperchaotic 4D Integral order system

The fractional order 4D Integral order hyperchaotic system (Deng et al. (2009)) is given by

$$\frac{d^q y_1}{dt^q} = a y_1 - y_2$$

$$\frac{d^q y_2}{dt^q} = y_1 - y_2 y_3^2$$

$$\frac{d^q y_3}{dt^q} = -b_1 y_2 - b_2 y_3 - b_3 y_4 \qquad (3.7)$$

$$\frac{d^q y_4}{dt^q} = y_3 + c y_4 ,$$

where y_1, y_2, y_3 and y_4 are states variables and a, b_1, b_2, b_3 and c are the constant parameters. The phase portraits of (3.7) in $y_1 - y_2 - y_3$, $y_1 - y_2 - y_4$ spaces are depicted through Fig. 3.2 for $q = 0.95$ at $a = 0.56, b_1 = 1.0, b_2 = 1.0, b_3 = 6.0$ and $c = 0.8$. Here also lowest value of q is 0.95 for which the system is hyperchaotic.

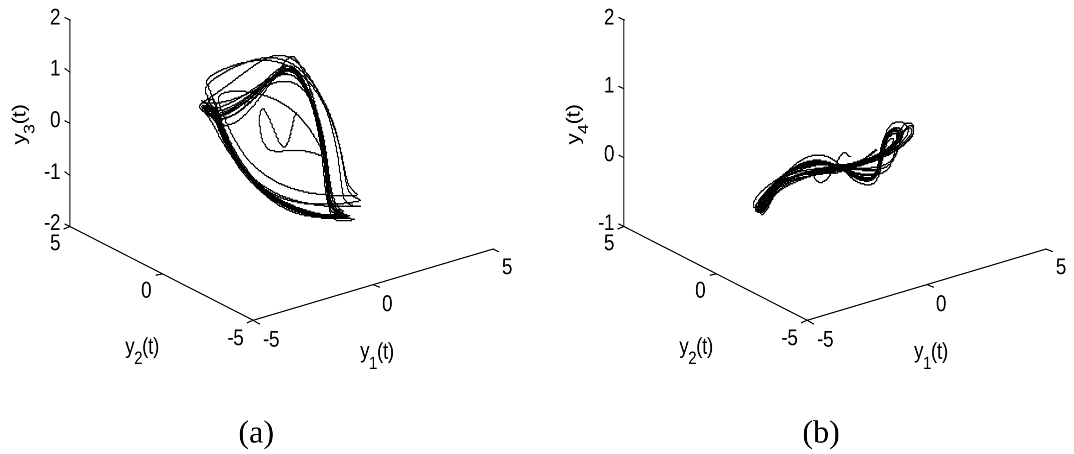


Fig. 3.2 Phase portraits of 4D Integral order hyperchaotic system for $q = 0.95$: (a) in $y_1 - y_2 - y_3$ space; (b) in $y_1 - y_2 - y_4$ space.

3.5 Phase synchronization between fractional order uncertain hyperchaotic Lu and 4D Integral order systems using nonlinear active control method

In this section the phase synchronization between fractional order Lu and 4D Integral order hyperchaotic systems is studied, which still remain hyperchaotic in presence of uncertainties and disturbances (Figs. 3.3-3.4). The uncertain fractional order Lu hyperchaotic system is defined as a master system as

$$\frac{d^q x_1}{dt^q} = a_1(x_2 - x_1) + x_4 + 0.8x_3 + \sin(20t)$$

$$\frac{d^q x_2}{dt^q} = -x_1 x_3 + a_3 x_2 - 0.3x_4 + \cos(20t)$$

$$\frac{d^q x_3}{dt^q} = x_1 x_2 - a_2 x_3 + 0.5x_1 + 2\sin(20t) \tag{3.8}$$

$$\frac{d^q x_4}{dt^q} = x_1 x_2 + a_4 x_4 - 0.9x_2 + 2\cos(20t),$$

where uncertain parameter $\Delta A_1 = \begin{pmatrix} 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & -0.3 \\ 0.5 & 0 & 0 & 0 \\ 0 & -0.9 & 0 & 0 \end{pmatrix}$ and disturbance term $d_1(t) = \begin{pmatrix} \sin(20t) \\ \cos(20t) \\ 2\sin(20t) \\ 2\cos(20t) \end{pmatrix}$. Fig. 3.3 shows the phase portraits of the fractional order Lu hyperchaotic system with uncertainties and disturbances in $x_1 - x_2 - x_3$, $x_1 - x_2 - x_4$ spaces for the order of the derivative $q = 0.95$.

