Chapter 2

Stability analysis, chaos control of fractional order Vallis and El-Nino systems and their synchronization

2.1 Introduction

The chaotic system is a nonlinear deterministic system which posseses complex dynamical behaviors which are extremely sensitive to initial conditions and having bounded trajectories in the phase space. The study of dynamic behavior in nonlinear fractional order systems has become an interesting topic to the scientists and engineers. Fractional calculus is playing an important role for the analysis of nonlinear dynamical systems. Through fractional calculus approach many systems in interdisciplinary fields can be described by the fractional differential equation such as dielectric polarization, viscoelastic system, electrode-electrolyte polarization and electronic wave (Bagley and Calico (1991), Koeller (1984), Koeller (1986), Heaviside (1971)). Another importance of fractional calculus is that it provides an excellent tool for the description of memory and hereditary properties, for which it is used in various physical areas of science and engineering.

Effect of chaos in nonlinear dynamics is studied during last few decades by the researchers from different parts of the world. This effect is most common and has been detected in a number of dynamical systems of various types of physical nature. In practice it is usually undesirable and restricts the operating range of many mechanical and electrical devices. This type of control of dynamical system has attracted a great

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deal of attention by the researchers in society of engineering. The chaos control of systems can be divided into two categories, first one is to suppress the chaotic dynamical behaviors and second one is to generate or enhance chaos in nonlinear systems known as chaotincation or anti-control of chaos. So far various types of methods and techniques have been proposed for control of chaos such as feedback and non-feedback control (Chen and Dong (1993), Yassen (2003a), Yassen (2005)), adaptive control(Yassen (2003b), Liao and Lin (1999)) and backstepping method (Lu and Zhang (2001)) etc. Synchronization of two dynamical systems is the phenomenon where one dynamical system behaves according to the behavior of the other dynamical system. In chaos synchronization, two or more chaotic systems are coupled or one chaotic system drives another system. Pecora and Carroll (1990) were first to introduce a method to synchronize drive and response systems of two identical or non-identical systems with different initial conditions.

In this chapter, the chaos control and stability analysis of Vallis and El-Nino systems in fractional order system, and also the synchronization between the considered systems are studied. A nonlinear control method is used for chaos control of fractional order Vallis and El-Nino systems, and also during their synchronization. Both the systems were proposed by J. Vallis in 1986 for the description of temperature fluctuations in the western and eastern parts of equatorial ocean, which have a strong influence on the Earth's global climate. The first model Vallis system does not allow trade winds, whereas the second model El-Nino system describes the nonlinear interactions of the atmosphere, and trade winds in the equatorial part of pacific ocean. The main feature in this chapter is that the study of time of synchronization between the systems through numerical simulation for different particular cases as systems' pair approaches fractional order from integer order.

2.2 Design of controller for fractional order chaotic system using nonlinear control method

Consider the fractional order chaotic system as the master system as

$$D_t^q x = Px + Qf(x), \tag{2.1}$$

where $0 < q < 1$ is the order of the fractional time derivative, $x = [x_1, x_2, ...x_n]^T \in R^n$ is the state vector, P and Q are the $n \times n$ matrices consisting of the system parameters and $f : R^n \to R^n$ is a nonlinear function of the system.

Consider another fractional order chaotic system as a slave system described as

$$D_t^q y = P_1 y + Q_1 g(y) + u(t), \tag{2.2}$$

where $y = [y_1, y_2, ...y_n]^T \in R^n$ is the state vector of the system, P_1 and Q_1 are the $n \times n$ matrices of the system parameters, $g : R^n \to R^n$ is a nonlinear part of the function of the system and $u(t)$ is the controller of the system (2.2).

During synchronization, defining the error as $e = y - x$, the error dynamical system is obtained as

$$D_t^q e = P_1 e + Q_1 g(y) + (P_1 - P)x - Qf(x) + u(t). \tag{2.3}$$

During the synchronization, the aim is to find the appropriate feedback controller $u(t)$, so that the error dynamics (2.3) is stabilized in order to get $\lim_{t \to \infty} \|e(t)\| = 0$, $\forall \; e(0) \in R^n$.

Now, defining the following Lyapunov function as

$$V(e) = \frac{1}{2} e^T e,$$

whose $q-th$ order fractional derivative w. r. to t is

$$\frac{d^q V(e)}{dt^q} = \frac{1}{2} \frac{d^q (e^T e)}{dt^q} = \frac{1}{2} \frac{d^q}{dt^q} (e_1^2 + e_2^2 + ... + e_n^2)$$

$$\leq (e_1 \frac{d^q e_1}{dt^q} + e_2 \frac{d^q e_2}{dt^q} + ... + e_n \frac{d^q e_n}{dt^q}), \text{ (using Lemma 1.1)} \tag{2.4}$$

Substituting the values of $\frac{d^q e_1}{dt^q}$, $\frac{d^q e_2}{dt^q}$, ….. $\frac{d^q e_n}{dt^q}$ and choosing appropriate control function $u(t)$, the $q-th$ order derivative of the Lyapunov function $V(e)$ becomes negative i.e., $\frac{d^q V(e)}{dt^q} < 0$, which helps to get synchronization between the systems (2.1) and (2.2).

2.3 Systems' descriptions and its stability

2.3.1 Fractional order Vallis system

The Vallis model (Magnitskii and Sidorov (2006), Magnitskii, and Sidorov (2007)) is described by

$$\frac{dx}{dt} = \mu y - ax$$

$$\frac{dy}{dt} = xz - y \tag{2.5}$$

$$\frac{dz}{dt} = 1 - xy - z,$$

where x is the speed of water molecules on the surface of ocean, $y = (T_w - T_e)/2$, $z = (T_w + T_e)/2$, T_w and T_e are temperatures accordingly in western and eastern parts of ocean, μ and a are positive parameters.

The fractional order Vallis system can be described as

$$\frac{d^q x}{dt^q} = \mu y - ax$$

$$\frac{d^q y}{dt^q} = xz - y \tag{2.6}$$

$$\frac{d^q z}{dt^q} = 1 - xy - z.$$

2.3.2 Equilibrium points and stability

To find the equilibrium points of the system (2.6), we have

$$D_t^q x = D_t^q y = D_t^q z = 0, \text{ where } D_t^q \equiv \frac{d^q}{dt^q}.$$

The equilibrium points are obtained as

$$E_1 = (0, 0, 1) \text{ and } E_{2,3} = \left(\pm\sqrt{\frac{(\mu - a)}{a}}, \pm\frac{\sqrt{a(\mu - a)}}{\mu}, \frac{a}{\mu} \right).$$

For convenience the point E_1 is shifted to the point of origin through the transformation $z \to z + 1$ and the system (2.6) reduces to

$$\frac{d^q x}{dt^q} = \mu y - ax$$

$$\frac{d^q y}{dt^q} = xz + x - y \qquad (2.7)$$

$$\frac{d^q z}{dt^q} = -xy - z.$$

For the parameters $\mu = 121$ and $a = 5$ and the initial condition $(0.1, 1.2, 0.5)$, the trajectories of the Vallis system are depicted through Figs. 2.1(a)-(d) for fractional order $q = 0.97$. Again for the same parameters' values and initial conditions, the Vallis system shows chaotic behaviour at the lowest fractional order $q = 0.981$, the trajectories of which are described through Figs. 2.2(a)-(d).

The equilibrium points of the system (2.7) are

$E_1 = (0, 0, 0)$, $E_2 = (4.8166, 0.1990, -0.9586)$ and $E_3 = (-4.8166, -0.1990, -0.9586)$.

The Jacobian matrix of the Vallis system (2.7) at the equilibrium point $\overline{E}(\bar{x},\ \bar{y},\ \bar{z})$ is

$$J(\overline{E}) = \begin{bmatrix} -a & \mu & 0 \\ \bar{z}+1 & -1 & \bar{x} \\ -\bar{y} & -\bar{x} & -1 \end{bmatrix}. \tag{2.8}$$

Putting the values of $a = 5$ and $\mu = 121$, the characteristic polynomial of the above Jacobian matrix will be

$$P(\lambda) = \lambda^3 + 7\lambda^2 - (-\bar{x}^2 + 121\bar{z} + 110)\lambda - 121\bar{z} + 5\bar{x}^2 + 121\overline{xy} - 116\,. \tag{2.9}$$

At the equilibrium point $E_1 = (0,\ 0,\ 0)$, the equation (2.9) becomes

$$P(\lambda) = \lambda^3 + 7\lambda^2 - 110\lambda - 116\,. \tag{2.10}$$

The eigenvalues of the equation (2.10) are $\lambda_1 = -14.1803,\ \ \lambda_2 = 8.1803,\ \ \lambda_3 = -1.0000$.

It is seen that the equilibrium point E_1 is a saddle point of index 1 and from definition 1.3, it is unstable for $0 < q < 1$.

At the equilibrium point $E_2 = (4.8166,\ 0.1990,\ -0.9586)$, the equation (2.9) becomes

$$P(\lambda) = \lambda^3 + 7\lambda^2 + 29.1902\lambda + 231.9676\,. \tag{2.11}$$

The eigenvalues of equation (2.11) are $\lambda_1 = -7.3331,\ \ \lambda_{2,3} = 0.1666 \pm 5.6219i$. The equilibrium point E_2 is the saddle point of index 2 (definition 1.4). E_2 is stable for $0 < q < 0.981$. Similarly the equilibrium point $E_3 = (-4.8166,\ -0.1990,\ -0.9586)$ is also stable for $0 < q < 0.981$.

