Chapter 1

Introduction

1.1 History of fractional calculus

The theory of fractional calculus is the generalization of the differentiation and integration of arbitrary order and its origin with classical integral and differential calculus. The beginning of the fractional calculus (Miller and Ross (1993)) is the answer to the G. W. Leibniz's letter to L' Hospital in 1695 raising the following question: “Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?”. In addition, L’ Hospital replied to Leibniz by another question: “What if the order will be 1/2?”, then Leibniz replied through another letter dated September 30, 1695, the exact birth of the fractional calculus! “It will lead to a paradox, from which one day useful consequences will be drawn”. After this question, the fractional derivative has been an ongoing topic for more than 300 years. Many mathematicians viz., J. Liouville, B. Riemann, H. Weyl, J. Fourier, N. H. Abel, S. F. Lacroix, G. W. Leibniz, A. K. Grunwald and A. V. Letnikov contributed to this theory over the years.

Nowadays, the theory of fractional calculus becomes a hot topic of research among scientists, mathematicians, researchers and engineers and it has been successfully applied by them in various scientific and engineering field such as viscoelasticity (Bagley and Calico (1991), Koeller (1984)), fluid mechanics (Kulish and Lage (2002), Das et. al (2010)), material science (Carpinteri et al. (2004)), quantum mechanics

(Yildirim (2011)), colored noise, dielectric polarization (Sun et al. (1984)), electromagnetic wave (Heaviside (1971)), bioengineering (Magin (2004)), biological model (Magin (2010), Gokdogan et al. (2012)), electromechanical system (Sabatier et al. (2004)), etc.

The theory of fractional calculus gives us flexibility for the generalization of the order of the derivative and integration from integer to any real number. Nevertheless, the name “fractional calculus” is kept for the general theory. Again due to the non-local property of fractional order differential operator, it takes into account the fact that the future state depends upon the present state as well as all of the history of its previous states. For this realistic property, the fractional order systems are becoming popular. Another reason behind using fractional order derivatives is that these are naturally related to the systems with memory which prevails for most of the physical and scientific system models. The fractional derivative of a function depends on the values of the function over the entire interval. Thus it is suitable for modelling of the systems with long range interactions both in space and time. Fractional derivative has the flexibility which allows incorporating different types of information. The fractional calculus which was in earlier stage considered as mathematical curiosity now becomes the object for the extensive development of fractional order partial differential equations for its applications in various physical areas of science and engineering. Geometric and physical interpretations of fractional differentiation and fractional integration can be found in Podlubny's work (Podlubny (1999)).

Lemma 1.1 (Norelys et al. (2014)) Let $x(t) \in R$ be a continuous and derivable function. Then for any time instant $t \geq t_0$,

$$\frac{1}{2}\,{}_{t_0}^{c}D_t^q x^2(t) \le x(t)\,{}_{t_0}^{c}D_t^q x(t),\ \forall\ q\in(0,1).$$

1.2 Some definitions of frcational derivative

1.2.1 Grunwald-Letnikov fractional derivative

The Grunwald-Letnikov fractional derivative of order $q>0$ of a function $f(x)$ is defined as

$$ {}_aD_t^q f(x) = \lim_{\substack{h\to 0\\ nh=x-a}} h^{-q}\sum_{k=0}^{n}(-1)^k \binom{q}{k} f(x-kh),$$

where

$$\binom{q}{k} = \frac{q(q-1)(q-2)....(q-k+1)}{k!} = \frac{\Gamma(q+1)}{k!\Gamma(q-k+1)}.$$

1.2.2 Riemann-Liouville fractional integral

The Riemann-Liouville fractional integral operator of order $q>0$ of a function $f(x)$ is defined as

$$J_x^q f(x) = \frac{1}{\Gamma(q)}\int_0^x (x-\xi)^{q-1} f(\xi)\,d\xi,\ \ x>0, \tag{1.1}$$

$$J_x^0 f(x) = f(x).$$

1.2.3 Riemann-Liouville fractional derivative

The Riemann-Liouville fractional derivative operator of order $q>0$ of a function $f(x)$ is defined as

$$D_x^q f(x) = \frac{d^n}{dx^n} J_x^{n-q} f(x),\ \ n-1<q<n\ ,\ n\in \mathrm{N}. \tag{1.2}$$

The basic properties of the Riemann-Liouville fractional integral operator J_x^q for $f \in C_\mu$, $\mu \geq -1$, $p, q \geq 0$ and $\gamma \geq -1$ are given as follows:

(1) $J_x^p J_x^q f(x) = J_x^{p+q} f(x)$,

(2) $J_x^p J_x^q f(x) = J_x^q J_x^p f(x)$,

(3) $J_x^p x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(p+\gamma+1)} x^{p+\gamma}$.

Definition 1.1 A function $f(x)$, $x > 0$ is said to be in the space C_μ^n ,$n \in N_0 = N \cup \{0\}$ iff $f^{(n)} \in C_\mu$.

1.2.4 Caputo fractional derivative

In 1967, Caputo given a definition of fractional derivative of a function $f(x)$, $x > 0$ is defined as

$$
{}^cD_x^q f(x) = \frac{d^q f(x)}{d\,x^q} = J_x^{n-q} \frac{d^n}{dx^n} f(x)
$$

$$
= \frac{1}{\Gamma(n-q)} \int_0^x (x-\xi)^{n-q-1} f^{(n)}(\xi)\, d\xi,\ \ n-1 < q < n\ ,\ n \in \mathrm{N}\ . \tag{1.3}
$$

There are following two basic properties of the Caputo fractional order derivative:

(1) Let $f \in C_{-1}^n$, $n \in N_0$, then ${}^cD_x^q f(x)$, $0 < q \leq n$ is well defined and ${}^cD_x^q f(x) \in C_{-1}$.

(2) Let $n-1 \leq q \leq n, n \in N$ and $f \in C_\mu^n$, $\mu \geq -1$, then

$$
(J_x^q\, {}^cD_x^q)\, f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!},\ x \geq 0.
$$

Definition 1.2 A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \Re$, if there exists a real number $p > \mu$, such that $f(x) = x^p\, f_1(x)$, where $f_1(x) \in C[0,\infty)$.

1.3 Dynamical system

A dynamical system is a system which changes over time according to a set of fixed rules that determine how one state of the system moves to another state. The dynamical system can be described as a system of *n* first-order differential equations as system of motion as

$$\frac{dx_i}{dt} = f_i(x_1,......,x_n,r),\ i = 1,........,n, \tag{1.4}$$

where $x_i(t)$ are dynamical quantities whose time dependence is generated by the equation (1.4), starting from specified initial conditions $x_i(0)$, $i = 1,.......,n$, and *t* is the independent variable which can be read as time. The f_i are nonlinear functions of the $x_i(t)$ and are characterized by the parameter(s) *r*.

Examples of dynamical systems are the solar system (sun and planets), Hamiltonian equations of motion in classical mechanics, the weather, the rate equations for chemical reactions or the evolution equations in population dynamics.