The uncertain fractional order 4D Integral order hyperchaotic system is considered as a slave system as

$$\frac{d^q y_1}{dt^q} = ay_1 - y_2 + 0.2y_3 - 0.1y_4 + 0.1\sin(100t) + u_1(t)$$

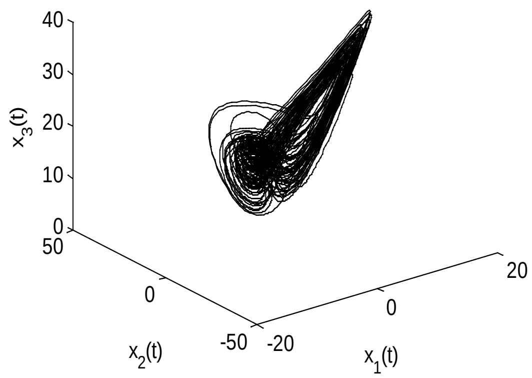
$$\frac{d^q y_2}{dt^q} = y_1 - y_2 y_3^2 - 0.01y_3 + 0.1\cos(100t) + u_2(t)$$

$$\frac{d^q y_3}{dt^q} = -b_1 y_2 - b_2 y_3 - b_3 y_4 - 0.03y_2 - 0.2\sin(100t) + u_3(t) \tag{3.9}$$

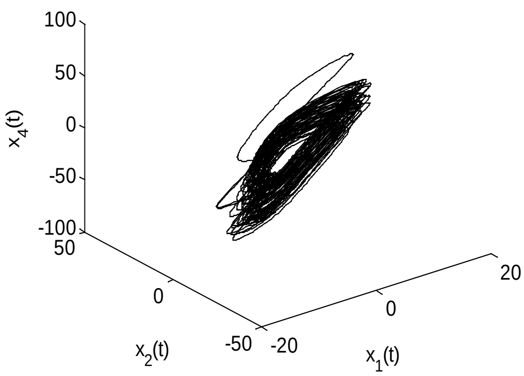
$$\frac{d^q y_4}{dt^q} = y_3 + cy_4 + 0.04y_1 - 0.02y_3 - 0.2\cos(100t) + u_4(t),$$

where uncertain parameter $\Delta A_2 = \begin{pmatrix} 0 & 0 & 0.2 & -0.1 \\ 0 & 0 & -0.01 & 0 \\ 0 & -0.03 & 0 & 0 \\ 0.04 & 0 & -0.02 & 0 \end{pmatrix}$ and disturbance term

$d_2(t) = \begin{pmatrix} 0.1\sin(100t) \\ 0.1\cos(100t) \\ -0.2\sin(100t) \\ -0.2\cos(100t) \end{pmatrix}$ and $u(t) = [u_1(t), u_2(t), u_3(t), u_4(t)]^T$ is the controller to be designed. The phase portraits of fractional order 4D Integral order hyperchaotic system with uncertainties and disturbances in $y_1 - y_2 - y_3$, $y_1 - y_2 - y_4$ spaces are depicted through Fig. 3.4.

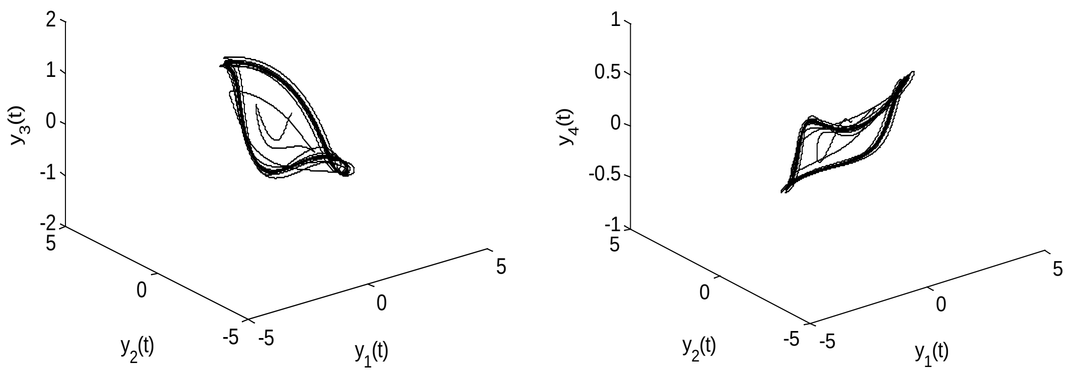


(a)



(b)

Fig. 3.3 Phase portraits of Lu hyperchaotic system with uncertainties and disturbances for $q = 0.95$: (a) in $x_1 - x_2 - x_3$ space; (b) in $x_1 - x_2 - x_4$ space.



(a) (b)

Fig. 3.4 Phase portraits of 4D Integral order hyperchaotic system with uncertainties and disturbances for $q = 0.95$: (a) in $y_1 - y_2 - y_3$ space; (b) in $y_1 - y_2 - y_4$ space.

From equations (3.8) and (3.9) we get the following error dynamical systems as

$$
\begin{aligned}
\frac{d^q e_1}{dt^q} &= ae_1 - e_2 + 0.2e_3 - 0.1e_4 + (a + a_1)x_1 - (a_1 + 1)x_2 - 0.6x_3 - 1.1x_4 + 0.1\sin(100t) \\
&\quad - \sin(20t) + u_1(t) \\
\frac{d^q e_2}{dt^q} &= e_1 - 0.01e_3 + x_1 - a_3 x_2 - 0.01x_3 + 0.3x_4 + x_1 x_3 - y_2 y_3^2 + 0.1\cos(100t) \\
&\quad - \cos(20t) + u_2(t) \\
\frac{d^q e_3}{dt^q} &= -(b_1 + 0.03)e_2 - b_2 e_3 - b_3 e_4 - 0.5x_1 - (b_1 + 0.03)x_2 + (a_2 - b_2)x_3 - b_3 x_4 - x_1 x_2 \\
&\quad - 0.2\sin(100t) - 2\sin(20t) + u_3(t) \\
\frac{d^q e_4}{dt^q} &= 0.04e_1 + 0.98e_3 + ce_4 + 0.04x_1 + 0.9x_2 + 0.98x_3 + (c - a_4)x_4 - x_1 x_3 - 0.2\cos(100t) \\
&\quad - 2\cos(20t) + u_4(t),
\end{aligned}
\qquad 3.10)
$$

where $e_i = y_i - x_i$, $i = 1, 2, 3, 4$ are error states.

