Chapter 7

Dual combination synchronization of the fractional order complex chaotic systems

7.1 Introduction

Due to advent of high speed computational facilities and the development of fractional order real systems, integer order complex systems, fractional order chaotic systems, many interesting and important results have been found during synchronization of the chaotic systems which have important applications in various fields such as physical systems (Lakshmanan and Murali (1996)), ecological systems (Blasius et al. (1999)), chemical systems, modelling brain activity, system identification (Cuomo and Oppenheim (1993)), pattern recognition phenomena and secure communication (Murali and Lakshmanan (2003)). Recently, finite-time synchronization for high dimensional chaotic systems has been applied in secure communication by Liu et al. (2015), which shows that the systems can realise monotonous synchronization and the information signal can be recovered undistorted. Wu et al. (2016) have used linearized method during chaos synchronization and have shown the potential role of fractional chaotic maps for secure communication. Synchronization of two identical and non-identical fractional order chaotic systems using active control method was studied by Golmankhaneh et al. (2015). The fractional order complex Lorenz system and its complete synchronization were studied by Luo and Wang

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(2013a). The fractional order complex Lu system was introduced by Jiang et al. (2014) where the authors realised its anti-synchronization. The fractional order complex T system was presented, and its functional projective synchronization was achieved by Liu et al. (2014). The fractional order complex Chen system was studied by Luo and Wang (2013a) and applied its hybrid synchronization to secure digital communication. Complex modified hybrid projective synchronization was investigated between the fractional order complex chaos and real hyperchaos by Liu (2014). The problem of hybrid projective synchronization of fractional order complex chaotic systems with time delays was considered by Velmurugan and Rakkiyapan (2016).

Dual synchronization is a special circumstance in synchronization in which two different pairs of chaotic systems, i.e., two master systems and two slave systems are synchronized. In the combination synchronization, three chaotic systems, i.e., two master systems and one slave system are synchronized. The dual synchronization in integer order systems was first proposed by Liu and Davis (2000), where a pair of master systems was synchronized with another pair of slave systems. Xiao et al. (2013) have constructed a theory frame about dual synchronization and the method was successfully used to design a synchronization controller to achieve synchronization of fractional order chaotic systems. Jiang et al. (2015) proposed a generalised combination complex synchronization taking two fractional order complex chaotic systems as master systems and one fractional order complex system as slave system. The importance of combination synchronization in secure communication has already been established by Runzi et al. (2011) by splitting the transmitted signals into several parts, each part loaded in different master systems or by means of splitting time into intervals so that the signals in different intervals are loaded in different master systems in order to ensure that the transmitted signals have stronger anti-attack and anti-translated capability. For the three different pairs, i.e., four master systems and two slave systems, the dual combination synchronization was studied by Sun et al. (2016a) for the integer order real chaotic systems. Motivated by the above discussion, the authors have investigated the dual combination synchronization for the six fractional order complex chaotic systems which has not yet been explored. Since in the present scenario, the numbers of variables are increased in the complex space, it will be more secure and interesting to transmit and receive signals in the application of communication. As the fractional order complex systems are in complex variables, it provides the best instrument to describe a variety of physical phenomena such as amplitudes of the electromagnetic field, thermal convection of liquid flow, detuned laser system etc.

To implement this scheme, there are many effective techniques which have been successfully applied to achieve the chaos synchronization viz., OGY method, active control method, adaptive control method, sliding mode control method, linear and nonlinear feedback method, time-delay feedback approach, backstepping approach, etc. It is noticed that the Lyapunov stability theory is effective and convenient to synchronize or anti-synchronize since the Lyapunov exponents are not required for these calculations. Since the method has been well tested to many practical systems, the authors have successfully applied this approach in the present article. Numerical simulation results, which are carried out using Adams-Bashforth-Mounton method show the method is easy to implement and reliable for synchronizing the fractional order complex Lorenz and T systems.

The rest of the chapter is organised as follows. The scheme for the dual combination synchronization is introduced in Section 7.2. The stability of the proposed scheme is studied in Section 7.3. In Section 7.4, the descriptions of the fractional order systems viz., complex Lorenz and T systems and their chaotic nature are given. The illustration of the scheme and numerical simulation results are carried out in Section 7.5 and Section 7.6 respectively. Finally, the conclusion is drawn in Section 7.7.