1.3.1 Mathematical definition of dynamical system

A continuously differentiable function $f : R \times R^n \to R^n$ is called dynamical system if it satisfies the following properties

(i) $f(0,x) = x$, for all $x \in R^n$

(ii) $f(t, f(s, x)) = f(t+s, x)$, for all $x \in R^n$ and $t, s \in R$, where R^n is the state space, a member $x \in R^n$ is a state of the system, and $f(t, x)$ is the state, to which the system arrives after time t starting from the state x.

1.4 Classification of dynamical system

There are following types of dynamical systems.

1.4.1 Linear system

Linear systems must verify the following two properties

(i) $f(x+y) = f(x) + f(y)$

(ii) $f(k\,x) = k\,f(x)$.

These two properties are called the superposition and homogeneity respectively for given two different inputs x and y in the domain of the function f and for any real number k.

Any function which does not satisfy superposition and homogeneity conditions is called nonlinear.

The dynamics of linear systems are also can be written in the form of

$$\dot{X} = A(t)\,x, \tag{1.5}$$

where $A(t)$ is an $n \times n$ matrix.

1.4.2 Nonlinear system

A nonlinear dynamical system is described by a set of nonlinear differential equations in the following form

$$\dot{X} = f(x, t), \tag{1.6}$$

where x is the $n \times 1$ state vector, and f is the $n \times 1$ nonlinear vector function. The number of states n is called the order of the system.

1.4.3 Autonomous system

The set of nonlinear differential equations (1.6) is said to be autonomous if f does not depend explicitly on time. In this case system's state equation can be written as

$$\dot{X} = f(x) . \tag{1.7}$$

If f depends on time, then the system is called non-autonomous system.

1.4.4 Discrete time dynamical system

A continuously differentiable function $f : Z \times R^n \to R^n$ is called discrete time dynamical system if it is satisfies the following properties

(i) $f(0, x) = x$, for all $x \in R^n$

(ii) $f(k, f(m, x)) = f(k+m, x)$, for all $x \in R^n$ and $k, m \in Z$,

A discrete time dynamical systems will be derived from a difference equation and the difference equation can be defined by a map. Let $g : R^n \to R^n$ be a continuous function and consider a difference equation

$$y_{k+1} = g(y_k)$$

with initial condition $y_0 = x$. The function $f(k, x) = y_k$ satisfies the properties of the definition of discrete time dynamical system. Thus a difference equation determines a discrete time dynamical system.

1.5 Lyapunov exponent

Lyapunov exponents measure the rate of divergence or convergence of two nearby trajectories. It is a quantitative measure of sensitive dependence on initial condition, which is the silent feature of the Lyapunov exponents.

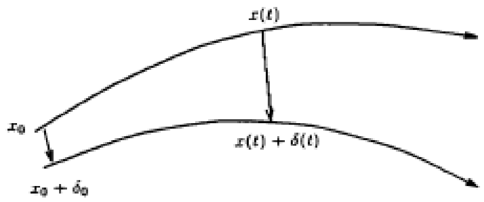


Fig. 1.1 Trajectories starting from two nearby points

Consider initially between two trajectories, at time $t = 0$, there is a point x_0 and a nearby point $x_0 + \delta_0$, where δ_0 is the initial separation between the trajectories and assume extremely small. Suppose $\delta(t)$ represents the separation of trajectories after a period of time t (Fig. 1.1). The exponential growth rate of $\|\delta(t)\|$ is a number λ defined as

$$\|\delta(t)\| \approx \|\delta_0\| e^{\lambda t}.$$

Taking the natural logarithm on both sides of above equation, we obtain

$$\lambda = \frac{1}{t}\ln\left(\frac{\|\delta(t)\|}{\|\delta(0)\|}\right).$$

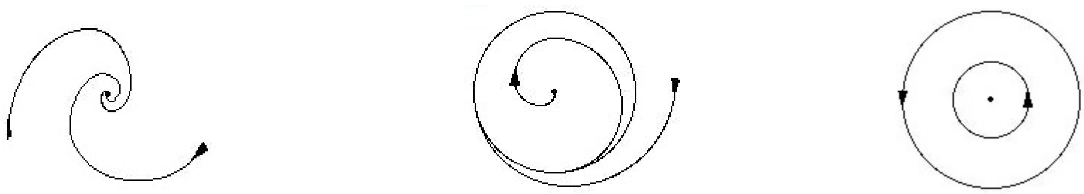
Then λ is treated as a Lyapunov exponent. Now,

(i) If $\lambda < 0$, the trajectory attracts to a fixed stable point or stable to periodic orbit. The Negative Lyapunov exponents are characteristic of dissipative or non-conservative systems, such types of systems exhibit asymptotic stability, if the Lyapunov exponent is more negative, then the systems have the greater stability.

If $\lambda = -\infty$ the systems have super stable fixed points and super stable periodic points.

(ii) If $\lambda = 0$, the trajectories will be a neutral fixed point or an eventually fixed point. Zero Lyapunov exponent indicates that the system is in some sort of steady state mode. A physical system with Lyapunov exponent Zero is conservative. This type of system exhibits Lyapunov stability.

(iii) If $\lambda > 0$, the trajectories will be unstable and chaotic.



(a) (Dissipative)
Trajectories attracting
a fixed point $\lambda < 0$

(b) (Dissipative)
Trajectories attracting
to a periodic orbit $\lambda < 0$

(c) (Conservative)
Neutral Fixed point to
and orbits $\lambda = 0$

A system has many Lyapunov exponents as per the number of dimensions of the phase space. Any system contains atleast one positive Lyapunov exponent is called a chaotic system and if system has more than one positive Lyapunov exponent is called hyperchaotic system.

A more specific and useful formula for the Lyapunov exponent λ can be derived as follows:

If vector form of a dynamical system written as

$$\frac{d\dot{X}(t)}{dt} = f(X(t), r), \tag{1.8}$$

where $f = [f_1, f_2, f_3]^T$ and $X = [x_1(t), x_2(t), x_3(t)]^T$ is the state space vector, r is a set of parameters. The equations for small deviations δX from the trajectory X are

$$\delta \dot{X}(t) = J_{ij}(X(t), r)\, \delta X(t), \qquad i,\ j = 1, 2, 3,$$

where $J_{ij} = \dfrac{\partial f_i}{\partial x_j}$ is the Jacobian matrix. Then an equation for Lyapunov exponents λ_i of the dynamical system (1.8) is defined by

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \log \frac{\left\| \delta x_i(t) \right\|}{\left\| \delta x_i(0) \right\|}.$$

For the calculation of Lyapunav exponents, a simple procedure has been developed by Benettin et al. (1980) which estimates the largest Lyapunov exponent directly from the equations governing the system. Wolf et al. (1985) generalized Bennetin's method to time series data, known as Wolf's method. Although Wolf's article only discussed the computation of non-negative Lyapunov exponents, it can be used effectively to compute the largest Lyapunov exponent of a chaotic system. Frank et al. (1990) improved Wolf's method to compute the largest Lyapunov exponent in a wider range of chaotic systems. After that Janaki et al. (1998) had given different methods of computing Lyapunov exponents for continuous-time dynamical systems. If a dynamical system has positive Lyapunov exponents, then nearby trajectories separate exponentially fast and are unpredictable. Therefore, positive Lyapunov exponent for a particular dynamical system is a quantitative measure of chaos.