In order to determine the controller, the Lyapunov function $V(e)$ is defined as

$$
V(e) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2), \qquad (3.11)
$$

whose $q-th$ order derivative w. r. to t is

$$\frac{d^q V(e)}{dt^q} = \frac{1}{2}\left(\frac{d^q e_1^2}{dt^q} + \frac{d^q e_2^2}{dt^q} + \frac{d^q e_3^2}{dt^q} + \frac{d^q e_4^2}{dt^q}\right)$$

$$\leq \left(e_1 \frac{d^q e_1}{dt^q} + e_2 \frac{d^q e_2}{dt^q} + e_3 \frac{d^q e_3}{dt^q} + e_4 \frac{d^q e_4}{dt^q}\right). \tag{3.12}$$

Substituting the values of $\frac{d^q e_i}{dt^q}, \; i = 1, 2, 3, 4$ from equation (3.10) in equation (3.12) and choosing the controller as

$$u_1(t) = -e_1 - ae_1 + e_2 - 0.2e_3 + 0.1e_4 - (a + a_1)x_1 + (a_1 + 1)x_2 + 0.6x_3 + 1.1x_4 - 0.1\sin(100t) + \sin(20t)$$

$$u_2(t) = -e_1 - e_2 + 0.01e_3 - x_1 + a_3 x_2 + 0.01x_3 - 0.3x_4 - x_1 x_3 + y_2 y_3^2 - 0.1\cos(100t) + \cos(20t)$$

$$u_3(t) = (b_1 + 0.03)e_2 + b_3 e_4 + 0.5x_1 + (b_1 + 0.03)x_2 - (a_2 - b_2)x_3 + b_3 x_4 + x_1 x_2 + 0.2\sin(100t) + 2\sin(20t)$$

$$u_4(t) = -0.04e_1 - 0.98e_3 - ce_4 - e_4 - 0.04x_1 - 0.9x_2 - 0.98x_3 - (c - a_4)x_4 + x_1 x_3 + 0.2\cos(100t) + 2\cos(20t),$$

we get the $q-th$ order derivative of the Lyapunov function $V(e)$ as

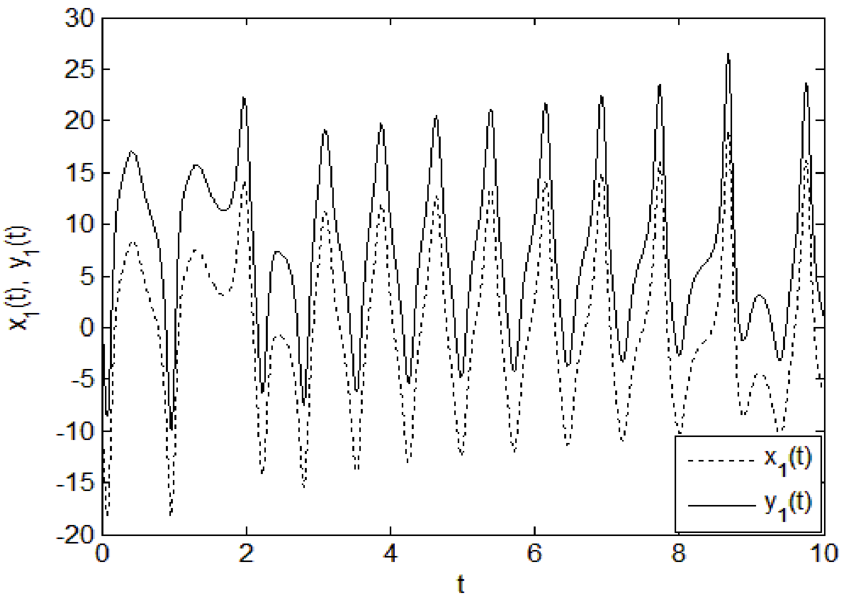
$$\frac{d^q V(e)}{dt^q} \leq -e_1^2 - e_2^2 - b_2 e_3^2 - e_4^2 < 0\,.$$

Thus it is concluded that $\lim_{t\to\infty}\|e(t)\| = 0$, and hence the synchronization between master and slave systems is achieved.