7.2 The scheme for dual combination synchronization

In this section, the dual combination synchronization among four master and two slave fractional order systems is designed. Let first two master systems are given as

$$D^q X_1 = f_1(X_1), \tag{7.1}$$

$$D^q X_2 = f_2(X_2), \qquad 0 < q \le 1, \tag{7.2}$$

where $X_1 = [x_{11}, x_{12}, \cdots, x_{1n}]^T$ and $X_2 = [x_{21}, x_{22}, \cdots, x_{2m}]^T$ are the two state complex vector spaces of uncoupled master systems (7.1) and (7.2) ; $f_1 : C^n \to C^n$ and $f_2 : C^m \to C^m$ are the two known complex vector valued functions. The coupled master system M_1 generates a complex vector signal in the form

$$\begin{aligned} M_1 &= [a_{11}x_{11}, a_{12}x_{12}, \cdots, a_{1n}x_{1n}, a_{21}x_{21}, a_{22}x_{22}, \cdots, a_{2m}x_{2m}]^T \\ &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = AX , \end{aligned} \tag{7.3}$$

where $A_1 = diag[a_{11}, a_{12}, \cdots, a_{1n}]$ and $A_2 = diag[a_{21}, a_{22}, \cdots, a_{2m}]$ are the two known matrices; a_{1i} and a_{2j} cannot be zero at the same time $(i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m)$.

Next two master systems are considered as

$$D^q Y_1 = g_1(Y_1), \tag{7.4}$$

$$D^q Y_2 = g_2(Y_2), \tag{7.5}$$

where $Y_1 = [y_{11}, y_{12}, \cdots, y_{1n}]^T$ and $Y_2 = [y_{21}, y_{22}, \cdots, y_{2m}]^T$ are the two state complex vector spaces of the uncoupled master systems (7.4) and (7.5); $g_1 : C^n \to C^n$ and $g_2 : C^m \to C^m$ are the two known complex vector valued functions. Hence the coupled master system M_2 generates a complex vector signal in the form

$$M_2 = [b_{11}y_{11}, b_{12}y_{12}, \cdots, b_{1n}y_{1n}, b_{21}y_{21}, b_{22}y_{22}, \cdots, b_{2m}y_{2m}]^T$$

$$= \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = BY \ , \tag{7.6}$$

where $B_1 = diag\left[b_{11}, b_{12}, \cdots, b_{1n}\right]$ and $B_2 = diag\left[b_{21}, b_{22}, \cdots, b_{2m}\right]$ are the two known matrices; b_{1i} and b_{2j} cannot be zero simultaneously $(i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m)$.

Now the corresponding two slave systems are described as

$$D^q Z_1 = h_1(Z_1) + U_1 \ , \tag{7.7}$$

$$D^q Z_2 = h_2(Z_2) + U_2 \ , \tag{7.8}$$

where $Z_1 = [z_{11}, z_{12}, \cdots, z_{1n}]^T$ and $Z_2 = [z_{21}, z_{22}, \cdots, z_{2m}]^T$ are the state vectors of two uncoupled slave systems (7.7) and (7.8) ; $h_1 : C^n \to C^n$ and $h_2 : C^m \to C^m$ are the two known complex vector valued functions; $U_1 : C^n \times C^n \times C^n \to C^n$ and $U_2 : C^m \times C^m \times C^m \to C^m$ are controller vector valued functions to be designed. Then the coupled slave system S_1 is generated a complex vector signal in the form

$$S_1 = [c_{11}z_{11}, c_{12}x_{12}, \cdots, c_{1n}z_{1n}, c_{21}z_{21}, c_{22}z_{22}, \cdots, c_{2m}z_{2m}]^T$$

$$= \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = CZ \ , \tag{7.9}$$

where $C_1 = diag\left[c_{11}, c_{12}, \cdots, c_{1n}\right]$ and $C_2 = diag\left[c_{21}, c_{22}, \cdots, c_{2m}\right]$ are the two known matrices; c_{1i} and c_{2j} cannot be zero at the same time $\left(i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m\right)$.

The error vector signal for dual combination synchronization is defined as

$$e = PM_1 + QM_2 - RS_1$$

$$= PAX + QBY - RCZ \ ,$$

where P, Q and R are called scaling matrices.