1.6 Stability of the systems

Consider an autonomous system of the following form

$$\frac{dx(t)}{dt} = f(x(t)) \text{ with } x(0) = x_0, \tag{1.9}$$

where $x(t) \in D \in R^n$ is the state vector of the system, D is the neighborhood of the equilibrium point. and $f : D \to R^n$ continuous function on D. Let x_e is the equilibrium point of the function f so $f(x_e) = 0$, then

(i) The equilibrium point x_e is said to be Lyapunov stable, if for every $\varepsilon > 0$ there exist a $\delta = \delta(\varepsilon)$ such that, if $\|x(0) - x_e\| < \delta$, then for every $t \geq 0$ we have $\|x(t) - x_e\| < \varepsilon$.

Lyapunov stability of system (1.9) of an equilibrium point x_e means that solutions of systems starting "close enough" to the equilibrium (within a distance δ from it) remain "close enough" forever (within a distance ε from it). It must be true for any ε which one may want to choose.

(ii) The equilibrium point x_e of the system (1.9) is said to be asymptotically stable, if it is Lyapunov stable and there exists $\delta > 0$ such that, if $\|x(0) - x_e\| < \delta$, then $\lim_{t \to \infty} \|x(t) - x_e\| = 0$.

Asymptotic stability of the system (1.9) means that solutions which start close enough not only remain close enough but also eventually converge to equilibrium.

(iii) The equilibrium point x_e of the system (1.9) is said to be exponentially stable, if it is asymptotically stable and there exists $\alpha > 0$, $\beta > 0$, $\delta > 0$ such that, if $\|x(0) - x_e\| < \delta$, then $\|x(t) - x_e\| \leq \alpha \|x(0) - x_e\| e^{-\beta t}$, for $t \geq 0$.

Exponential stability means, the solutions of systems not only converge, but in fact converge faster than or at least as fast as a rate $\alpha \|x(0) - x_e\| e^{-\beta t}$.

1.6.1 Lyapunov first method

Theorem 1.1. Let $x=0$ be an equilibrium point of a nonlinear system (1.9) and consider λ_i $(i=1,\ldots,n)$ are the eigenvalues of the matrix $A=\left.\frac{\partial f}{\partial x}\right|_{x=0}$, where A is the Jacobin matrix of the system.

(i) If $\operatorname{Re}\lambda_i<0$ for all i then $x=0$ is asymptotically stable for the nonlinear system.

(ii) If $\operatorname{Re}\lambda_i>0$ for one or more i, then $x=0$ is unstable for the nonlinear system.

(iii) If $\operatorname{Re}\lambda_i<0$ for all i and at least one $\operatorname{Re}\lambda_i=0$, then $x=0$ may be either stable, asymptotically stable or unstable for the nonlinear system.

1.6.2 Lyapunov second method (Lyapunov direct method)

Theorem 1.2 Let $x=0$ be an equilibrium point of a nonlinear system (1.9). Let $V(x):D\to R$ be a positive definite continuously differentiable function on a neighborhood D of $x=0$, such that $\dot{V}(x)\le 0$ in D. Then, the equilibrium point $x=0$ is stable. Moreover, if $\dot{V}(x)<0$ in $D-\{0\}$, then the point $x=0$ is said to be asymptotically stable. $V(x)$ is called a Lyapunov function.

Theorem 1.3 Let $x=0$ be an equilibrium point of a nonlinear system (1.9). Consider $V:R^n\to R$ be a positive definite continuously differentiable function, such that $V(x)\to\infty$ as $\|x\|\to\infty$ and $\dot{V}(x)\le 0$, $\forall\ x\ne 0$, then $x=0$ is globally asymptotically stable.

1.7 Stability of the fractional order systems

Consider a fractional order dynamical system as

$$D_t^q x(t) = f_1(x,y,z)$$
$$D_t^q y(t) = f_2(x,y,z) \tag{1.10}$$
$$D_t^q z(t) = f_3(x,y,z),$$

where $q \in (0,1)$ and D_t^q is the Caputo derivative. The Jacobian matrix at equilibrium points of the above system is

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix} \tag{1.11}$$

Theorem 1.4 (Matignon (1996), Li and Ma (2013)) The system (1.9) is locally asymptotically stable if all the eigenvalues of the Jacobian matrix at its equilibrium point satisfy the condition

$$|\arg(\lambda)| > \frac{q\pi}{2}. \tag{1.12}$$

The characteristic equation of the Jacobian matrix (1.11) at their equilibrium points will be

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 \tag{1.13}$$

and its discriminant is given by

$$D(P) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3a_1^3 - 4a_2^3 - 27a_3^2. \tag{1.14}$$

The fractional order Routh-Hurwitz conditions (Srivastava et al. (2014a), Ahmed et al. (2006)) for the system are given as

(a) If $D(P) > 0,$ then the necessary and sufficient conditions for the equilibrium point to be locally asymptotically stable if $a_1 > 0, a_3 > 0, a_1 a_2 - a_3 > 0.$

(b) If $D(P) < 0, a_1 \geq 0, a_2 \geq 0, a_3 > 0,$ then the equilibrium point is locally asymptotically stable for $q < 2/3.$ However, if $D(P) < 0, a_1 < 0, a_2 < 0, q > 2/3,$ then all the roots of equation (1.13) satisfy the condition $|\arg(\lambda)| < \frac{q\pi}{2}.$

(c) If $D(P) < 0, a_1 > 0, a_2 > 0, a_1 a_2 - a_3 = 0,$ then the equilibrium point is locally asymptotically stable for all $0 < q < 1.$

(d) The necessary condition for the equilibrium point to be locally asymptotically stable if $a_3 > 0.$

In studying the existence of chaotic attractors and the synchronization between fractional order systems, the previous stability results play an important role.

The fractional differential system will be stable, if all the eigenvalues of the Jacobian matrix are calculated at equilibrium point of the outside of the sector region. According to the above results, stable and unstable regions with order q are shown in Fig. 1.2. From the figure it is clear that stability sector region for the fractional order system is greater than for integer order system. The stability region for an integer system will be only left side of imaginary axis while for fractional order system, it will be in the right side also.

Definition 1.3 (Faieghi and Delavari (2012)) An equilibrium point E of a system is called a saddle point of index 1 if the Jacobian matrix at point E has one eigenvalue with a non-negative real part (i.e., unstable).

Definition 1.4 An equilibrium point E of a system is called a saddle point of index 2 if the Jacobian matrix at point E has two unstable eigen values.