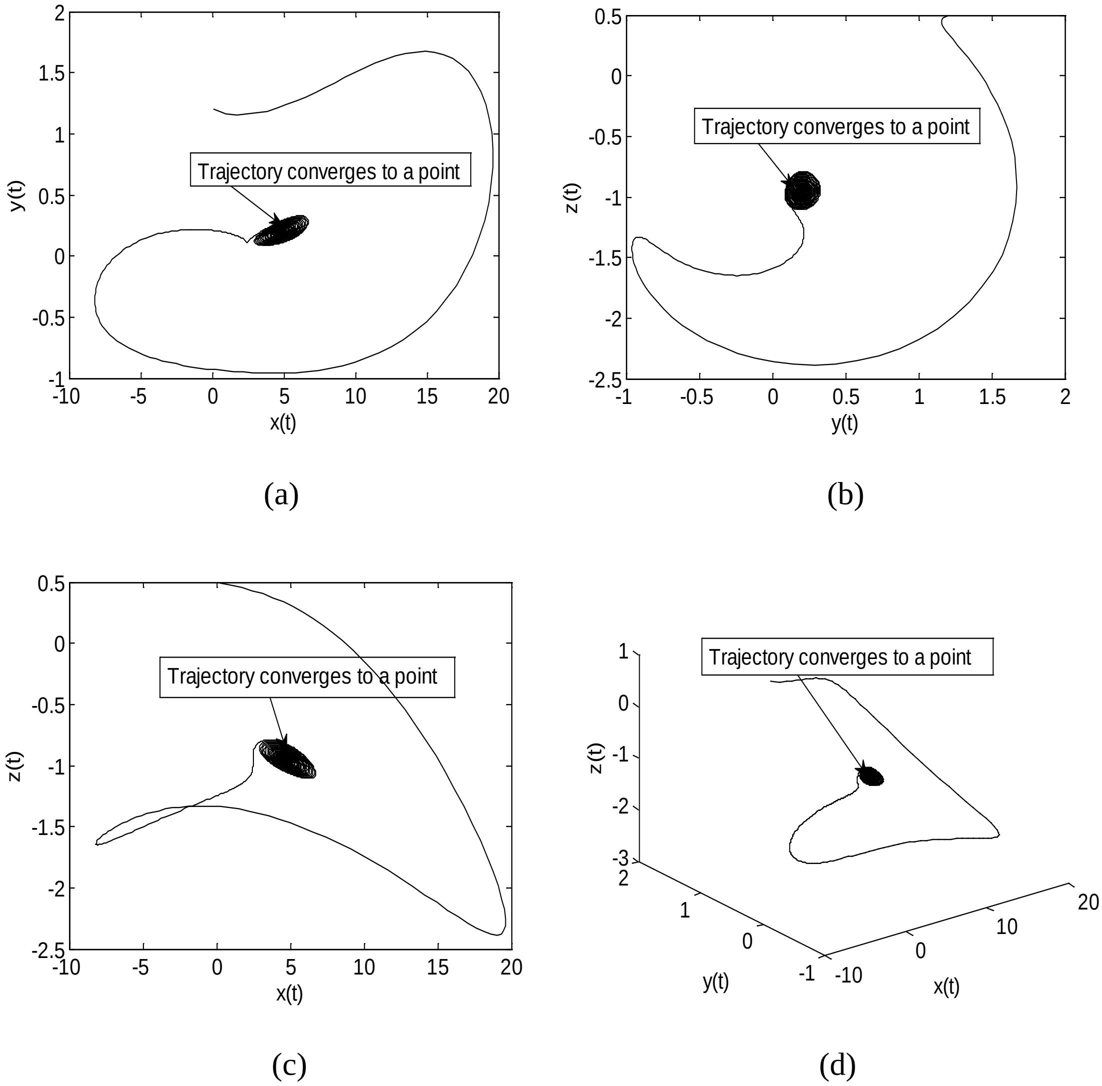
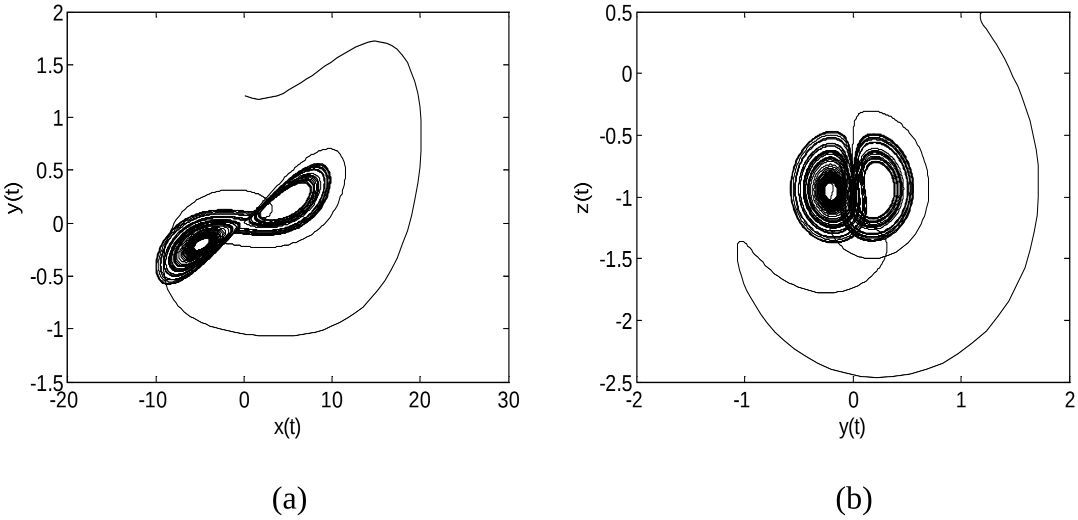
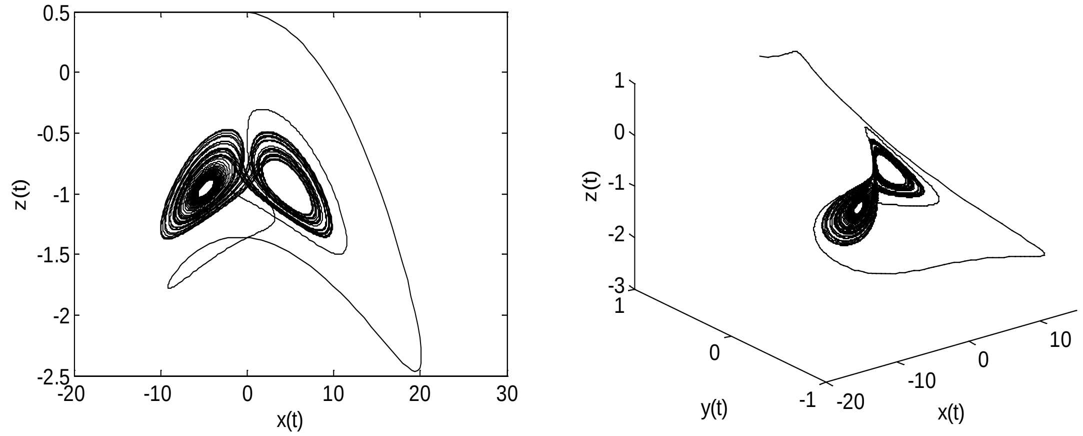


Fig. 2.1 Phase portraits of fractional order Vallis system for fractional order $q = 0.97$.





(c) (d)

Fig. 2.2 Phase portraits of fractional order Vallis system for fractional order $q = 0.981$.

2.3.3 Control of chaos using nonlinear control method

Let the fractional order Vallis system is taken as a controlled system with control functions $u_i(t),\ i = 1,\ 2,\ 3$ to stabilize unstable periodic orbit or fixed point as given in equation (2.7).

Let $(\bar{x},\ \bar{y},\ \bar{z})$ is the solution of the system (2.7), then we have

$$\frac{d^q\bar{x}}{dt^q} = \mu\bar{y} - a\bar{x}$$

$$\frac{d^q\bar{y}}{dt^q} = \overline{xz} + \bar{x} - \bar{y} \tag{2.12}$$

$$\frac{d^q\bar{z}}{dt^q} = -\overline{xy} - \bar{z}\ .$$

Defining error functions as $e_1 = x - \bar{x}$, $e_2 = y - \bar{y}$ and $e_3 = z - \bar{z}$, the error system is defined as

$$\frac{d^q e_1}{dt^q} = \mu e_2 - ae_1 + u_1(t)$$

$$\frac{d^q e_2}{dt^q} = e_1 - e_2 + xz - \overline{xz} + u_2(t) \tag{2.13}$$

$$\frac{d^q e_3}{dt^q} = -e_3 - xy + \overline{xy} + u_3(t).$$

To stabilize the error system, define the Lyapunov function as

$$V = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 + \frac{1}{2}e_3^2,$$

whose q-th order fractional derivative is

$$\frac{d^q V}{dt^q} = \frac{1}{2}\frac{d^q e_1^2}{dt^q} + \frac{1}{2}\frac{d^q e_2^2}{dt^q} + \frac{1}{2}\frac{d^q e_3^2}{dt^q}$$

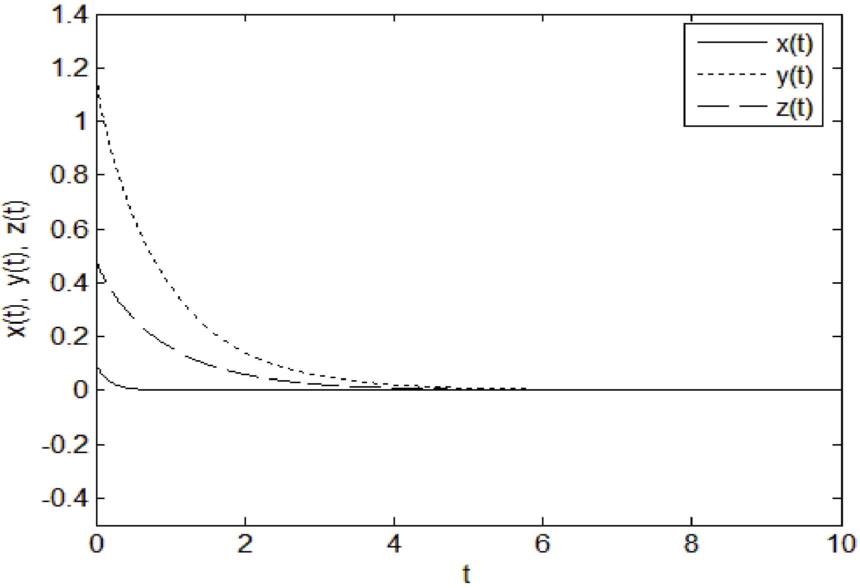
$$\leq e_1\frac{d^q e_1}{dt^q} + e_2\frac{d^q e_2}{dt^q} + e_3\frac{d^q e_3}{dt^q}, \quad \text{(from Lemma 1.1)}$$

i.e., $\leq e_1[\mu e_2 - ae_1 + u_1(t)] + e_2[e_1 - e_2 + xz - \overline{xz} + u_2(t)] + e_3[-e_3 - xy + \overline{xy} + u_3(t)]$.