For the convenience of the ensuing discussion, let us assume P, Q and R are diagonal matrices. Then e is reduced to

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} P_1 A_1 X_1 + Q_1 B_1 Y_1 - R_1 C_1 Z_1 \\ P_2 A_2 X_2 + Q_2 B_2 Y_2 - R_2 C_2 Z_2 \end{bmatrix}, \tag{7.10}$$

where $e = [e_1^T, e_2^T]^T$, $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$, $Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$ and $R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$ such that $e_1 = [e_{11}, e_{12}, \cdots, e_{1n}]^T$, $e_2 = [e_{21}, e_{22}, \cdots, e_{2m}]^T$, $P_1 = diag\left[p_{11}, p_{12}, \cdots, p_{1n}\right]$, $P_2 = diag\left[p_{21}, p_{22}, \cdots, p_{2m}\right]$, $Q_1 = diag\left[q_{11}, q_{12}, \cdots, q_{1n}\right]$, $Q_2 = diag\left[q_{21}, q_{22}, \cdots, q_{2m}\right]$, $R_1 = diag\left[r_{11}, r_{12}, \cdots, r_{1n}\right]$ and $R_2 = diag\left[r_{21}, r_{22}, \cdots, r_{2m}\right]$.

Definition 7.1 Dual combination synchronization of the master systems (7.1), (7.2), (7.4) and (7.5), and the slave systems (7.7) and (7.8) is achieved, if $\lim_{t\to\infty}\|e(t)\| = 0$, where $\|\cdot\|$ denotes matrix norm.

Remark 7.2 If the matrices $A = B = C = I$, then it is easy to verify that the proposed scheme is applicable for

(1) combination synchronization if $P_1 = Q_1 = R_1 = 0$ or $P_2 = Q_2 = R_2 = 0$.

(2) projective synchronization if (i) $Q_1 = Q_2 = 0$; $P_1 = R_1 = 0$, $P_2 = I$, $R_2 = D$; or $P_2 = R_2 = 0$, $P_2 = I$, $R_1 = D$; or (ii) $P_1 = P_2 = 0$; $Q_1 = R_1 = 0$, $Q_2 = I$, $R_2 = D$; or $Q_2 = R_2 = 0$, $Q_1 = I$, $R_1 = D$.

(3) complete synchronization if (i) $P_1 = Q_1 = R_1 = 0$; $P_2 = 0$, $Q_2 = R_2 = I$; or $P_2 = R_2 = I$, $Q_2 = 0$; or (ii) $P_2 = Q_2 = R_2 = 0$; $P_1 = 0$, $Q_1 = R_1 = I$; or $P_1 = R_1 = I$, $Q_1 = 0$.

(4) chaos control problem if $A = B = 0$ or $P = Q = 0$.

Remark 7.3 If $P = Q = R = I$, the proposed scheme is applicable for the dual synchronization for $A = C = I$, $B_1 = B_2 = 0$ or $A_1 = A_2 = 0$, $B = C = I$ and also applicable for all kinds of synchronization mentioned in Remark 7.2 and many more cases with appropriate choices of matrices.

7.3 Stability analysis

In order to achieve the dual combination synchronization, let us design the control vector functions as

$$u_{1i} = -h_{1i} + c_{1i}^{-1} r_{1i}^{-1} p_{1i} a_{1i} \ f_{1i} + c_{1i}^{-1} r_{1i}^{-1} q_{1i} b_{1i} \ g_{1i} + k_1 c_{1i}^{-1} r_{1i}^{-1} e_{1i} \ , \quad i = 1, 2, \cdots, n$$

$$u_{2j} = -h_{2j} + c_{2j}^{-1} r_{2j}^{-1} p_{2j} a_{2j} \ f_{2j} + c_{2j}^{-1} r_{2j}^{-1} q_{2j} b_{2j} \ g_{2j} + k_2 c_{2j}^{-1} r_{2j}^{-1} e_{2j} \ , \quad j = 1, 2, \cdots, m \quad (7.11)$$

where $U_1 = [u_{11}, u_{12}, \cdots, u_{1n}]^T$, $U_2 = [u_{21}, u_{22}, \cdots, u_{2m}]^T$, $f_1 = [f_{11}, f_{12}, \cdots, f_{1n}]^T$, $f_2 = [f_{21}, f_{22}, \cdots, f_{2m}]^T$, $g_1 = [g_{11}, g_{12}, \cdots, g_{1n}]^T$, $g_2 = g_{21}, g_{22}, \cdots, g_{2m}]^T$, $h_1 = [h_{11}, h_{12}, \cdots, h_{1n}]^T$, $h_2 = [h_{21}, h_{22}, \cdots, h_{2m}]^T$ and k_1 and k_2 are constants.