The scrolls are generated only around the saddle points of index 2. Saddle point of index 1 is responsible only for connecting scrolls.

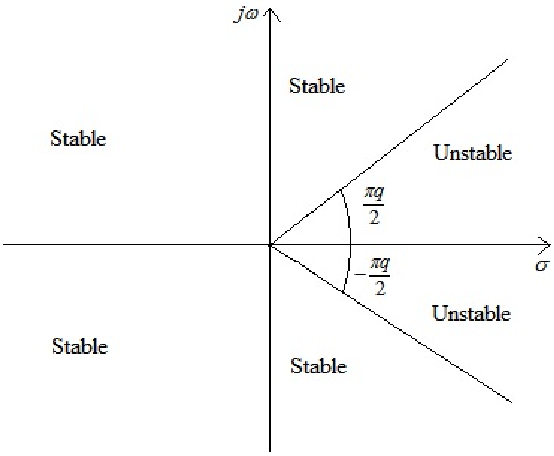


Fig. 1.2 Stability region of linear fractional-order system with order q.

1.8 Definition of chaos

1.8.1 Devaney's definition of chaos (Devaney (1989))

Let X be a set. A continuous map $f: X \to X$ is said to be chaotic on X if

(i) f has sensitive dependence on initial conditions.

(ii) f is transitive.

(iii) the periodic points of f are densed in X.

A map $f: J \to J$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in J$ and any neighborhood N of x, there exists $y \in N$ and $n \geq 0$ such that $\left| f^n(x) - f^n(y) \right| > \delta$. The sensitive dependence on initial conditions means that if there exists a positive real number δ (a sensitive constant) such that for every point x in X and every neighborhood N of x there exists a point y in N and a non-negative integer n such that the n^{th} iterates $f^n(x)$ and $f^n(y)$ of x and y are more than δ distance.

A map $f : J \to J$ is said to topologically transitive if for any pairs of open sets $U, V \subset J$ there exists $k > 0$ such that $f^{(k)}(U) \cap V \neq \phi$.

f is transitive means for all non-empty subsets U, V of J there exists a natural number $k > 0$ such that $f^{(k)}(U) \cap V$ is nonempty. Consequently, the dynamical systems cannot be decomposed into two disjoint open sets which are invariant under the map. It will be notable that if a map possesses a dense orbit, then it is clearly topological transitive.

Definition 1.5

In 1994 Strogatz mentioned chaos by following quotation as “Chaos is a periodic long-term behaviour, in a deterministic system that exhibits sensitive dependence on initial condition”.

Aperiodic long-term behaviour means that there are trajectories which do not settle down to fixed points, periodic orbits, or quasi-periodic orbits as time tends to infinity. Deterministic means that the system has no random or noisy inputs or parameters. This irregular behaviour arises from the system's nonlinearity, rather than from noisy driving forces. Sensitive dependence on initial conditions means that a small change in the initial state will lead to progressively larger changes in later system states or each point in such a system is arbitrarily closely approximated by other points with significantly different future trajectories.

1.9 Historical background of chaos

The chaos theory was started in 19-th century in the study of problem of Henri Poincaré on the motion of three objects in mutual gravitational attraction (e.g. a star and two planet) the so-called three-body problem. By considering the behavior of the orbits arising from sets of initial points, Poincare was able to show that orbits are aperiodic, and yet not increasing infinitely (deterministic) nor approaching any fixed point or limit cycles.

In 1898 J. Hadamard observed general divergence of trajectories in spaces in terms of negative curvature. He was able to show that all trajectories are unstable. In that case all particle trajectories diverge exponentially from one another, with a positive Lyapunov exponent.

In 1963, the meteorologist E. N. Lorenz (Lorenz (1963)) discovered the butterfly effect while trying to forecast the weather. He did a computer simulation of a set of simplified differential equations for fluid convection in which he saw complicated behavior that seemed to depend sensitively on initial conditions. Lorenz also showed that the solution settled down in a fascinating butterfly shaped set of points, which caused the consideration of this high level sensitive dependence on initial conditions to become commonly known as “butterfly effect”. He concluded that the earth’s weather is a chaotic and therefore a long-range prediction is an impossible task.

In 1971, David Ruelle and Floris Takens described "strange attractors" in an alternative mathematical explanation of the turbulence in fluid dynamics based (Ruelle and Takens (1971)).

In 1975, Li and Yorke (1975) explained the sustained aperiodic and unpredictable behaviors arising in deterministic nonlinear maps. In the research article, they described the term chaos for the various phenomena that demonstrated aperiodicity along with sensitive dependence on initial condition.

Currently, the chaos theory becomes an active research topic for last few decades to the researchers working in the area of nonlinear dynamical system and has many useful applications in many areas of engineering such as digital communication, secure communication, power electric and power quality, biological systems, chemical reactions analysis and design and information processing etc.

1.10 Synchronization

Synchronization of chaotic systems is the phenomenon that may occur when two or more chaotic oscillators are coupled. It is also a process in which two or more chaotic systems (identical or non-identical) adjust a given property of their motion to a common behavior, due to coupling or forcing. This is a difficult problem due to their extremely sensitive dependence on initial conditions. Any initial correlation presents between identical systems, starting from very close initial conditions exponentially decrease to zero with time. Mathematically synchronization is achieved when the difference of state vectors of master and slave systems converges to zero when time approaches to infinity i.e. $\lim_{t\to\infty} \|x(t) - y(t)\| = 0$, where $x(t)$ and $y(t)$ are the state vectors of the master and slave systems respectively.

1.11 Types of synchronization

The different types of synchronization are given below:

1.11.1 Synchronization/Complete synchronization

In complete synchronization, the differences of state variables of synchronized systems with different initial values converge to zero when time approaches to infinity. This is observed in coupled chaotic systems and chaotic systems with noise perturbation.

Mathematically, consider two chaotic systems as

$$\dot{x}(t) = f(x(t)) \tag{1.15}$$

$$\dot{y}(t) = g(y(t)) + u(x(t),\ y(t)), \tag{1.16}$$

where $x(t),\ y(t) \in R^n$ are the state vectors, $u(x(t),\ y(t))$ is the controller and $f, g : R^n \to R^n$ is continuous nonlinear vector function.

The systems (1.15) and (1.16) are said to be synchronized when $\lim_{t\to\infty} \|y(t) - x(t)\| = 0$, for initial conditions of $x(0)$ and $y(0)$.

1.11.2 Anti-synchronization

Two chaotic systems are said to be anti-synchronized, when the respective states of chaotic systems have the same magnitude but opposite in sign.

Mathematically the anti-synchronization of two systems is achieved, when $\lim_{t\to\infty} \|y(t) + x(t)\| = 0$, where $x(t)$ and $y(t)$ are the state vectors of the drive and response systems respectively.