If we take $u_1(t) = -\mu e_2$, $u_2(t) = -e_1 - xz + \overline{xz}$ and $u_3(t) = xy - \overline{xy}$, it becomes $\frac{d^q V}{dt^q} \leq -ae_1^2 - e_2^2 - e_3^2 < 0$. This shows that the trajectories $(x(t),\ y(t),\ z(t))$ converge to the point $(\bar{x}, \bar{y}, \bar{z})$.

2.3.4 Stabilizing the points E_1, E_2 and E_3

It is clear from Figs. 2.3(a)-(c) that at $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 0) = E_1$, the system (2.7) is stable at the point E_1 for the order $0 < q < 1$. Similarly for $(\bar{x}, \bar{y}, \bar{z}) = (4.8166, 0.1990, -0.9586) = E_2$ and $(\bar{x}, \bar{y}, \bar{z}) = (-4.8166, -0.1990, -0.9586) = E_3$, the system (2.7) is also stable for the order $0 < q < 1$. The plots of the control functions $u_1(t), u_2(t), u_3(t)$ used to stabilize the fractional order chaotic system are depicted through Fig. 2.3(d), which clearly show that the chosen functions tend to zero as time approaches infinity at the equilibrium point E_1. It can be shown that the nature of the above functions at other two equilibrium points E_2 and E_3 are similar.


x(t)
y(t)
z(t)
x(t), y(t), z(t)
t
1.4
1.2
1
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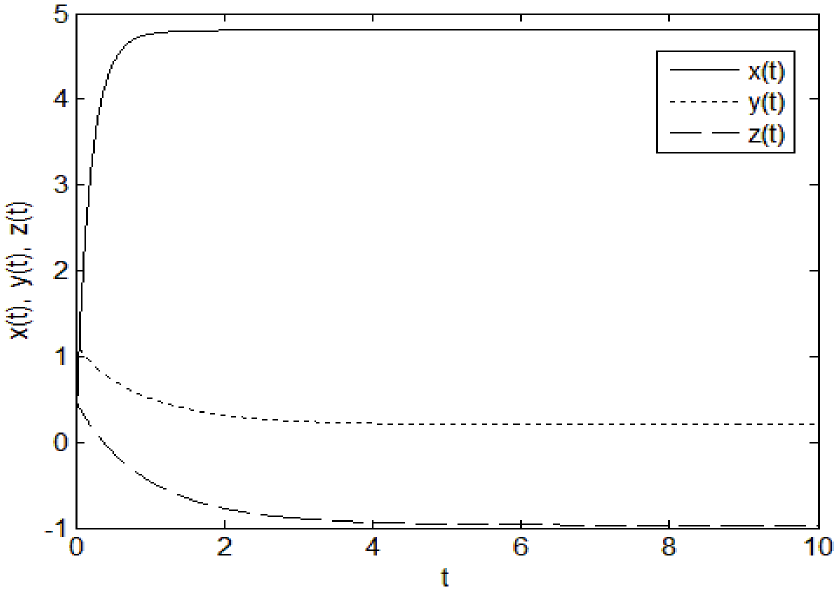
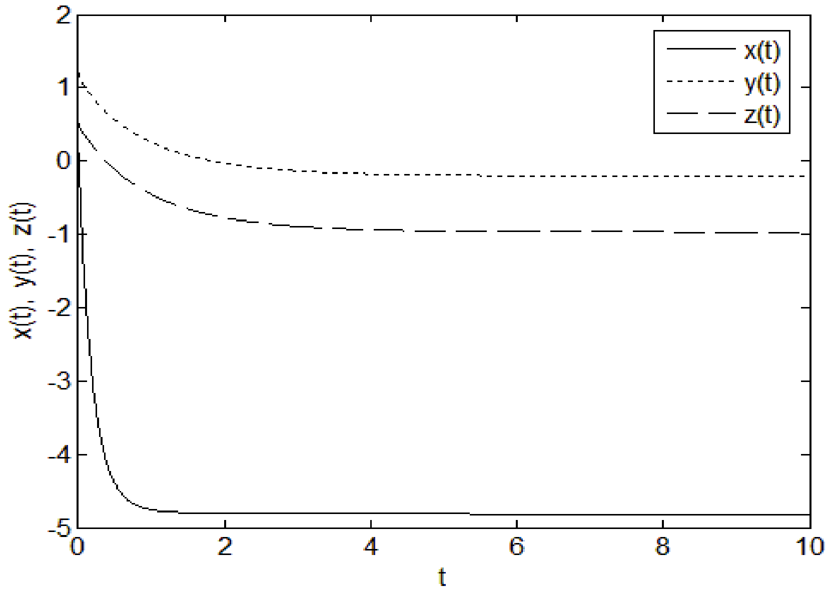
0.6
0.4
0.2
0
-0.2
-0.4
0
2
4
6
8
10

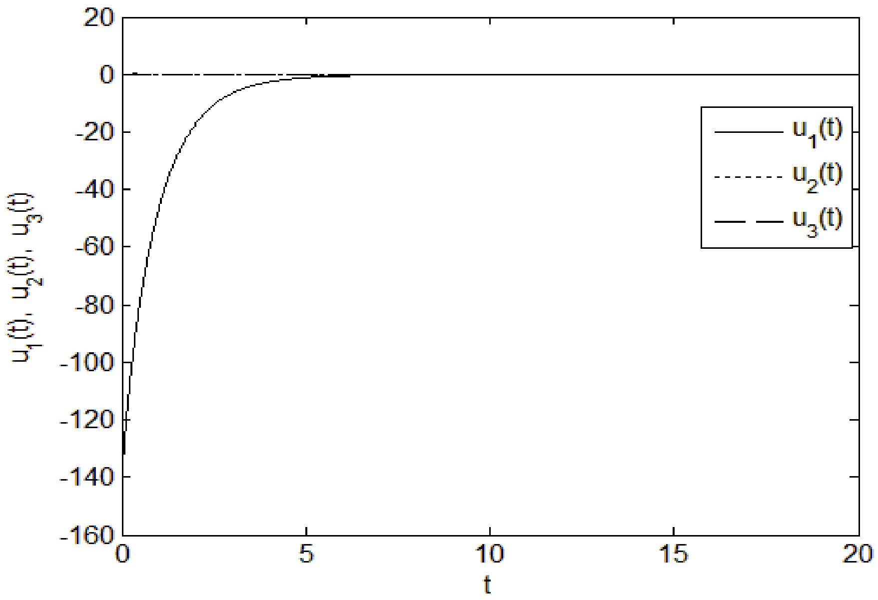
(a)


x(t)
y(t)
z(t)
x(t), y(t), z(t)
t
5
4
3
2
1
0
-1
0
2
4
6
8
10

(b)



(c)



(d)

Fig. 2.3 Plots of $x(t), y(t), z(t)$ of the controlled system (2.7): (a) at equilibrium point E_1; (b) at the equilibrium point E_2; (c) at the equilibrium point E_3; (d) plots of control functions $u_1(t), u_2(t), u_3(t)$ at E_1.

2.3.5. Fractional order El-Nino system

El-Nino system is nonlinear and non-autonomous system represented by three differential equations as (Magnitskii and Sidorov (2006), Magnitskii, and Sidorov (2007))

$$\frac{dx}{dt} = \mu'(y-z) - b(x - f(t))$$

$$\frac{dy}{dt} = xz - y + c \tag{2.14}$$

$$\frac{dz}{dt} = -xy - z + c,$$

where x, y and z are the speed of surface ocean current, temperature of water accordingly on western and eastern bounds of water pool respectively, $f(t)$ is the periodic function considering influence of trades winds.

Taking $f(t) \equiv 0$ to make an autonomous system as

$$\frac{dx}{dt} = \mu'(y-z) - bx$$

$$\frac{dy}{dt} = xz - y + c \tag{2.15}$$

$$\frac{dz}{dt} = -xy - z + c\,.$$

The fractional order El-Nino system is described by

$$\frac{d^q x}{dt^q} = \mu'(y-z) - bx$$

$$\frac{d^q y}{dt^q} = xz - y + c \tag{2.16}$$

$$\frac{d^q z}{dt^q} = -xy - z + c\,.$$

2.3.6 Equilibrium points and stability

To find the equilibrium points of the system (2.16), we have

$\mu'(y-z)-bx=0$

$xz-y+c=0$

$-xy-z+c=0$.

The equilibrium points are obtained as

$$P_1=(0,\,c,\,c) \text{ and } P_2=\left(\sqrt{\frac{2\mu c}{b}-1},\ \frac{b+\sqrt{2\mu bc-b^2}}{2\mu},\ \frac{b-\sqrt{2\mu bc-b^2}}{2\mu}\right)$$

$$P_3=\left(-\sqrt{\frac{2\mu c}{b}-1},\ \frac{b-\sqrt{2\mu bc-b^2}}{2\mu},\ \frac{b+\sqrt{2\mu bc-b^2}}{2\mu}\right).$$

Making a shifting through $y \to y+c$ and $z \to z+c$, the system (2.16) will be reduced to the following form

$$\frac{d^q x}{dt^q}=\mu'(y-z)-bx$$

$$\frac{d^q y}{dt^q}=xz+xc-y \qquad (2.17)$$

$$\frac{d^q z}{dt^q}=-xy-xc-z\,.$$

For the parameters $\mu'=83.6,$ $b=10$ and $c=12$ and the initial condition $(-2,3,5)$, the El-Nino system shows chaotic behaviour at $q=0.934,$ the lowest fractional order (see Figs. 2.5 (a)-(d)). For the same values of parameters and initial conditions the trajectories of the system at $q=0.93$ are described through Figs. 2.4(a)-(d).

The equilibrium points of the system (2.17) are calculated as

$P_1 = (0, 0, 0)$, $P_2 = (14.1294, -11.0951, -12.7852)$ and $P_3 = (-14.1294, -12.7852, -11.0951)$.