Theorem 7.4 Dual combination synchronization of the considered systems is achieved if $k_1 > 0$ and $k_2 > 0$.

Proof: The $q-$th order derivative of the Equation (7.10) is given by

$$\begin{bmatrix} D^q e_1 \\ D^q e_2 \end{bmatrix} = \begin{bmatrix} P_1 A_1 [D^q X_1] + Q_1 B_1 [D^q Y_1] - R_1 C_1 [D^q Z_1] \\ P_2 A_2 [D^q X_2] + Q_2 B_2 [D^q Y_2] - R_2 C_2 [D^q Z_2] \end{bmatrix},$$

and the dynamical error system is obtained as

$$\begin{bmatrix} D^q e_1 \\ D^q e_2 \end{bmatrix} = \begin{bmatrix} P_1 A_1 [f_1(X_1)] + Q_1 B_1 [g_1(Y_1)] - R_1 C_1 [h_1(Z_1) + U_1] \\ P_2 A_2 [f_2(X_2)] + Q_2 B_2 [g_2(Y_2)] - R_2 C_2 [h_2(Z_2) + U_2] \end{bmatrix},$$

which can be written as

$$D^q e_{1i} = p_{1i} \ a_{1i} \ f_{1i} + q_{1i} \ b_{1i} \ g_{1i} - r_{1i} \ c_{1i} (h_{1i} + u_{1i}) \ , \qquad i = 1, 2, \cdots, n$$

$$D^q e_{2j} = p_{2j} \ a_{2j} \ f_{2j} + q_{2j} \ b_{2j} \ g_{2j} - r_{2j} \ c_{2j} (h_{2j} + u_{2j}) \ , \quad j = 1, 2, \cdots, m \ . \quad (7.12)$$

Let us design the Lyapunov candidate as

$$V=\frac{1}{2}e^{T}e=\frac{1}{2}e_{11}^{2}+\frac{1}{2}e_{12}^{2}+\cdots+\frac{1}{2}e_{1n}^{2}+\frac{1}{2}e_{21}^{2}+\frac{1}{2}e_{22}^{2}+\cdots+\frac{1}{2}e_{2m}^{2} ,$$

whose $q-\text{th}$ derivative using Lemma 1.10 is obtained as

$$D^{q}V\le e_{11}\,D^{q}e_{11}+e_{12}\,D^{q}e_{12}+...+e_{1n}\,D^{q}e_{1n}+e_{21}\,D^{q}e_{21}+e_{22}\,D^{q}e_{22}+...+e_{2m}\,D^{q}e_{2m} .$$

The above equation with the aid of Equation (7.11) and Equation (7.12) gives rise to

$$\begin{aligned}D^{q}V&\le-[k_{1}(e_{11}^{2}+e_{12}^{2}+\cdots+e_{1n}^{2})+k_{2}(e_{21}^{2}+e_{22}^{2}+\cdots+e_{2m}^{2})]\\&<0 .\end{aligned}$$

Therefore according to Lyapunov stability theory, the error dynamical system (7.12) is asymptotically stable i.e., $\lim_{t\to\infty}\|e\|=0$. Hence the dual combination synchronization is achieved.

7.4 Systems' descriptions

7.4.1 The fractional order complex Lorenz system

The fractional order complex Lorenz system (Luo and Wang (2013b)) is given as

$$\begin{aligned}D^{q}l_{1}&=a_{11}(l_{2}-l_{1}) ,\\D^{q}l_{2}&=a_{12}l_{1}-l_{2}-l_{1}l_{3} ,\\D^{q}l_{3}&=\frac{1}{2}(\overline{l_{1}}\,l_{2}+l_{1}\,\overline{l_{2}})-a_{13}l_{3} ,\qquad 0<q<1,\end{aligned}\tag{7.13}$$

where $l = [l_1, \quad l_2, \quad l_3]^T$ is the state vector variable, $l_1 = x_{11} + i\,x_{12}$ and $l_2 = x_{13} + i\,x_{14}$ are complex variables; $l_3 = x_{15}$ is real variable; a_{11}, a_{12}, a_{13} are parameters and $i = \sqrt{-1}$. One can obtain the real version of the system (7.13) as