1.11.3 Phase synchronization

In phase synchronization the coupled chaotic systems keep their phase difference bounded by a constant while their amplitudes remain uncorrelated. The phase synchronization, usually applied upon two waveforms of the same frequency with identical phase angles with each cycle. However, it can be applied if there is an integer relationship of frequency such that the cyclic signals share a repeating sequence of phase angles over consecutive cycles.

1.11.4 Hybrid synchronization

Hybrid synchronization for drive and response systems of three states are defined in this way, that first and third states of the two systems are synchronized, and the second state of the systems are anti-synchronized so in hybrid synchronization, synchronization and anti- synchronization co-exist in the system.

1.11.5 Projective synchronization

A synchronization technique in which the master and slave systems synchronize up to a constant scaling factor $\beta,$ known as Projective synchronization. Consider the following master system and slave system with control function as

$$\dot{x} = F(x) \tag{1.17}$$

$$\dot{y} = G(y) + u(x, y), \tag{1.18}$$

where $x, y \in R^n$ are the state vectors, $u(x, y)$ is the vector controller and $f, g : R^n \to R^n$ are continuous nonlinear vector functions. Next define the error system as

$$e(t) = y(t) - \beta\, x(t), \tag{1.19}$$

where β is the real constant. Then the systems (1.17) and (1.18) are said to be projective synchronized if the error system (1.19) is asymptotically stable i.e., $\lim_{t\to\infty}\|e(t)\| = 0$.

In particular, if $\beta = 1$ and $\beta = -1$, the projective synchronization is further simplified to complete synchronization and anti-phase synchronization respectively.

In 1999 Mainieri and Rehacek (1999) introduced the Projective synchronization in partially linear systems, where the responses of two identical systems synchronize up to a constant scaling factor.

1.11.6 Complex projective synchronization

Let us consider two complexes nonlinear systems of which first one is master (drive) system as

$$\dot{x}_m = Ax_m + f(x_m),$$

where $x_m = x_{m_1} + jx_{m_2}$ is complex state variable with

$$\dot{x}_{m_1} = Ax_{m_1} + f_1(x_{m_1})$$

$$\dot{x}_{m_2} = Ax_{m_2} + f_2(x_{m_2}) \tag{1.20}$$

and the second is response (slave) system as

$$\dot{y}_s = By_s + g(y_s) + u(t),$$

where $y_s = y_{s_1} + jy_{s_2}$ is complex state variable and $u(t) = u_1^r(t) + ju_2^i(t)$ is control function with

$$\dot{y}_{s_1} = By_{s_1} + g_1(y_{s_1}) + u_1^r(t)$$

$$\dot{y}_{s_2} = By_{s_2} + g_2(y_{s_2}) + u_2^i(t)\,, \tag{1.21}$$

where $u_1^r(t) = (u_1, u_3, ..., u_{2n-1})^T$ and $u_2^i(t) = (u_2, u_4, ..., u_{2n})^T$. Now complex error function e_r between drive and response systems are defined for complex projective synchronization as

$e_r = e_{r_1} + je_{r_2} = y_s - Mx_m,$

where $M = M_1 + jM_2 = diag(\phi_1, \phi_2, ..., \phi_n)$, $\phi_l = m_l^r + jm_l^i$, $l = 1, 2, ..., n$.

Then the error function will be

$e_{r_1} = y_{s_1} - M_1 x_{m_1} + M_2 x_{m_2}$

$e_{r_2} = y_{s_2} - M_1 x_{m_2} + M_2 x_{m_1}$.

Case-I: If $\lim_{t\to\infty} e_{r_1} = \lim_{t\to\infty} \left\| y_{s_1} - M_1 x_{m_1} + M_2 x_{m_2} \right\| = 0$ and $\lim_{t\to\infty} e_{r_2} = \lim_{t\to\infty} \left\| y_{s_2} - M_1 x_{m_2} - M_2 x_{m_1} \right\| = 0$, $e_{r_1} = (e_1, e_3, ..., e_{2n-1})^T$ and $e_{r_2} = (e_2, e_4, ..., e_{2n})^T$, then the complex projective synchronization is obtained between the systems (1.20) and (1.21).

Case-II: When $\phi_1 = \phi_2 = ... = \phi_n = j$, then complex complete synchronization between master and response systems can be obtained.

Case-III: If we take $\phi_1 = \phi_2 = ... = \phi_n =$ a real number, then the projective synchronization between complex systems (1.20) and (1.21) can be obtained.

1.11.7 Function projective synchronization

The Function projective synchronization is generally known as FPS in which drive and response systems are synchronized up to a desired scaling function. FPS is one of the important synchronization methods that has been widely investigated to obtain faster communication with its proportional feature.

Consider the following master system and slave system with control function as

$$\dot{x} = F(x) \tag{1.22}$$

$$\dot{y} = G(y) + u(x, y) \ , \tag{1.23}$$

where $x, y \in R^n$ are the state vectors, $u(x, y)$ is the vector controller and $f, g : R^n \to R^n$ are continuous nonlinear vector functions, defining the error system as

$$e(t) = y(t) - \beta(t)\,x(t)\,, \tag{1.24}$$

where $\beta(t)$ is the continuously differentiable function with $\beta(t) \neq 0 \;\forall\; t$.

Then the systems (1.22) and (1.23) are said to be function projective synchronized, if there exists a scaling function $\beta(t)$ such that $\lim_{t\to\infty}\|e(t)\| = 0$.

1.11.8 Dual synchronization

We consider first two master (drive) system as

Master systems-I:

$$\dot{X} = f(X), \tag{1.25}$$

where X is state variable.

Master system-II:

$$\dot{Y} = g(Y), \tag{1.26}$$

where Y is state variable.

The linear combination of the master systems I & II, gives rise to

$$V_m = \sum_{i=1}^{n} a_i X_i + \sum_{i=1}^{n} b_i Y_i = [a_1, a_2, ...a_n]X + [b_1, b_2, ...b_n]Y = A^T X + B^T Y = [A^T \; B^T]\begin{bmatrix} X \\ Y \end{bmatrix} + C^T \xi,$$

where $A = [a_1, a_2, ...a_n]^T$ and $B = [b_1, b_2, ...b_n]^T$ are known and $C = [A^T \;\; B^T]^T$.

The next two response (slave) system as

Response system-I:

$$\dot{x} = f(x) + u^{(1)}, \tag{1.27}$$

where x is state variable with

Responce system-II:

$$\dot{y} = g(y) + u^{(2)}, \tag{1.28}$$

where y is state variable and $u^{(1)}(t)$, $u^{(2)}(t)$ are control functions, $u^{(i)}(t) = [u_1^{(i)}, u_2^{(i)}, \ldots, u_n^{(i)}]^T$, $i = 1, 2$

The linear combination gives

$$V_s = \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i y_i = [a_1, a_2, \ldots. a_n]x + [b_1, b_2, \ldots. b_n]y = A^T x + B^T y = [A^T \; B^T] \begin{bmatrix} X_m \\ Y_m \end{bmatrix} + C^T \eta.$$

The goal to obtain the dual synchronization between master and slave systems. Now defining the error function between the master (1.25), (1.26) and response (slave) systems (1.27), (1.28) as

$$e = V_s - V_m,$$

which gives rise to $\lim_{t\to\infty}\|e\| = \lim_{t\to\infty}\|\eta - \xi\| = 0.$

The master systems (1.25), (1.26) and response systems (1.27), (1.28) are said to be dual function projective synchronized if $\lim_{t\to+\infty}\|e\| = 0$, where $\|\cdot\|$ denotes matrix norm.