The Jacobian matrix of the El-Nino system (2.17) at the equilibrium point $\overline{P}(\bar{x}, \bar{y}, \bar{z})$ is

$$J(\overline{P}) = \begin{bmatrix} -b & \mu' & -\mu' \\ \bar{z}+c & -1 & \bar{x} \\ -\bar{y}-c & -\bar{x} & -1 \end{bmatrix}.$$

Putting the values of $\mu' = 83.6$, $b = 10$ and $c = 12$, we obtain characteristic polynomial of the above Jacobian matrix as

$$P(\lambda) = \lambda^3 + 12\lambda^2 - (-\bar{x}^2 + 83.6\bar{y} + 83.6\bar{z} + 1985.40)\lambda + 10\bar{x}^2 - 83.6\bar{y} - 83.6\bar{z} + 83.6\bar{x}\,\bar{y} - 83.6\bar{x}\,\bar{z} - 1996.40.$$

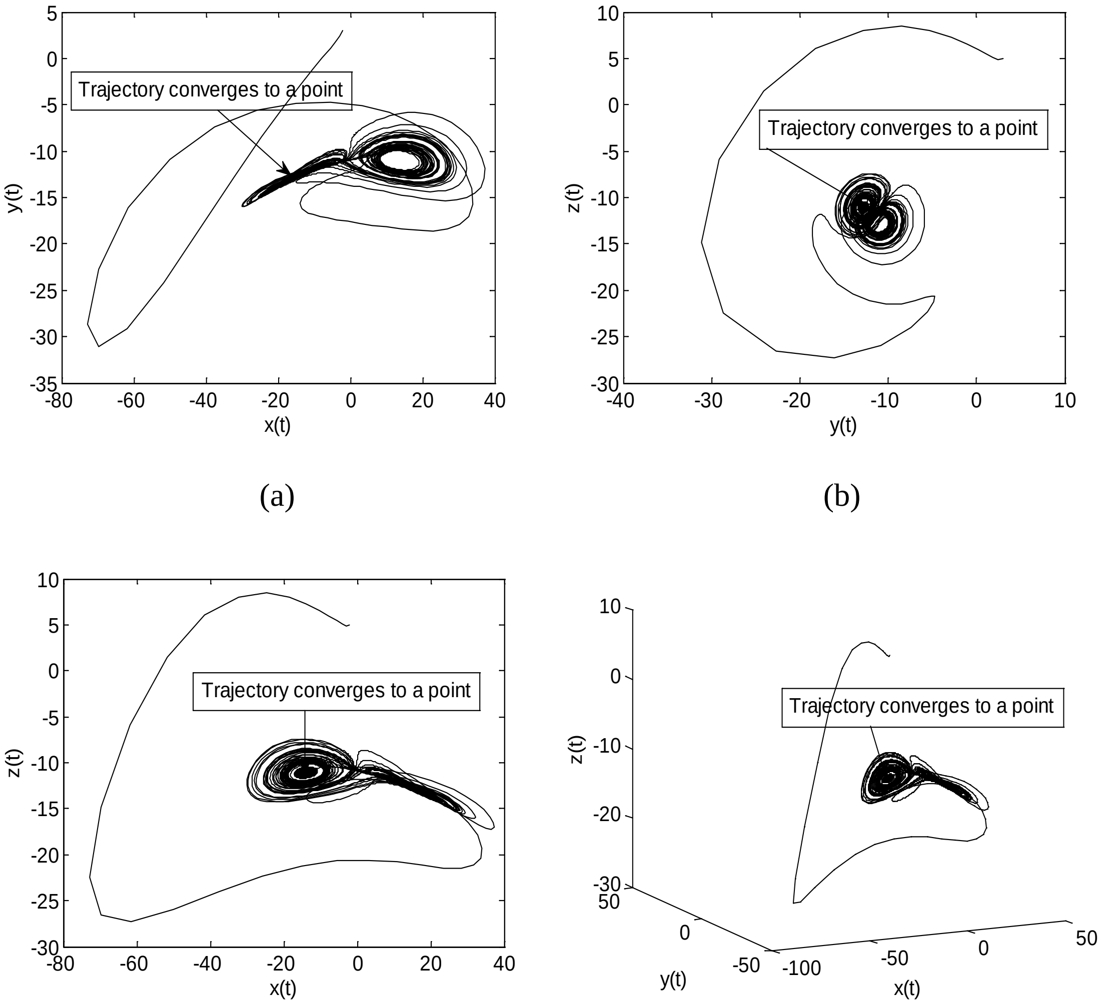
At the equilibrium point $P_1 = (0, 0, 0)$,

$$P(\lambda) = \lambda^3 + 12\lambda^2 - 1985.40\lambda - 1996.40.$$

Solving $P(\lambda) = 0$, we get $\lambda_1 = -50.5183$, $\lambda_2 = 39.5183$, $\lambda_3 = -1.0000$.

Now P_1 is a saddle point of index 1 and from definition 1.3, it is unstable for $0 < q < 1$.

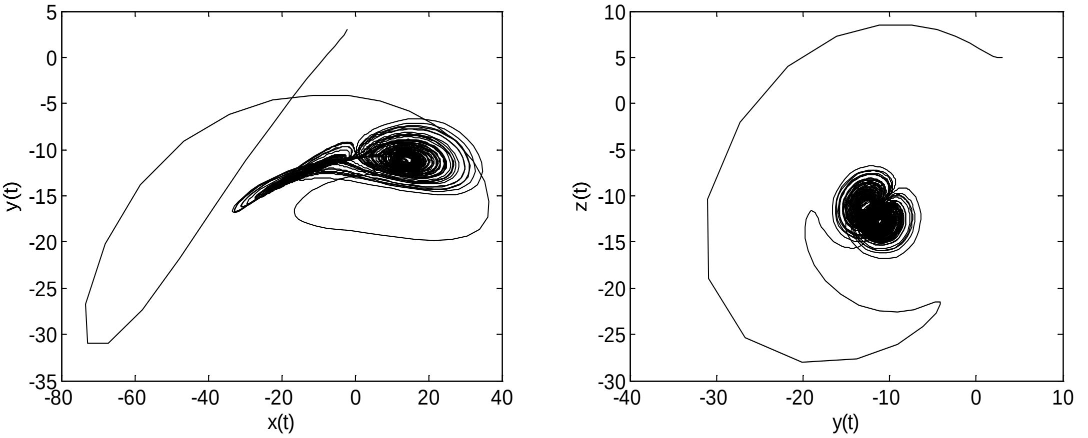
At the equilibrium point $P_2 = (14.1294, -11.0951, -12.7852)$, the polynomial becomes $P(\lambda) = \lambda^3 + 12\lambda^2 + 210.6330\lambda + 3992.7687$ and thus the eigenvalues are $\lambda_1 = -15.2956$, $\lambda_{2,3} = 1.6478 \pm 16.0725i$. P_2 is the saddle point of index 2 (definition 1.4). So P_2 is stable for $0 < q < 0.934$. Similarly at $P_3 = (-14.1294, -12.7852, -11.0951)$, the eigenvalues are obtained as $\lambda_1 = -15.2956$, $\lambda_{2,3} = 1.6478 \pm 16.0725i$, and this shows that P_3 is stable for $0 < q < 0.934$.



(a) (b)

(c) (d)

Fig. 2.4 Phase portraits of fractional order El-Nino system for fractional order $q = 0.93$.



(a) (b)

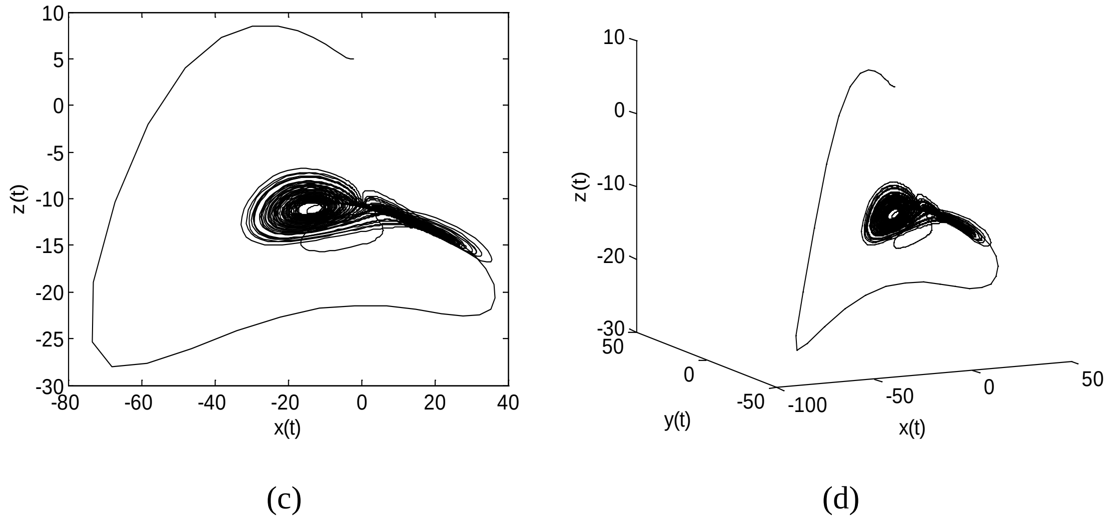


Fig. 2.5 Phase portraits of fractional order El-Nino system for fractional order $q = 0.934$.