$$D^q x_{11} = a_{11}(x_{13} - x_{11}) ,$$

$$D^q x_{12} = a_{11}(x_{14} - x_{12}) ,$$

$$D^q x_{13} = a_{12}x_{11} - x_{13} - x_{11}x_{15} ,$$

$$D^q x_{14} = a_{12}x_{12} - x_{14} - x_{12}x_{15} ,$$

$$D^q x_{15} = x_{11}x_{13} + x_{12}x_{14} - a_{13}x_{15} . \tag{7.14}$$

Figure 7.1 depicts the chaotic behaviour of the system in various three dimensional combinations of the state spaces at fractional order $q = 0.95$ for the values of parameters $a_{11} = 10$, $a_{12} = 180$, $a_{13} = 1$ and the initial condition $l(0) = [2+3i, \quad 5+6i \quad 9]^T$.

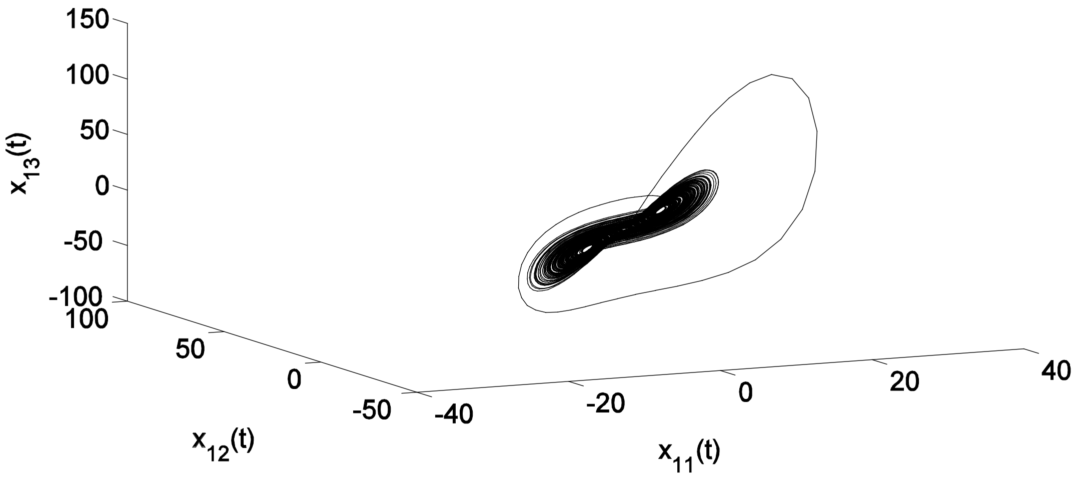

Figure 7.1 (a)

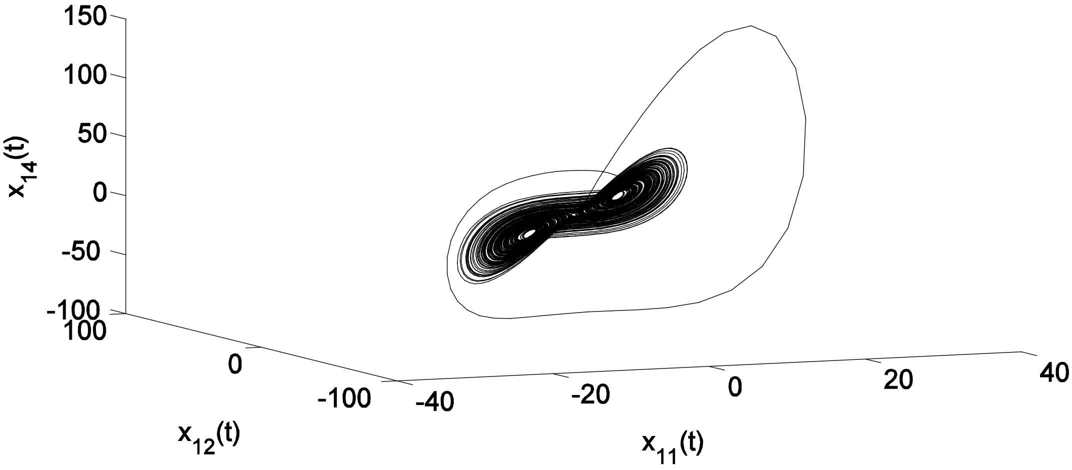

Figure 7.1 (b)