1.11.9 Combination synchronization

In combination synchronization, we are assuming two drive systems and one response system.

The master systems are considered as

$$\dot{x}_1 = f_1(x_1) \tag{1.29}$$

$$\dot{x}_2 = f_2(x_2) \tag{1.30}$$

and the response system is taken as

$$\dot{y} = f(y) + U(x_1, x_2, y), \tag{1.31}$$

where $x_1 = (x_1^1, x_2^1, ..., x_n^1)$, $x_2 = (x_1^2, x_2^2, ..., x_n^2)$ and $y = (y_1, y_2, ..., y_n)$ with $x_1, x_2, ..., x_{n-1}, x_n \in R^n$ are the state vectors of the chaotic systems. $f_1, f_2, f : R^n \to R^n$ are continuous vector functions and $U(x_1, x_2, y)$ is a controller.

The master systems (1.29), (1.30) and one response system (1.31) are said to be combination synchronization if there exists three constants matrixes called scaling matrixes $A_1, A_2, A_3 \in R^n$ and $A_3 \neq 0$ such that

$\lim_{n\to+\infty} \|A_1 x_1 + A_2 x_2 - A_3 y\| = 0$, where $\| \, . \, \|$ represents the matrix norm.

It is noted that if $A_1 \neq 0, A_2 = 0, A_n = I$ then this problem is reduced to the projective synchronization, where I is an $n \times n$ identity matrix. If the scaling matrix A_1 is considered as a function, then synchronization problem is reduced into function projective synchronization problem.

1.11.10 Dual combination synchronization

In this section the dual combination synchronization is proposed among four master and two slave systems. First two master systems are defined in equation (1.25) and (1.26). Next two master systems are defined as

Master systems-I:

$$\dot{X}' = f(X'), \tag{1.32}$$

where X' is state variable.

Master system-II:

$$\dot{Y}' = g(Y'), \tag{1.33}$$

where Y' is state variable.

The linear combination of the master systems I & II, gives rise to

$$\begin{aligned} V'_m &= \sum_{i=1}^{n} a_i X'_i + \sum_{i=1}^{n} b_i Y'_i = [a_1, a_2, ...a_n]X' + [b_1, b_2, ...b_n]Y' = A^T X + B^T Y \\ &= [A^T \; B^T] \begin{bmatrix} X' \\ Y' \end{bmatrix} + C^T \xi'. \end{aligned}$$

Now, the corresponding two response (slave) systems with control functions are described in equations (1.27) and (1.28).

Now defining the error function between the master systems (1.25), (2.26) and (1.32), (1.33) and slave systems (1.27), (1.28) as

$$e = V_s - V_m - V'_m,$$

two get $\lim_{t\to\infty}\|e\| = \lim_{t\to\infty}\|\eta - \xi - \xi'\| = 0$.

The master systems (1.25), (1.26), (1.32) and (1.33), and the slave systems (1.27) and (1.28) are said to be in dual combination synchronized, if $\lim_{t\to+\infty}\|e\| = 0$, where $\|\cdot\|$ denotes matrix norm.

1.12 Methods for synchronization

1.12.1 Active control method

The active control method was first proposed by E. W. Bai and K. E. Lonngren (Bai and Lonngren, (1997)) in 1997, and synchronizes the identical Lorenz chaotic system using active control method. After this, in 2000, Bai and Lonngren (2000) showed the sequential synchronization of two Lorenz systems using this method. In 2000, the active control method successfully applied for synchronization of two different chaotic systems viz., easy periodic system and Rossler system by Ho and Hung (2002). In 2007, Li and Yan (2007) investigated chaos synchronization of fractional order Lorenz, Rossler and Chen systems taking one system as master and other as slave system. In 2008, Vincent (2008) presented chaos synchronization between two nonlinear systems using two different techniques viz., active control and back stepping control in terms of transient analysis. In the same year, Zhou and Cheng (2008) showed synchronization between different fractional order chaotic systems viz., Rossler & Chen systems and Chua & Chen systems. Recently, Srivastava et. al. (2014b) have successfully applied the active control method for anti-synchronization between identical and non-identical fractional order chaotic systems. The active control method has received huge attention during the last few years.

The active control method for synchronization of two identical chaotic systems can be illustrated using chaotic Lu system. The Lu system is described by following set of nonlinear differential equations as

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= -xz+cy \\ \dot{z} &= xy-bz. \end{aligned} \tag{1.34}$$

The system (1.34) is chaotic for the parameter values $a=36, b=3$ and $c=20$. Here The aim is to make synchronization of system (1.34) by using active control method. The master (drive) system is taken as

$$\dot{x}_1 = a(y_1 - x_1)$$

$$\dot{y}_1 = -x_1 z_1 + c y_1 \tag{1.35}$$

$$\dot{z}_1 = x_1 y_1 - b z_1$$

and the slave (response) system as

$$\dot{x}_2 = a(y_2 - x_2) + u_1(t)$$

$$\dot{y}_2 = -x_2 z_2 + c y_2 + u_2(t) \tag{1.36}$$

$$\dot{z}_2 = x_2 y_2 - b z_2 + u_3(t),$$

where $u_1(t)$, $u_2(t)$ and $u_3(t)$ are control functions. For synchronization, defining the error states as

$$e_1 = x_2 - x_1$$

$$e_2 = y_2 - y_1 \tag{1.37}$$

$$e_3 = z_2 - z_1.$$

The error dynamical system is obtained from (1.35) and (1.36) as

$$\dot{e}_1 = a(e_2 - e_1) + u_1(t)$$

$$\dot{e}_2 = c e_2 + x_1 z_1 - x_2 z_2 + u_2(t) \tag{1.38}$$

$$\dot{e}_3 = -b e_3 + x_2 y_2 - x_1 y_1 + u_3(t).$$

The active control function is $u_1(t)$, $u_2(t)$ and $u_3(t)$ are defined as

$$u_1(t) = v_1(t)$$

$$u_2(t) = -x_1 z_1 + x_2 z_2 + v_2(t) \tag{1.39}$$

$$u_3(t) = -x_2 y_2 + x_1 y_1 + v_3(t).$$

Hence

$$\dot{e}_1 = a(e_2 - e_1) + v_1(t)$$

$$\dot{e}_2 = c e_2 + v_2(t) \tag{1.40}$$

$\dot{e}_3 = -be_3 + v_3(t)$,

where $v_1(t), v_2(t)$ and $v_3(t)$ are the linear control functions, which are the functions of e_1, e_2 and e_3. There are many possible choices for the control functions $v_1(t), v_2(t)$ and $v_3(t)$. Let us choose

$$\begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} = A \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \tag{1.41}$$

where A is a 3×3 constant matrix. For the system (1.40) to be asymptotically stable, the elements of the matrix A are properly chosen so that the closed loop system (1.40) will have all eigen values with negative real parts. There is no unique choice for matrix A. A good choice can be as follows

$$A = \begin{bmatrix} a-1 & -a & 0 \\ 0 & -c-1 & 0 \\ 0 & 0 & b-1 \end{bmatrix}.$$

For this particular choice, the closed loop system (1.40) has eigenvalues those are found to be $-1, -1$ and -1. This choice will lead to the error states e_1, e_2 and e_3 converge to zero as time t approaches to infinity and this implies that the synchronization of two chaotic Lu systems is achieved. The active control method realizes robust synchronization of two identical chaotic systems. This method is simple and easy to implement in practical applications.