2.3.7 Control of chaos using nonlinear control method

Consider the fractional order El-Nino system as a controlled system with control functions $u_1(t), u_2(t)$ and $u_3(t)$ for stabilizing unstable periodic orbit and $(\bar{x},\ \bar{y},\ \bar{z})$ be the solution of the system (2.17) so that

$$\frac{d^q\bar{x}}{dt^q} = \mu'(\bar{y} - \bar{z}) - b\bar{x}$$

$$\frac{d^q\bar{y}}{dt^q} = \overline{xz} + \bar{x}c - \bar{y} \qquad (2.18)$$

$$\frac{d^q\bar{z}}{dt^q} = -\overline{xy} - \bar{x}c - \bar{z}\,.$$

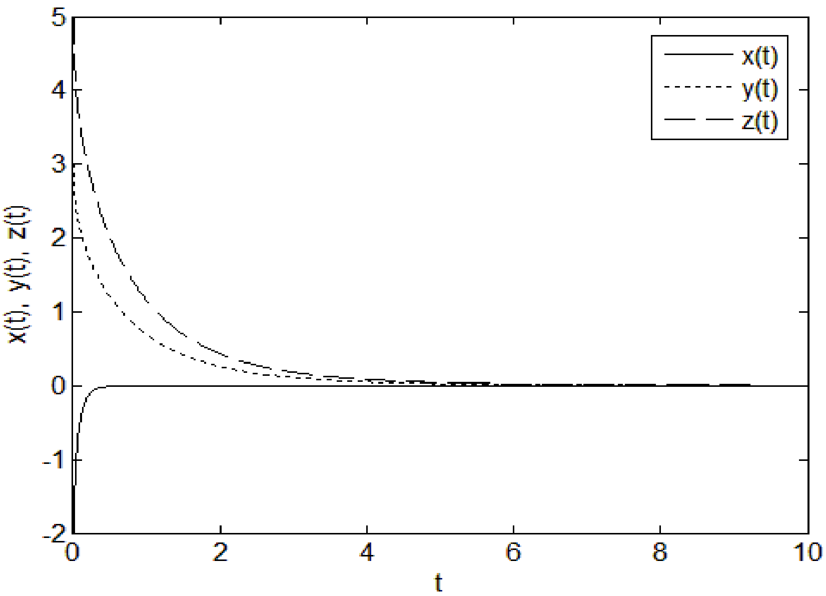
Defining the error function $e(t)$ and Lyapunov function V as in section 2.3.3 for stabilizing the error system, we get the q-th order derivative of V as

$$\frac{d^qV}{dt^q} \le e_1[\mu'(e_2 - e_3) - be_1 + u_1(t)] + e_2[ce_1 - e_2 + xz - \overline{xz} + u_2(t)] + e_3[-ce_1 - e_3 - xy + \overline{xy} + u_3(t)].$$

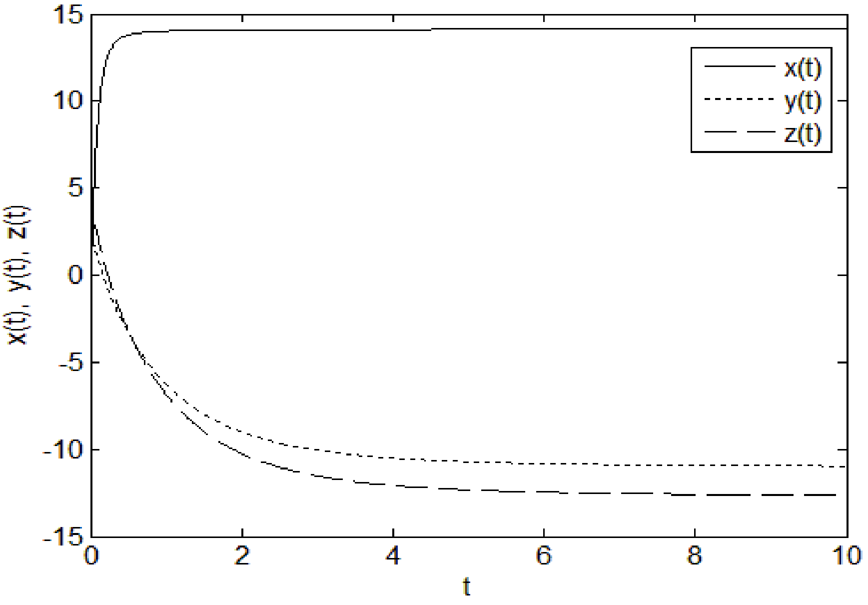
Taking $u_1(t) = -\mu'(e_2 - e_3)$, $u_2(t) = -ce_1 - xz + \overline{xz}$ and $u_3(t) = ce_1 + xy - \overline{xy}$, we get $\frac{d^qV}{dt^q} \le -be_1^2 - e_2^2 - e_3^2 < 0$, which implies the trajectories $(x(t),\ y(t),\ z(t))$ converge to $(\bar{x},\ \bar{y},\ \bar{z})$.

2.3.8 Stabilizing the points P_1, P_2 and P_3

It is seen from Figs. 2.6(a)-(c) that at $P_1 = (0, 0, 0)$, $P_2 = (14.1294, -11.0951, -12.7852)$ and $P_3 = (-14.1294, -12.7852, -11.0951)$, the system (2.17) is stable for the order $0 < q < 1$. Like previous system, the chosen control functions for this fractional order chaotic system converge to zero at all the equilibrium points P_1, P_2, P_3 as time approaches infinity. The plots at P_1 are shown through Fig. 2.6(d).



(a)


15
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5
0
-5
-10
-15
x(t), y(t), z(t)
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8
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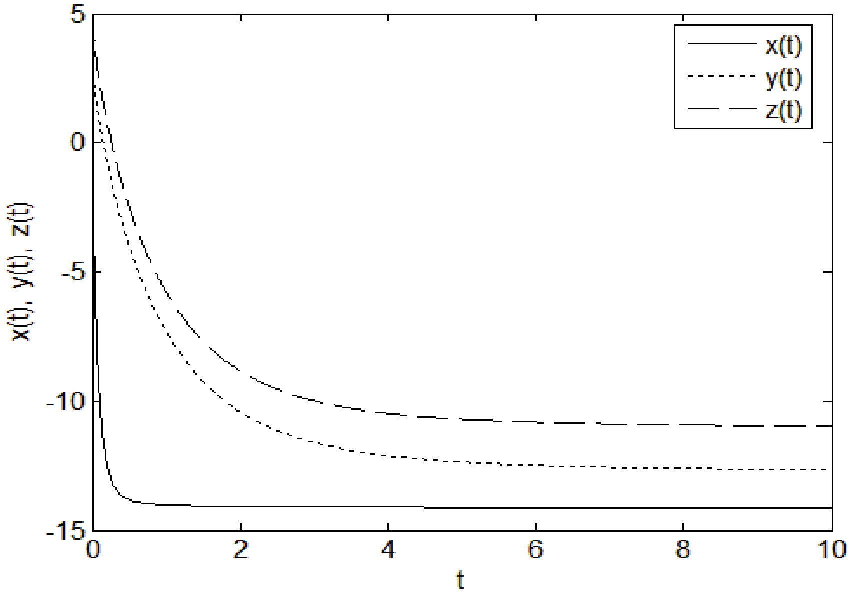
t
x(t)
y(t)
z(t)

(b)


5
0
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-15
x(t), y(t), z(t)
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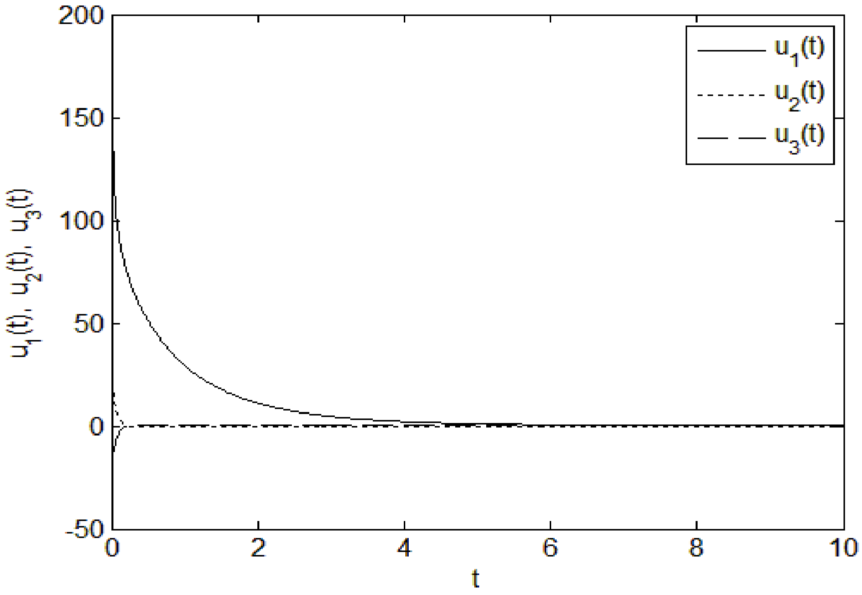
(c)



(d)

Fig. 2.6 Plots of $x(t),\ y(t),\ z(t)$ of the controlled system (2.17): (a) at equilibrium point P_1; (b) at the equilibrium point P_2; (c) at the equilibrium point P_3; (d) plots of control functions $u_1(t), u_2(t), u_3(t)$ at P_1.

2.4 Synchronization between fractional order Vallis and El-Nino systems using nonlinear control method

In this section to study the synchronization between fractional order Vallis and El-Nino systems, we consider the fractional order Vallis system as the master system as

$$\frac{d^q x_1}{dt^q} = \mu y_1 - a x_1$$

$$\frac{d^q y_1}{dt^q} = x_1 z_1 + x_1 - y_1 \tag{2.19}$$

$$\frac{d^q z_1}{dt^q} = -x_1 y_1 - z_1$$

and the fractional order El-Nino system as slave system as

$$\frac{d^q x_2}{dt^q} = \mu'(y_2 - z_2) - b x_2 + v_1(t)$$

$$\frac{d^q y_2}{dt^q} = x_2 z_2 + x_2 c - y_2 + v_2(t) \tag{2.20}$$

$$\frac{d^q z_2}{dt^q} = -x_2 y_2 - x_2 c - z_2 + v_3(t),$$

where $v_1(t)$, $v_2(t)$ and $v_3(t)$ are the control functions. Defining error functions as