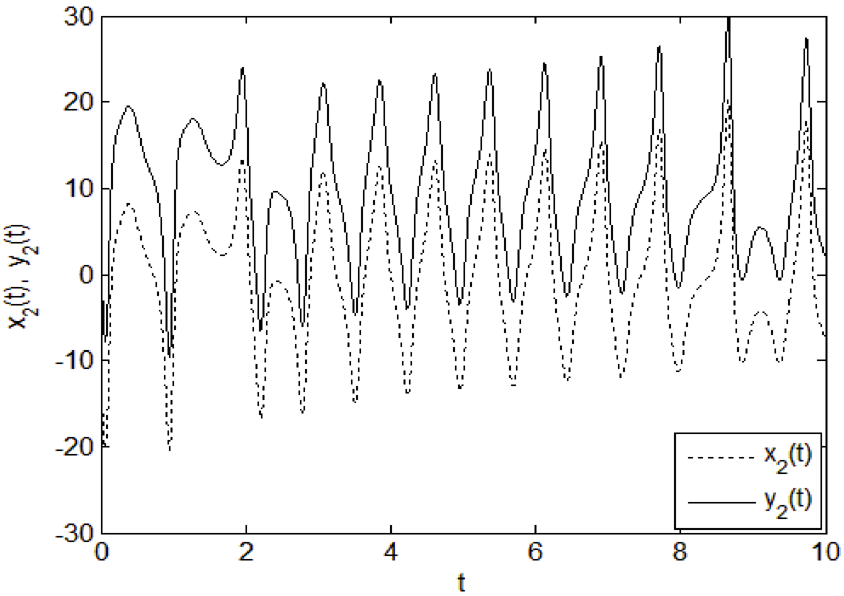
Also the error system is reduced to

$$\frac{d^q e_i}{dt^q} = -e_i\,, \; i = 1, 2, 3, 4, 5\,. \tag{3.13}$$

All the eigen values of the error systems are negative and hence satisfy the condition $|\arg(\lambda_i)| > 0.5\pi q,\ i = 1, 2, 3, 4$ (Cafagna and Grassi (2012b)) which will lead the system (3.13) asymptotically converge to zero as $t \to \infty$ and hence synchronization between systems (3.8) and (3.9) is achieved.



(a)



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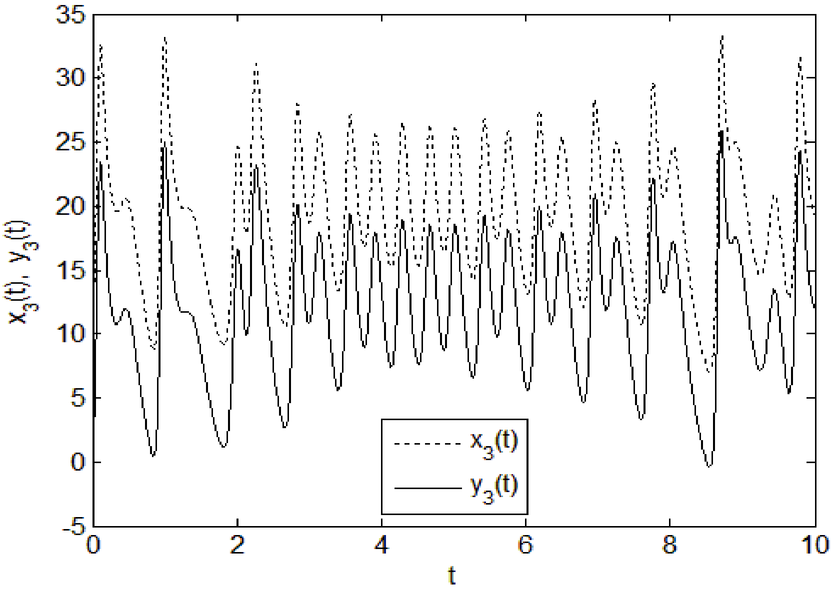
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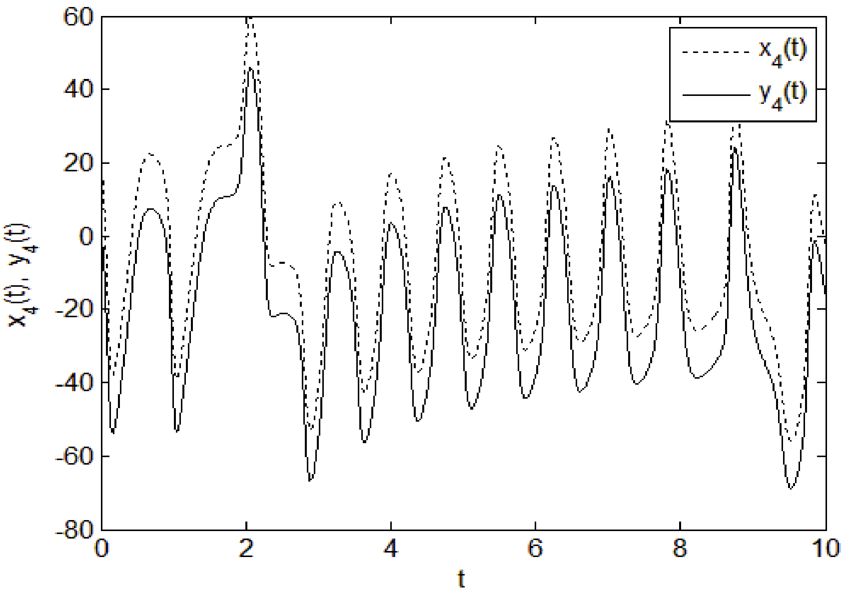
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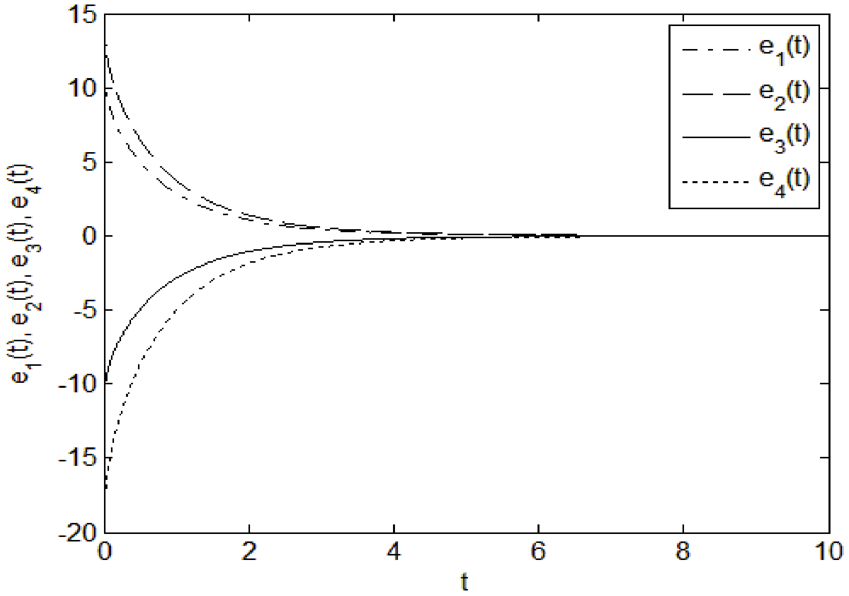