1.12.2 Nonlinear control method

In 2005, J. H. Park (Park (2005)) studied the chaos synchronization of chaotic systems via nonlinear control method. Dong et al. (2006) studied synchronization of the hyperchaotic Rossler system with uncertain parameter using the same method in the year 2006. In 2009, Xin (2009) proposed the projective synchronization using this method. The method was successfully used by Li and Ge (2011) during the study of pragmatical adaptive synchronization of different orders chaotic systems with uncertain parameters and also by Singh et al. (2014) during synchronization and ant-synchronization of chaotic systems.

The procedure of the method for synchronization is given below

First consider the fractional order chaotic system as the master system as

$$\dot{x}_i = Px_i + Qf(x_i),\ 0 < q < 1,\ i = 1, 2, ..., n, \tag{1.42}$$

where $x_i = [x_1, x_2, ...x_n]^T \in R^n$ is the state vectors, P and Q are the $n \times n$ matrix of the system parameters and $f : R^n \to R^n$ is a nonlinear function of the system.

Consider another fractional order chaotic system described by the dynamic as a slave system as

$$\dot{y}_i = P_1 y_i + Q_1 g(y_i) + u_i(t), \quad i = 1, 2, ..., n, \tag{1.43}$$

where $y_i = [y_1, y_2, ...y_n]^T \in R^n$, is the state vector of the system, P_1 and Q_1 are the $n \times n$ matrix of the system parameters, $g : R^n \to R^n$ is a nonlinear function of the system and $u_i(t)$ are the control function of the system.

Defining the error states as $e_i = y_i - x_i, i = 1, 2, ..., n$, the error system becomes

$$\dot{e}_i = P_1 e_i + Q_1 g(y_i) + (P_1 - P)x_i - Qf(x_i) + u_i(t). \tag{1.44}$$

During the synchronization the aim is to find the appropriate feedback controller $u_i(t)$ so that the error dynamics (1.44) can be stabilized in order to get $\lim_{t\to\infty} \|e(t)\| = 0$, for all $e(0) \in R^n$.

Now, defining the Lyapunov function as

$V(e_i)=\frac{1}{2}e_i^T e_i$, with $e_i(t)=[e_1,e_2,.....,e_n]^T$.

The derivative of $V(e_i)$ w. r. to t is

$$\frac{dV(e_i)}{dt}=\frac{1}{2}\frac{d(e_i^T e_i)}{dt}$$

$$=\frac{1}{2}\frac{d}{dt}(e_1^2+e_2^2+......+e_n^2)$$

$$=\sum_{i=1}^{n}\frac{1}{2}\frac{de_i^2}{dt}$$

$$=\sum_{i=1}^{n}e_i\frac{de_i}{dt}.$$

Choosing the control functions as $u_i(t)=-(P_1+1)e_i-(P_1-P)x_i-Q_1g(y_i)+Qf(x_i)$, we get

$$\frac{dV(e_i)}{dt}=-\sum_{i=1}^{n}e_i{}^2, \tag{1.45}$$

which shows that the Lyapunov function $V(e_i)$ becomes negative definite so as to get the required synchronization of the systems (1.42) and (1.43).

1.12.3 Backstepping method

The backstepping design is a recursive procedure which combines choice of Lyapunov function with the design of feedback control functions. There are many advantages of the method as it is a systematic procedure for controlling chaotic dynamic. It can be applied over circuits and systems. In 1999, Mascolo and Grassi (1999) have controlled chaotic dynamic using backstepping design while its application to the Lorenz system and Chua's circuit. In 2001, Wang and Ge (2001) proposed the Adaptive synchronization of uncertain chaotic systems via backstepping design. In the same year, Lu and Zhang (2001) controlled the Chen's chaotic attractors using backstepping design based on parameters identification. In the year 2003, Tan et al. (2003) synchronized the chaotic systems using backstepping design and again in the same year Yu and Zhang (2003) controlled the uncertain behavior of chaotic systems using backstepping design. Recently, Park (2006) and Wu et al. (2009) have shown that the backstepping method is very simple, reliable and powerful for controlling the chaotic behavior and synchronization of chaotic systems.

The backstepping design method for synchronization of two identical chaotic systems can be illustrated using chaotic Lorenz system. The Lorenz system is described by following set of nonlinear differential equations as

$$\begin{aligned} \dot{x}_1 &= a(y_1 - x_1) \\ \dot{y}_1 &= x_1(c - z_1) - y_1 \\ \dot{z}_1 &= x_1 y_1 - b z_1 . \end{aligned} \tag{1.46}$$

The system (1.46) is taken as master system and the slave system is described as

$$\begin{aligned} \dot{x}_2 &= a(y_2 - x_2) + u_1(t) \\ \dot{y}_2 &= x_2(c - z_2) - y_2 + u_2(t) \\ \dot{z}_2 &= x_2 y_2 - b z_2 + u_3(t) , \end{aligned} \tag{1.47}$$

where $u_1(t)$, $u_2(t)$ and $u_3(t)$ are control inputs. Here the aim is to investigate the synchronization of systems (1.46) and (1.47).

Now defining the error states as $e_1 = x_2 - x_1$, $e_2 = y_2 - y_1$ and $e_3 = z_2 - z_1$, one can obtain the following error dynamical system as

$$\dot{e}_1 = a(e_2 - e_1) + u_1(t)$$

$$\dot{e}_2 = e_1 - e_2 - (e_1 + x_1)e_3 - e_1 z_1 + u_2(t) \qquad (1.48)$$

$$\dot{e}_3 = -be_3 + (e_1 + x_1)e_2 + e_1 y_1 + u_3(t)\,.$$

Equations (1.48) can be considered in terms of control problem where the system is to be controlled by the control inputs $u_1(t)$, $u_2(t)$ and $u_3(t)$, which are functions of error vectors e_1, e_2 and e_3. As long as these control inputs can stabilize the system, the error vectors e_1, e_2, e_3 converge to zero as time t goes to infinity and as a result the systems (1.46) and (1.47) are synchronized with each other. Here the bckstepping design procedure consists three steps. At the *i*-th step, the virtual control function α_i can be obtained by constructing Lyapunov function V_i.