$e_1 = x_2 - x_1$, $e_2 = y_2 - y_1$ and $e_3 = z_2 - z_1$,

the error system is obtained as

$$\frac{d^q e_1}{dt^q} = \mu'(e_2 - e_3) - be_1 + (a-b)x_1 + (\mu' - \mu)y_1 - \mu' z_1 + v_1(t)$$

$$\frac{d^q e_2}{dt^q} = ce_1 - e_2 + (c-1)x_1 + x_2 z_2 - x_1 z_1 + v_2(t) \tag{2.21}$$

$$\frac{d^q e_3}{dt^q} = -ce_1 - e_3 - x_1 c - x_2 y_2 + x_1 y_1 + v_3(t).$$

In order to stabilize the error system, let us consider the Lyapunov function as

$$V(e) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2). \tag{2.22}$$

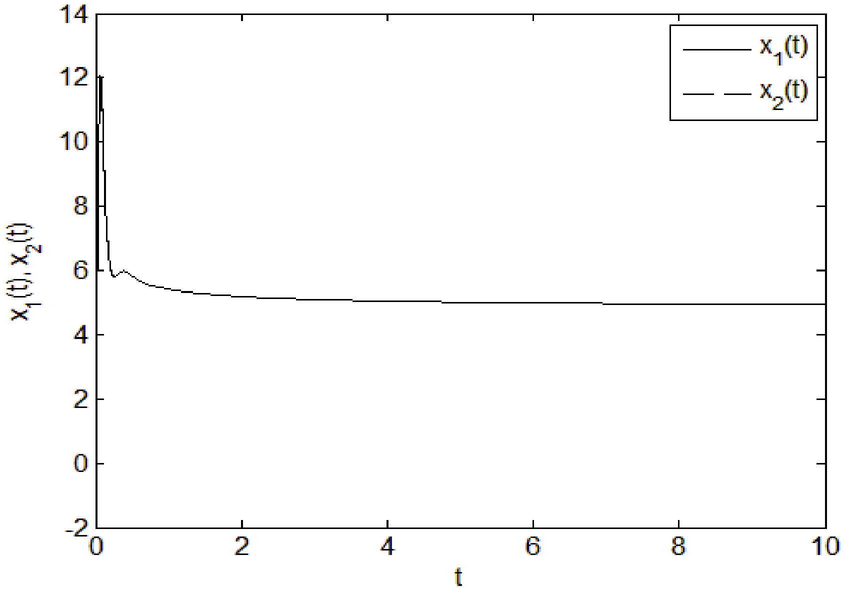
Choosing the control functions as

$v_1(t) = -\mu'(e_2 - e_3) - (a-b)x_1 - (\mu' - \mu)y_1 + \mu' z_1$,

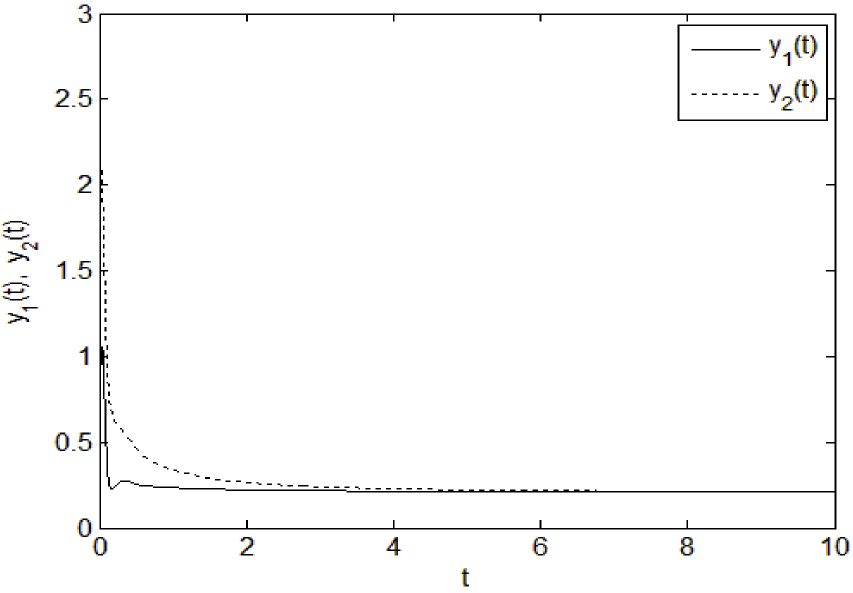
$v_2(t) = -ce_1 - (c-1)x_1 - x_2 z_2 + x_1 z_1$,

$v_3(t) = ce_1 + x_1 c + x_2 y_2 - x_1 y_1$,

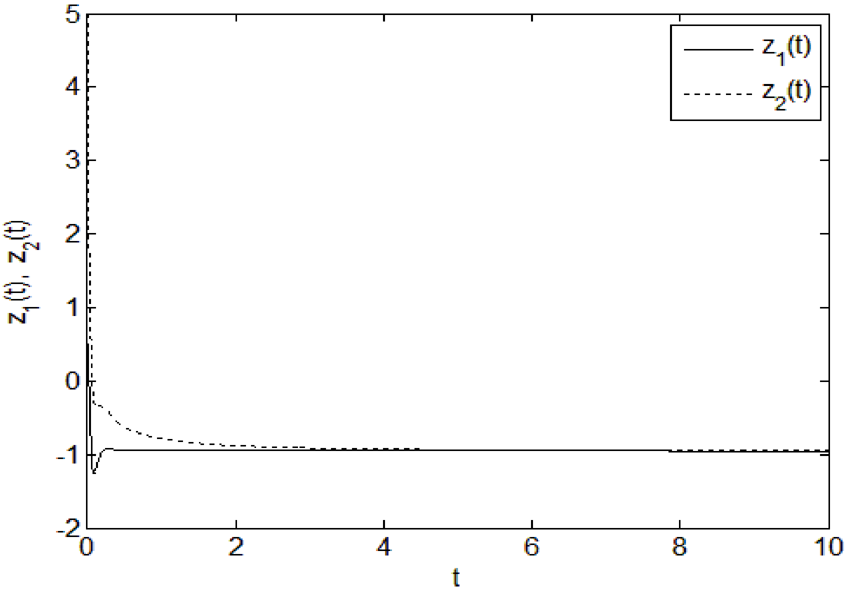
the $q-th$ order derivative of the Lyapunov function $V(e)$ becomes $\frac{d^q V(e)}{dt^q} \le -be_1^2 - e_2^2 - e_3^2 < 0$, which concludes that $\lim_{t\to\infty}\|e(t)\| = 0$, and hence the synchronization between master and response systems is achieved.


14
12
10
8
6
4
2
0
-2
0
2
4
6
8
10
t
x1(t), x2(t)
x1(t)
x2(t)

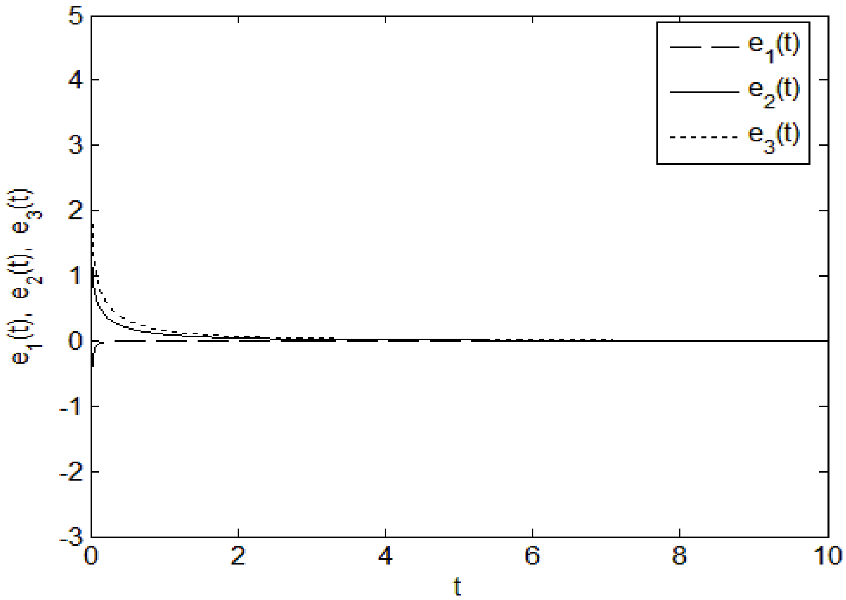
(a)


3
2.5
2
1.5
1
0.5
0
0
2
4
6
8
10
t
y1(t), y2(t)
y1(t)
y2(t)

(b)

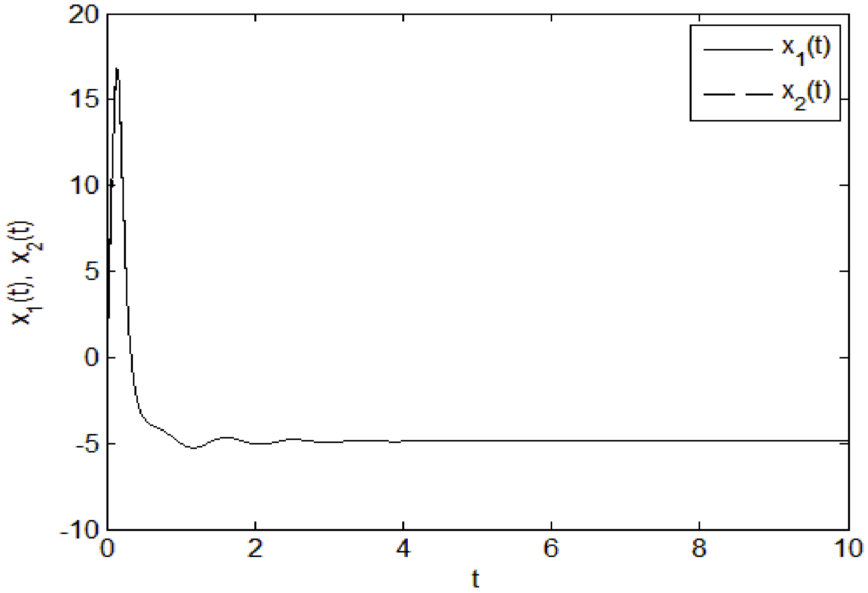


(c)

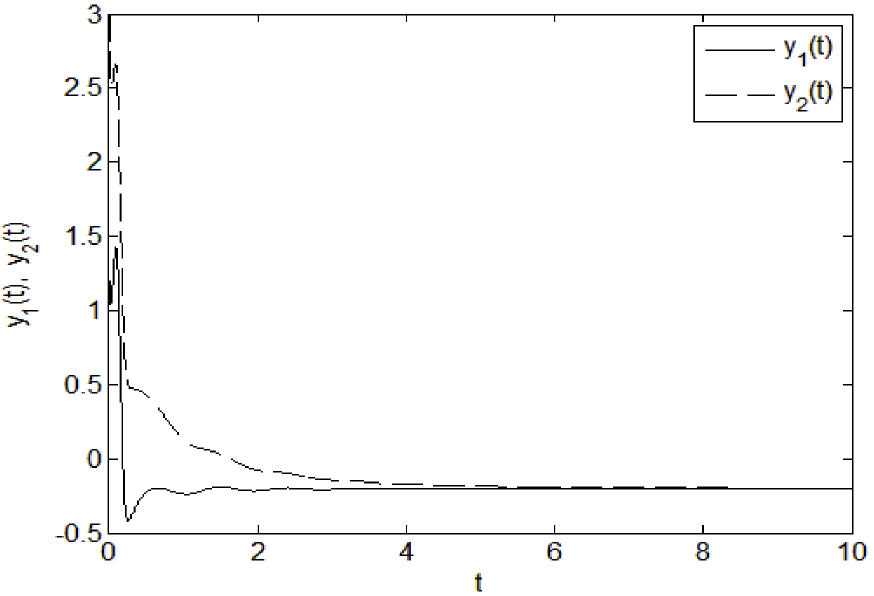


(d)