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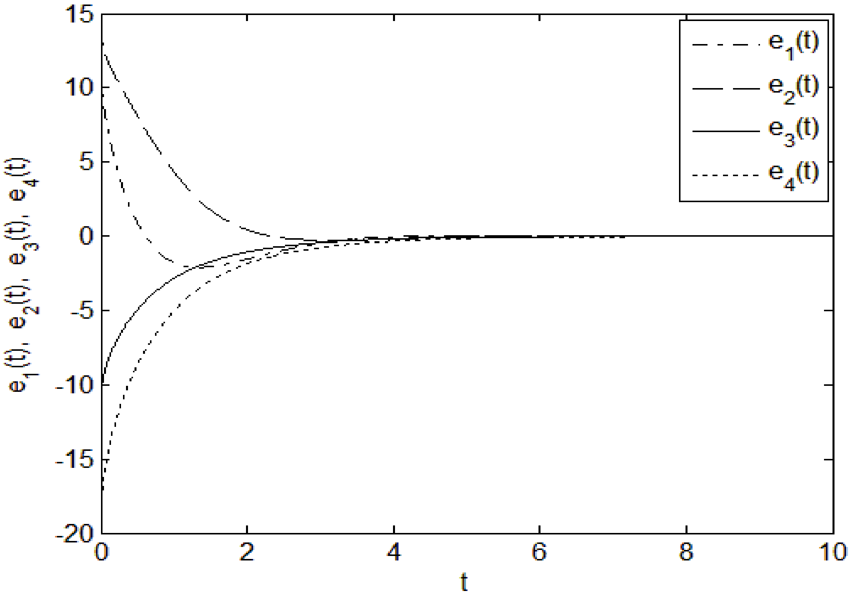


(f)

Fig. 3.5 Phase synchronization for signals: (a) between x_1 and y_1; (b) between x_2 and y_2 (c) between x_3 and y_3; (d) between x_4 and y_4; (e) The evolution of the error functions of uncertain hyperchaotic systems; (f) The evolution of the error functions of hyperchaotic systems, for fractional order derivative $q = 0.95$.

3.6 Anti-phase synchronization between fractional order uncertain hyperchaotic Lu and 4D Integral order systems using nonlinear active control method

In this section during the study of anti-phase synchronization, the fractional order Lu hyperchaotic system (3.8) is taken as master system and fractional order 4D Integral order hyperchaotic system (3.9) is considered as a slave system. Defining the error states as $e_i = y_i + x_i,\ i = 1, 2, 3, 4$ and proceeding as before, the error dynamical system becomes

$$\begin{aligned}
\frac{d^q e_1}{dt^q} &= ae_1 - e_2 + 0.2e_3 - 0.1e_4 - (a + a_1)x_1 + (a_1 + 1)x_2 + 0.6x_3 + 1.1x_4 + 0.1\sin(100t) \\
&\quad + \sin(20t) + u_1(t) \\
\frac{d^q e_2}{dt^q} &= e_1 - 0.01e_3 - x_1 + a_3 x_2 + 0.01x_3 - 0.3x_4 - x_1 x_3 - y_2 y_3^2 + 0.1\cos(100t) \\
&\quad + \cos(20t) + u_2(t) \\
\frac{d^q e_3}{dt^q} &= -(b_1 + 0.03)e_2 - b_2 e_3 - b_3 e_4 + 0.5x_1 + (b_1 + 0.03)x_2 + (a_2 - b_2)x_3 + b_3 x_4 \\
&\quad + x_1 x_2 - 0.2\sin(100t) + 2\sin(20t) + u_3(t) \\
\frac{d^q e_4}{dt^q} &= 0.04e_1 + 0.98e_3 + ce_4 - 0.04x_1 - 0.9x_2 - 0.98x_3 - (c - a_4)x_4 + x_1 x_3 \\
&\quad - 0.2\cos(100t) + 2\cos(20t) + u_4(t).
\end{aligned} \tag{3.14}$$