Step I: Defining $w_1 = e_1$, we get

$$\dot{w}_1 = \dot{e}_1 = a(e_2 - w_1) + u_1(t), \qquad (1.49)$$

where $e_2 = \alpha_1(w_1)$ is regarded as an virtual controller. For the designing of $\alpha_1(w_1)$ to stabilize w_1- subsystem, choosing the Lyapunov function V_1 as

$$V_1 = \frac{1}{2} w_1^2\,.$$

The derivative of V_1 w. r. to t is

$$\frac{dV_1}{dt} = \frac{1}{2}\frac{dw_1^2}{dt}$$

$$= w_1 \frac{dw_1}{dt}$$

$$= w_1[a(\alpha_1(w_1) - w_1) + u_1(t)]\,.$$

If $u_1(t)=0$ and $\alpha_1(w_1)=w_1-\dfrac{w_1}{a}$, then $\dfrac{dV_1}{dt}=-w_1^2<0$, which implies that w_1-subsystem (1.49) is asymptotically stable. Since virtual control function $\alpha_1(w_1)$ is an estimate function, defining the following error variable between e_2 and $\alpha_1(w_1)$ as

$$w_2=e_2-\alpha_1(w_1)$$

one can obtain the following (w_1,w_2)-subsystem as

$$\frac{dw_1}{dt}=aw_2-w_1$$

$$\frac{dw_2}{dt}=-aw_2+cw_1-(w_1+x_1)e_3-w_1z_1+u_2(t)\,, \tag{1.50}$$

where $e_3=\alpha_2(w_1,w_2)$ is regarded as an virtual controller.

Step II: In this step to stabilize (w_1,w_2) - subsystem (1.50), choose Lyapunov function as

$$V_2=V_1+\frac{1}{2}w_2^2=\frac{1}{2}w_1^2+\frac{1}{2}w_2^2.$$

The derivative of V_2 w. r. to t is

$$\frac{dV_2}{dt}=\frac{1}{2}\frac{dw_1^2}{dt}+\frac{1}{2}\frac{dw_2^2}{dt}$$

$$=w_1\frac{dw_1}{dt}+w_2\frac{dw_2}{dt}$$

$$=aw_1w_2-w_1^2+w_2[-aw_2+cw_1-(w_1+x_1)\alpha_2(w_1,w_2)-w_1z_1+u_2(t)].$$

If $\alpha_2(w_1,w_2)=0$ and $u_2(t)=(a-1)w_2-(c+a)w_1+w_1z_1$, then $\dfrac{dV_2}{dt}=-w_1^2-w_2^2<0$ makes subsystem (1.50) asymptotically stable.

Now defining the error variable w_3 as

$w_3 = e_3 - \alpha_2(w_1, w_2)$,

the (w_1, w_2, w_3) - subsystem is

$$\frac{dw_1}{dt} = aw_2 - w_1$$

$$\frac{dw_2}{dt} = -(w_1 + x_1)w_3 - w_2 - aw_1 \tag{1.51}$$

$$\frac{dw_3}{dt} = -bw_3 + (w_1 + x_1)(w_2 + w_1 - \frac{w_1}{a}) + w_1 y_1 + u_3(t) .$$

Step III: To stabilize the (w_1, w_2, w_3) - subsystem (1.51), choosing the following Lyapunov function V_3 as

$$V_3 = V_2 + \frac{1}{2}w_3^2 = \frac{1}{2}w_1^2 + \frac{1}{2}w_2^2 + \frac{1}{2}w_3^2 .$$

The derivative of V_3 is

$$\frac{dV_3}{dt} = \frac{1}{2}\frac{dw_1^2}{dt} + \frac{1}{2}\frac{dw_2^2}{dt} + \frac{1}{2}\frac{dw_3^2}{dt}$$

$$= w_1 \frac{dw_1}{dt} + w_2 \frac{dw_2}{dt} + w_3 \frac{dw_3}{dt}$$

$$= -w_1^2 - w_1 w_2 w_3 - w_2 w_3 x_1 - w_2^2 + w_3[-bw_3 + (w_1 + x_1)(w_2 + w_1 - \frac{w_1}{a}) + w_1 y_1 + u_3(t)].$$

Let, $u_3(t) = (b-1)w_3 + (\frac{1}{a} - 1)w_1^2 - w_1 x_1 + \frac{w_1 x_1}{a} - w_1 y_1$,

then $\frac{dV_3}{dt} = -w_1^2 - w_3^2 - w_3^2 < 0$ is negative definite. Thus the system (1.51) is asymptotically stable. For $w_1 = e_1, w_2 = e_2 - \alpha_1(w_1) = e_2 - e_1 + \frac{e_1}{a}$ and

$w_3 = e_3 - \alpha_2(w_1, w_2) = e_3$, e_i $(i = 1, 2, 3)$ tend to zero asymptotically which helps to obtain synchronization between fractional order Lorenz systems.

1.13 Numerical approximation method

Numerical methods used for solving ODEs have to be modified for solving fractional differential equations (FDEs). The predictor-corrector scheme is derived for drive-response systems. This scheme is the generalization of Adams–Bashforth–Moulton one (Diethelm et al. (2004), Diethelm and Ford (2004)). The approximate solution of nonlinear fractional-order differential equations is interpreted by means of this algorithm in the following way.

The following differential equation

$$D_t^q y(t)=f(t,y(t)), \qquad 0 \le t \le T, \tag{1.52}$$

$$y^{(k)}(0) = y_0^{(k)}, \qquad k = 0,1,.......,[q]$$

is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s))ds. \tag{1.53}$$

Setting, $h = T/N$, $t_n = nh, n = 0,1,........N \in Z^+$, The equation (1.53) can be discredited as

$$y_h(t_{n+1}) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^q}{\Gamma(q+2)} f(t_{n+1}, y_h^p(t_{n+1})) + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_h, y_h(t_j)), \tag{1.54}$$

$$a_{j,n+1}=\begin{cases} n^{q+1}-(n-q)(n+1)^{q}, & if \quad j=0, \\ (n-j+2)^{q+1}+(n-j)^{q+1}-2(n-j+1)^{q+1}, & if \quad 0\leqslant j\leqslant n, \\ 1, & if \quad j=n+1, \end{cases} \tag{1.55}$$

$$y_h^p(t_{n+1})=\sum_{k=0}^{[q]-1} y_0^{(k)}\frac{t_{n+1}^k}{k!}+\frac{1}{\Gamma(q)}\sum_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j)) \tag{1.56}$$

$$b_{j,n+1}=\frac{h^q}{q}((n+1-j)^q-(n-j)^q) \tag{1.57}$$

The error estimate is

$$\max_{j=0,1,\ldots..N}\left|y(t_j)-y_h(t_j)\right|=o(h^p) \text{ in which } p=\min(2,1+q). \tag{1.58}$$