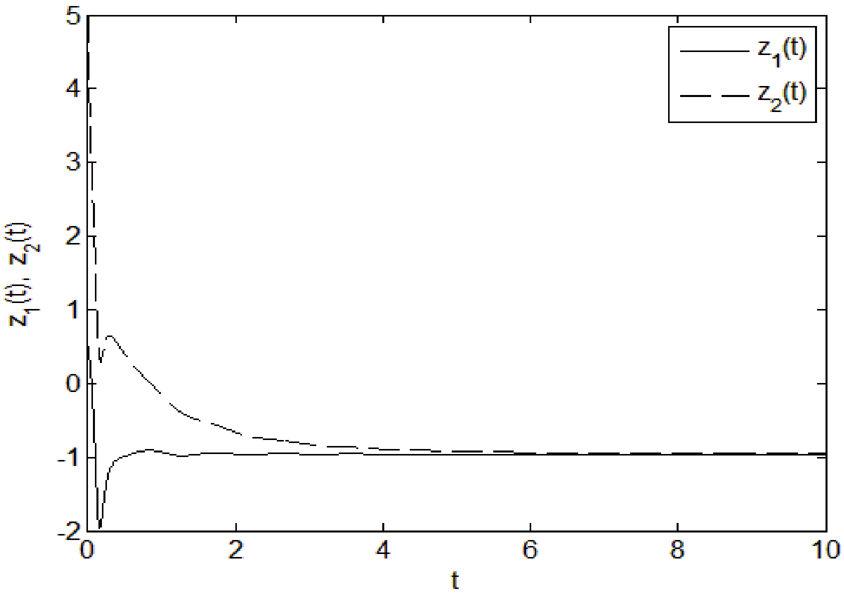
Fig. 2.7 State trajectories of master system (2.19) and slave system (2.20) for fractional order $q = 0.7$: (a) synchronization between x_1 and x_2; (b) synchronization between y_1 and y_2; (c) synchronization between z_1 and z_2; (d) the evolution of the error functions $e_1(t), e_2(t)$ and $e_3(t)$.


20
15
10
5
0
-5
-10
0
2
4
6
8
10
t
$x_1(t)$, $x_2(t)$
$x_1(t)$
$x_2(t)$

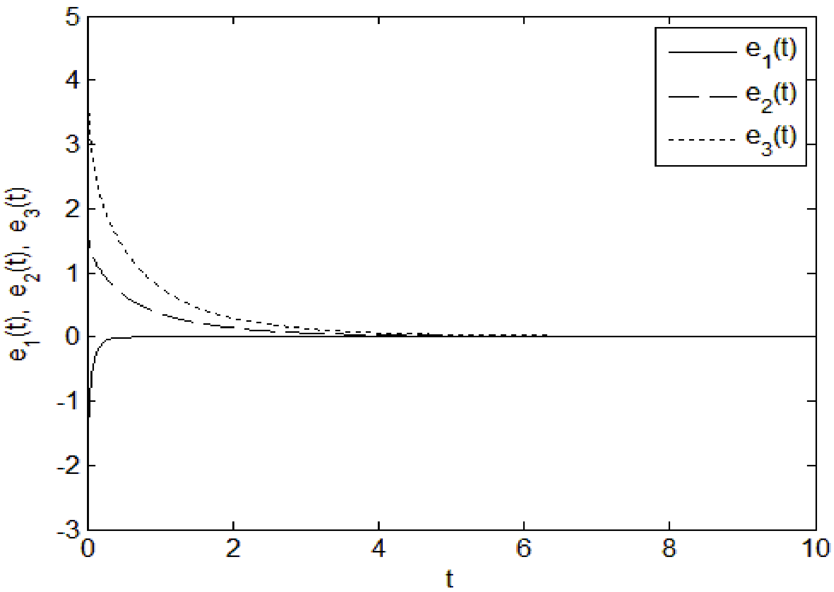
(a)


3
2.5
2
1.5
1
0.5
0
-0.5
0
2
4
6
8
10
t
$y_1(t)$, $y_2(t)$
$y_1(t)$
$y_2(t)$

(b)

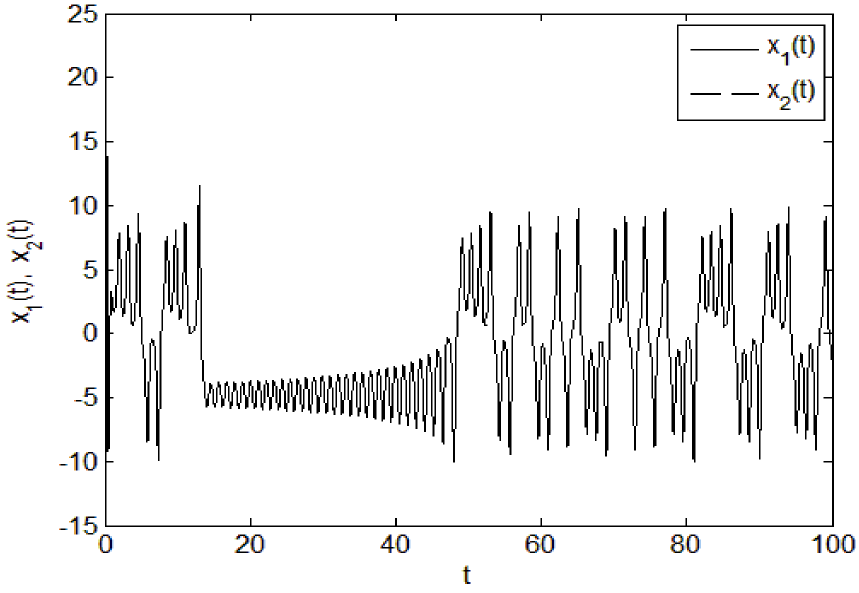


(c)

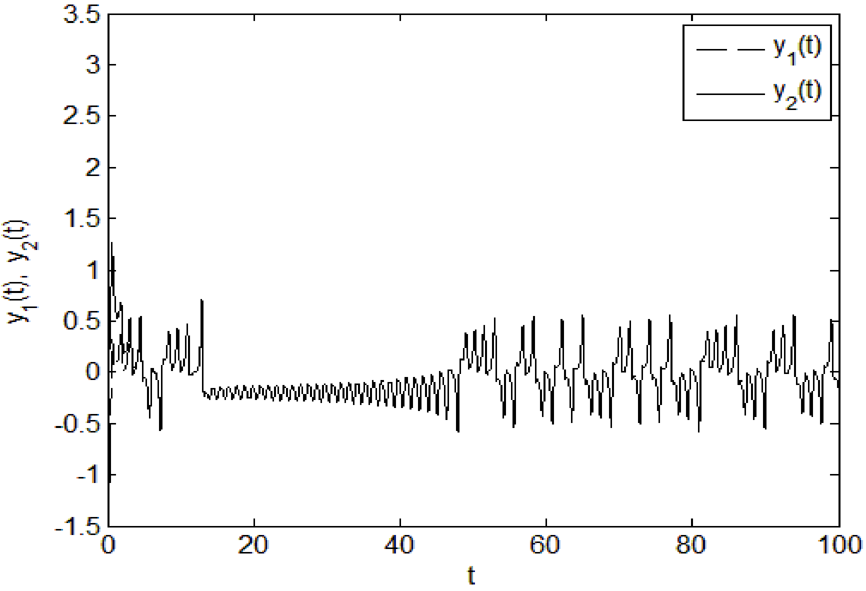


(d)

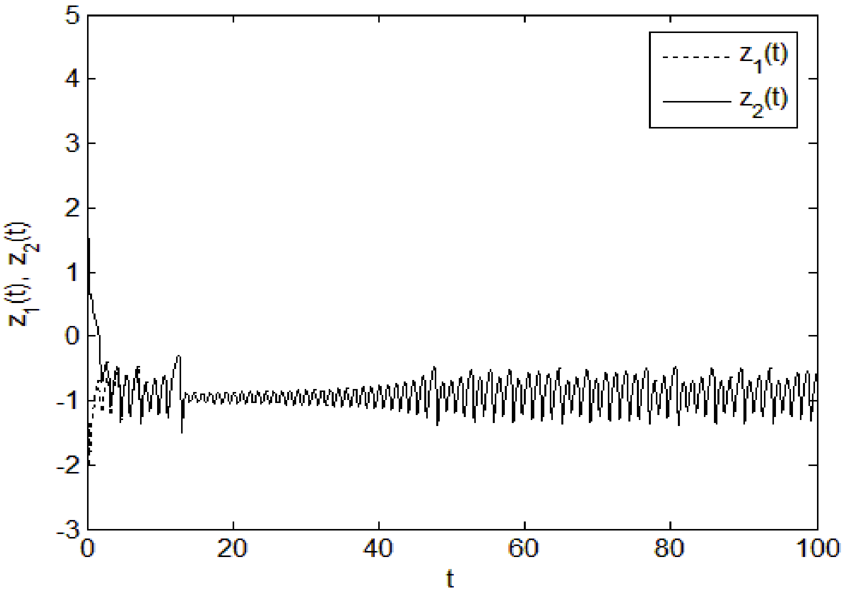
Fig. 2.8 State trajectories of the systems (2.19) and (2.20) for fractional order $q = 0.9$: (a) synchronization between x_1 and x_2; (b) synchronization between y_1 and y_2; (c) synchronization between z_1 and z_2; (d) the evolution of the error functions $e_1(t), e_2(t)$ and $e_3(t)$.