Defining the Lyapunov function $V(e)$ as given in equation (3.11) and choosing the controller as

$$\begin{aligned}
u_1(t) &= -e_1 - ae_1 + e_2 - 0.2e_3 + 0.1e_4 + (a + a_1)x_1 - (a_1 + 1)x_2 - 0.6x_3 - 1.1x_4 \\
&\quad - 0.1\sin(100t) - \sin(20t) \\
u_2(t) &= -e_1 - e_2 + 0.01e_3 + x_1 - a_3 x_2 - 0.01x_3 + 0.3x_4 + x_1 x_3 + y_2 y_3^2 - 0.1\cos(100t) \\
&\quad - \cos(20t) \\
u_3(t) &= (b_1 + 0.03)e_2 + b_3 e_4 - 0.5x_1 - (b_1 + 0.03)x_2 - (a_2 - b_2)x_3 - b_3 x_4 - x_1 x_2 \\
&\quad + 0.2\sin(100t) - 2\sin(20t)
\end{aligned}$$

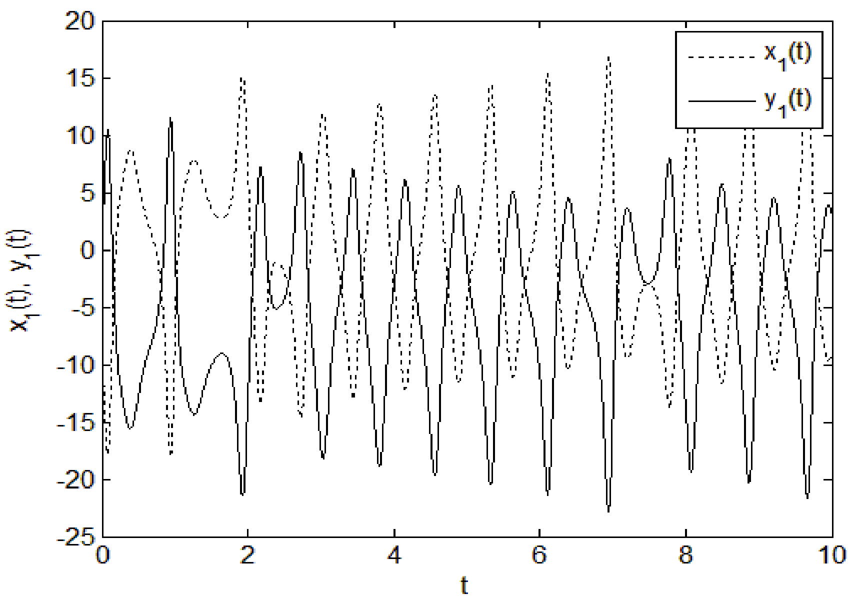
$$u_4(t) = -0.04e_1 - 0.98e_3 - ce_4 - e_4 + 0.04x_1 + 0.9x_2 + 0.98x_3 + (c - a_4)x_4 - x_1x_3$$
$$+ 0.2\cos(100t) - 2\cos(20t).$$

It is finally obtained $\frac{d^qV(e)}{dt^q} \leq -e_1^2 - e_2^2 - b_2e_3^2 - e_4^2 < 0$, which implies that the error system (3.14) is asymptotically stable and thus the anti-synchronization between fractional order Lu and 4D Integral order hyperchaotic systems is achieved.

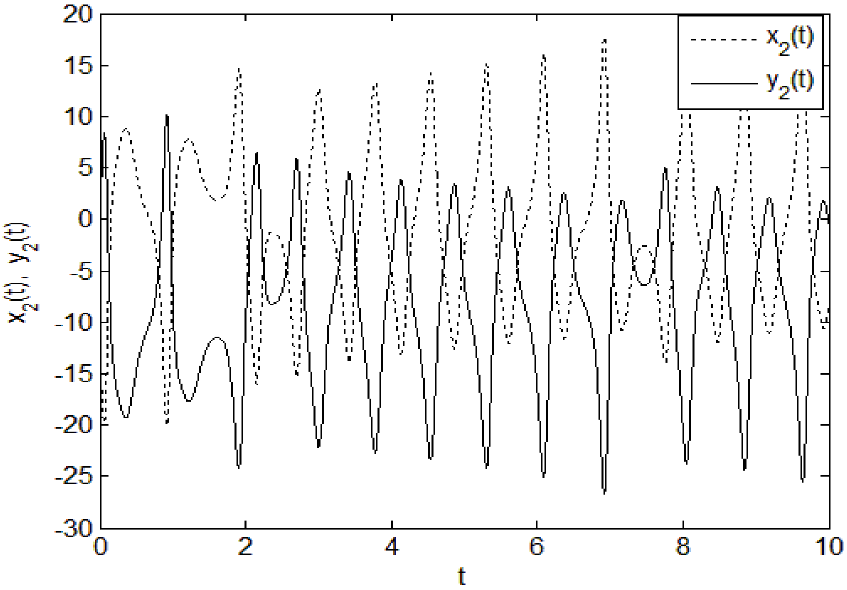
Again using the control functions, the error system will be reduced to