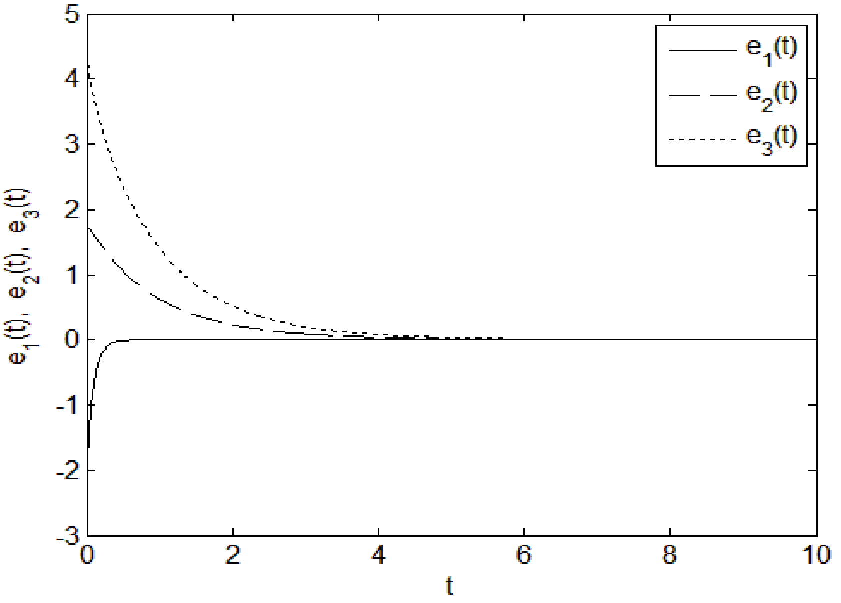
(a)



(b)

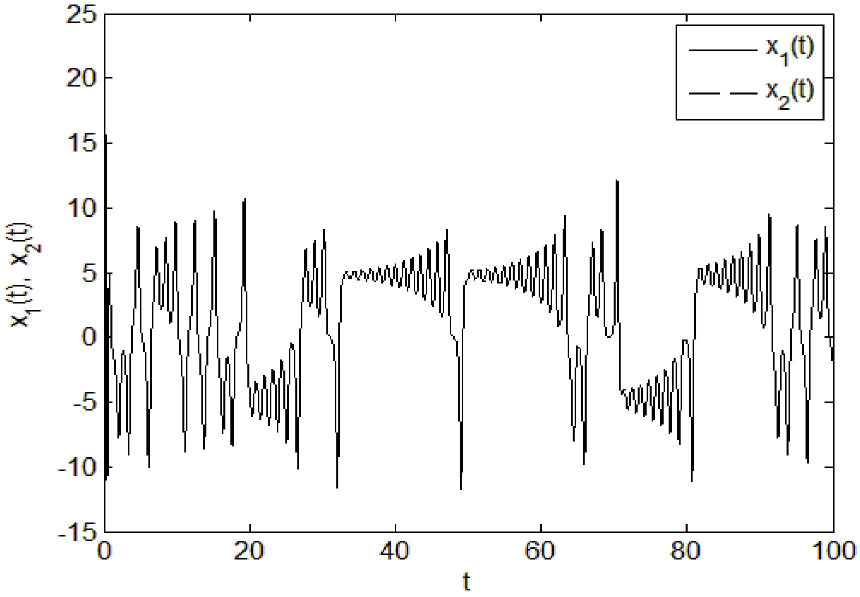


(c)

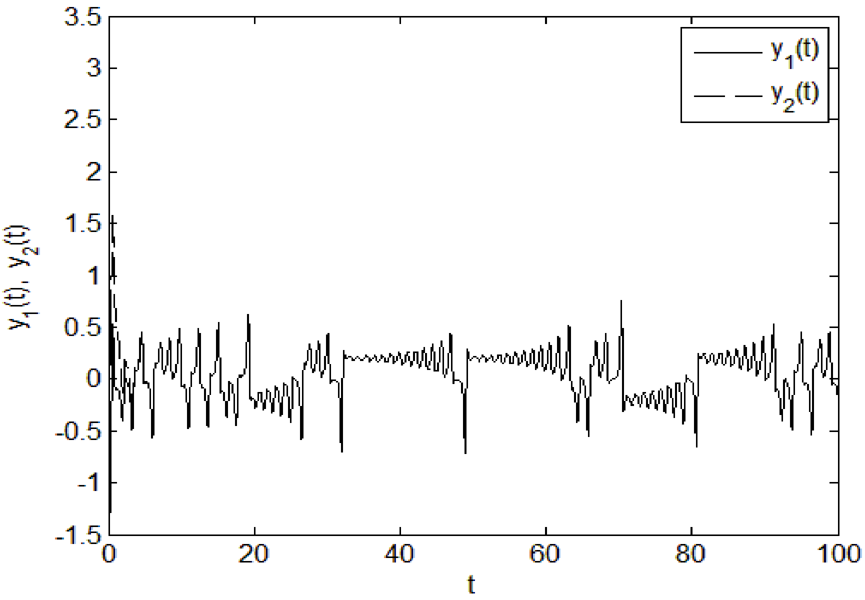


(d)

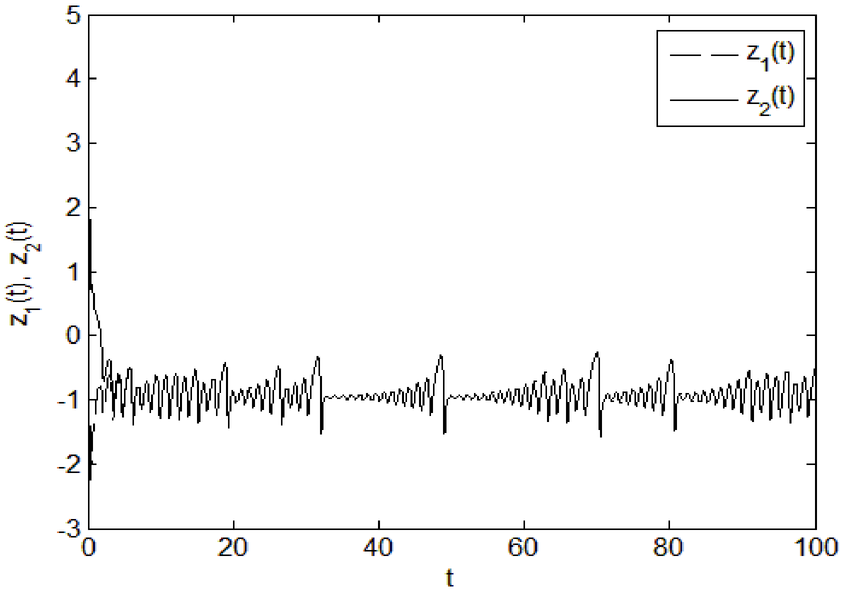
Fig. 2.9 State trajectories of the systems (2.19) and (2.20) for order $q = 0.981$: (a) synchronization between x_1 and x_2; (b) synchronization between y_1 and y_2; (c) synchronization between z_1 and z_2; (d) the evolution of the error functions $e_1(t), e_2(t)$ and $e_3(t)$.


x1(t)
x2(t)
x1(t), x2(t)
t
25
20
15
10
5
0
-5
-10
-15
0
20
40
60
80
100

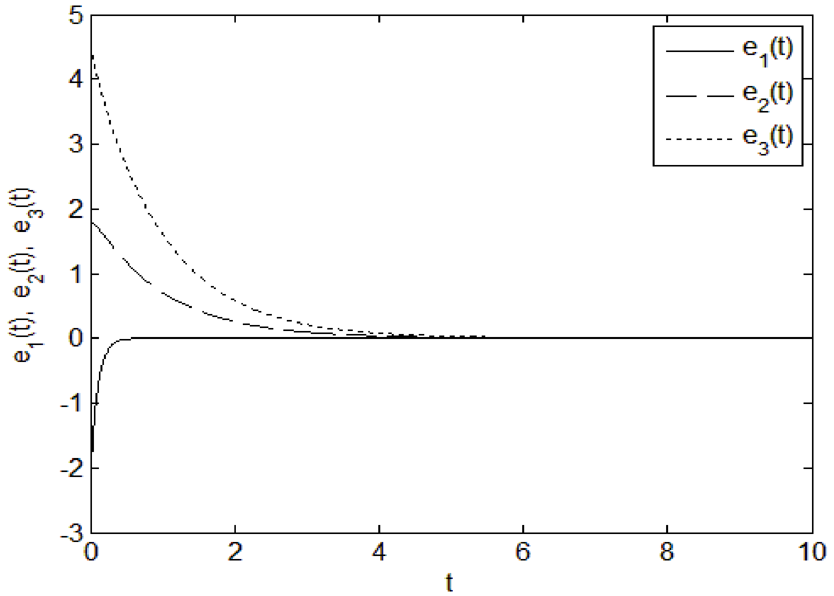
(a)


y1(t)
y2(t)
y1(t), y2(t)
t
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
20
40
60
80
100

(b)



(c)



(d)

Fig. 2.10 State trajectories of the systems (2.19) and (2.20) for $q = 1$: (a) synchronization between x_1 and x_2; (b) synchronization between y_1 and y_2; (c) synchronization between z_1 and z_2; (d) evolution of the error functions $e_1(t), e_2(t)$ and $e_3(t)$.

2.5 Numerical simulation and results

In this section, the earlier considered values of the parameters of systems are taken. The initial conditions of master and slave systems are $(x_1(0), y_1(0), z_1(0)) = (0.1, 1.2, 0.5)$ and $(x_2(0), y_2(0), z_2(0)) = (-2, 3, 5)$ respectively. Hence the initial conditions of error system will be $(e_1(0), e_2(0), e_3(0)) = (-2.1, 1.8, 4.5)$. During synchronization of the systems the time step size is taken as 0.005. The synchronization between $x_1 - x_2$, $y_1 - y_2$ and $z_1 - z_2$ are depicted through Figs. 2.7-2.10 at $q = 0.7,\ 0.9,\ 0.981,\ 1.0$ respectively. The time for synchronization of the considered fractional order chaotic systems clearly exhibits that it takes less time for synchronization when the order of the derivative approaches from standard order to the fractional order.

2.6 Conclusion

Four important goals have been achieved through the analysis of the present study. First one is the stability analysis to locate the range of fractional order beyond which the systems show chaotic behaviour. Second one is the synchronization between the considered fractional order systems and also chaos control of both the systems using nonlinear control method. The third one is the proper design of the control functions so that the error states decay to zero as time approaches infinity which helps to get the required time for synchronization. The most important part of the study is the comparison of time of synchronization through effective numerical simulation and graphical presentations for different particular cases as systems pair approaches from standard order to fractional order. The author believes that the outcome of the results will be appreciated and utilized by the scientists and engineers working in the field of atmospheric science and oceanography.