$$\frac{d^qe_i}{dt^q} = -e_i, \quad i = 1, 2, 3, 4.$$

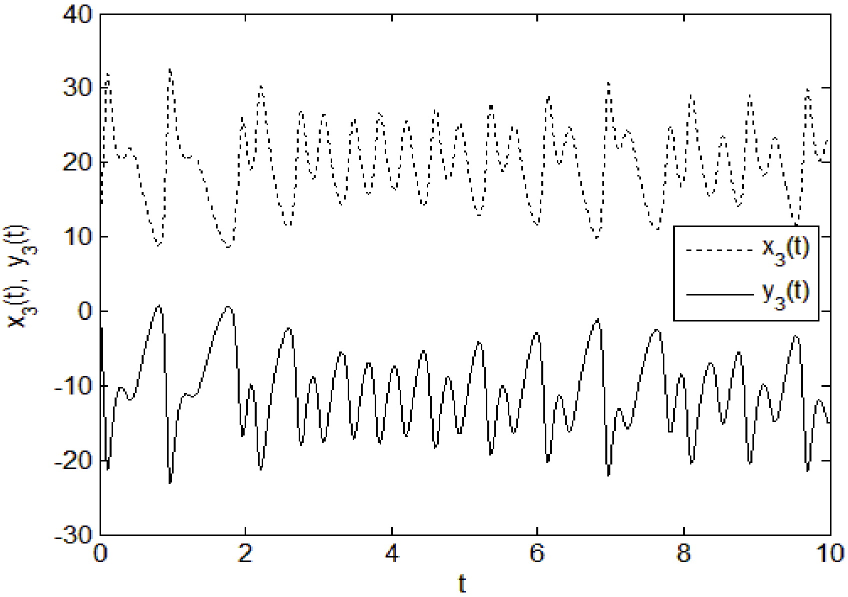
Since all the eigen values of the above system are negative, it may be concluded that the desired anti-synchronization between the systems is achieved.



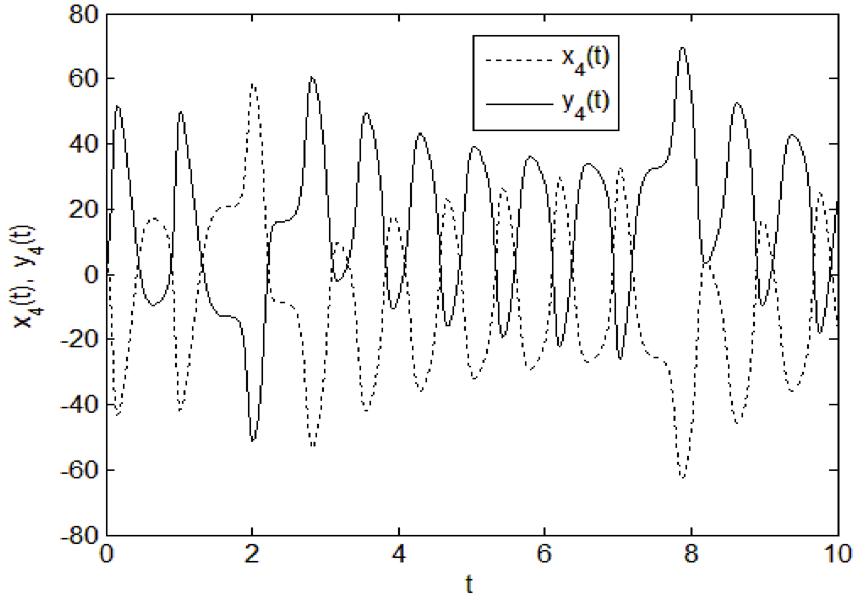
(a)


x_2(t)
y_2(t)
20
15
10
5
0
-5
-10
-15
-20
-25
-30
0
2
4
6
8
10
t
x_2(t), y_2(t)

(b)

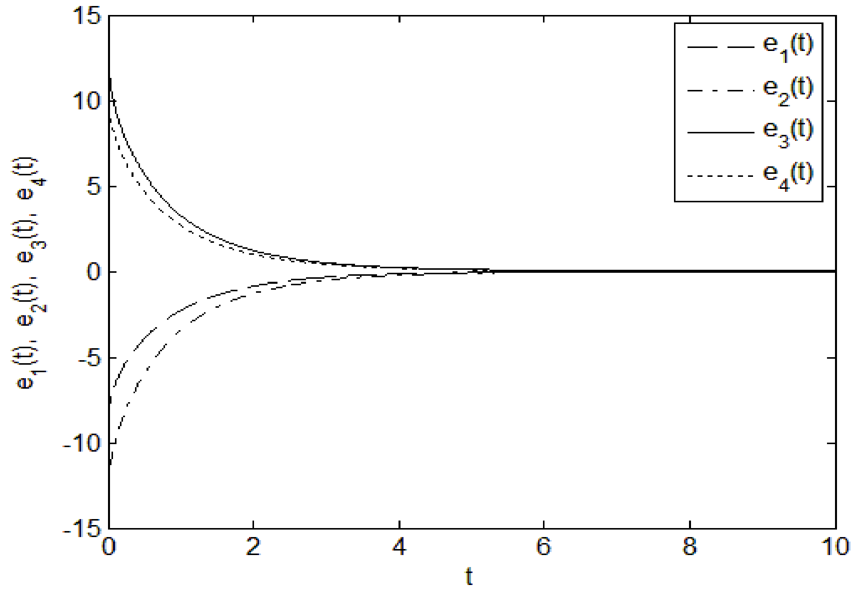

x_3(t)
y_3(t)
40
30
20
10
0
-10
-20
-30
0
2
4
6
8
10
t
x_3(t), y_3(t)

(c)



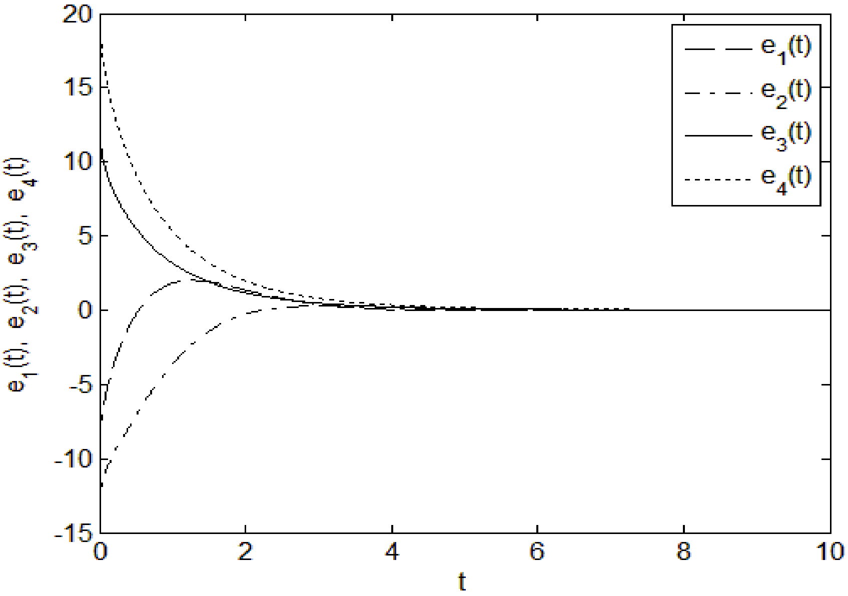
80
60
40
20
0
-20
-40
-60
-80
0
2
4
6
8
10
t
x4(t), y4(t)
x4(t)
y4(t)

(d)



15
10
5
0
-5
-10
-15
0
2
4
6
8
10
t
e1(t), e2(t), e3(t), e4(t)
e1(t)
e2(t)
e3(t)
e4(t)

(e)



(f)

Fig. 3.6. Anti-phase synchronization for signals : (a) between x_1 and y_1; (b) between x_2 and y_2; (c) between x_3 and y_3; (d) between x_4 and y_4; (e) The evolution of the error functions of uncertain hyperchaotic systems; (f) The evolution of the error functions of hyperchaotic systems, for fractional order derivative $q = 0.95$.

3.7 Numerical simulation and results

In the numerical simulation the parameters of the fractional order Lu and 4D Integral order hyperchaotic systems are taken as $a_1 = 36, a_2 = 3, a_3 = 20, a_4 = -1,$ and $a = 0.56, b_1 = 1.0, b_2 = 1.0, b_3 = 6.0, c = 0.8$ respectively. The initial conditions of the master and slave systems are taken as $(x_1(0), x_2(0), x_3(0), x_4(0)) = (-10, -14, 12, 10)$ and $(y_1(0), y_2(0), y_3(0), y_4(0)) = (1.2, 0.6, 0.8, 0.5)$ respectively. Hence the initial conditions of error system will be $(e_1(0), e_2(0), e_3(0), e_4(0)) = (11.2, 14.6, -11.2, -9.5)$. During synchronization of the systems the time step size is taken as 0.005.

Now choosing $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = -b_2, \lambda_4 = -1,$ the phase synchronization between signals x_1 and y_1 is achieved. It should be noted that, when $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = -b_2, \lambda_4 = -1,$ signals x_2 and y_2, x_3 and y_3, x_4 and y_4 become synchronized. If $\lambda_1 = -1, \lambda_2 = 0, \lambda_3 = -b_2, \lambda_4 = -1,$ $\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 0, \lambda_4 = -1,$

and $\lambda_1 = -1,\ \lambda_2 = -1,\ \lambda_3 = -b_2,\ \lambda_4 = 0,$ are taken, phase synchronizations between signals x_2 and y_2, x_3 and y_3, x_4 and y_4 are obtained respectively. State trajectories of the phase synchronization of master and slave systems are depicted through Fig. 3.5 for the order of the derivative $q = 0.95$.

During anti-synchronization, the initial conditions are taken as $(e_1(0), e_2(0), e_3(0), e_4(0)) = (\text{-}8.8, \text{-}13.4, 12.8, 10.5)$. Now proceeding as above, with proper choices of eigen values, the obtained state trajectories during the anti-phase synchronization of master and slave systems are displayed through Fig. 3.6 at $q = 0.95$.

Fig. 3.5(e) shows that time taken for synchronization between fractional uncertain Lu and 4D integral order hyperchaotic systems with disturbances is more as compared to that of the simple fractional order Lu and 4D integral order hyperchaotic systems depicted through Fig. 3.5(f). While during anti-synchronization between fractional order uncertain Lu and 4D integral order hyperchaotic systems, it is found that it takes less time during the first one (Fig. 3.6(e)) than the later one (Fig. 3.6(f)).

3.8. Conclusion

In this chapter the major contribution is that the successful use of nonlinear active control method to achieve perfect control of two fractional order hyperchaotic systems along a desired trajectory. The proper design of control function through a new lemma to achieve synchronization and anti-synchronization through error states tend to zero as time becomes large is another contribution of the study. It is worth mention that a comparative study of measurement of time of synchronization and anti-synchronization with and without the presence of uncertain and disturbance terms through numerical simulation and graphical presentation will definitely lead the researchers working in the field of fractional order dynamical systems towards a new direction.