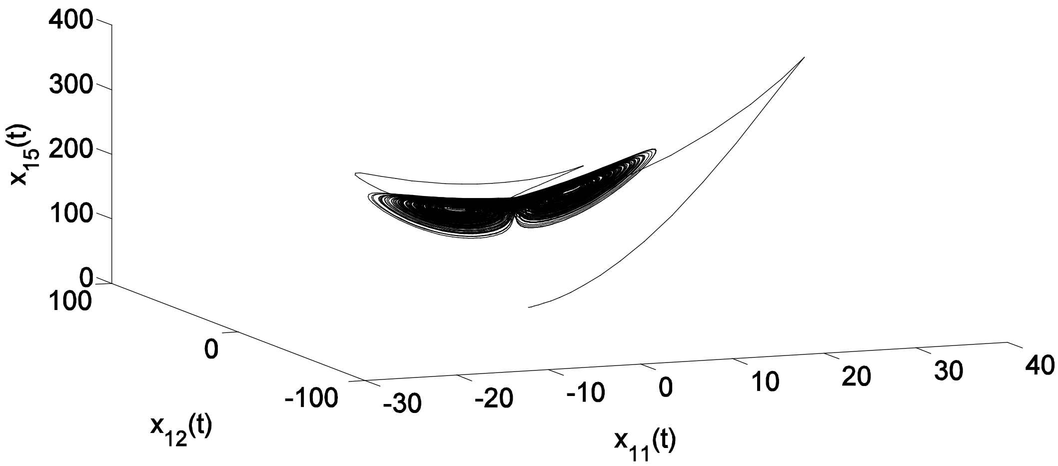

Figure 7.1 (c)

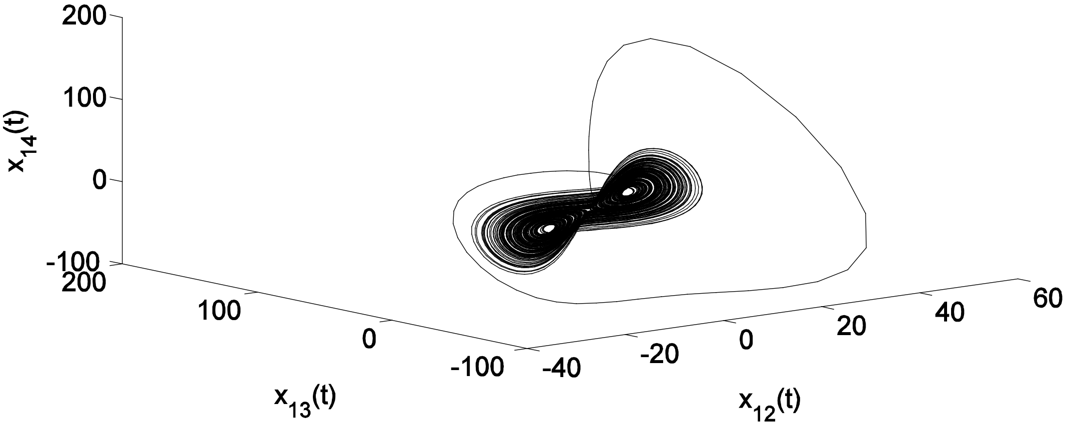

Figure 7.1 (d)

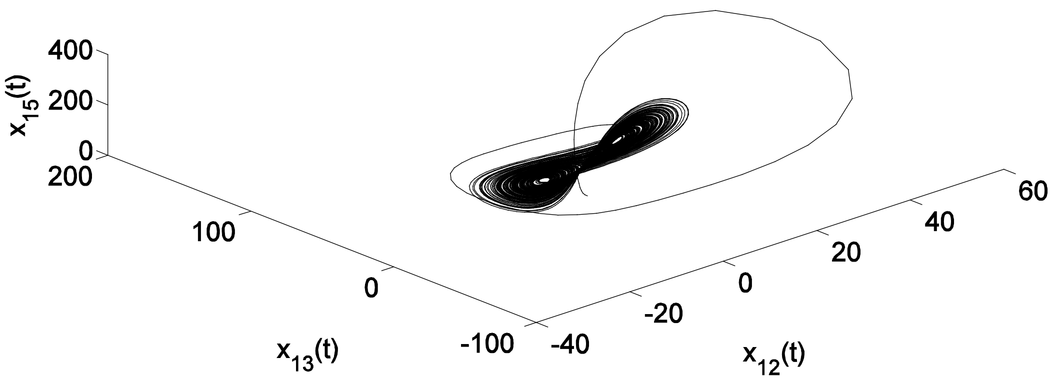

Figure 7.1 (e)

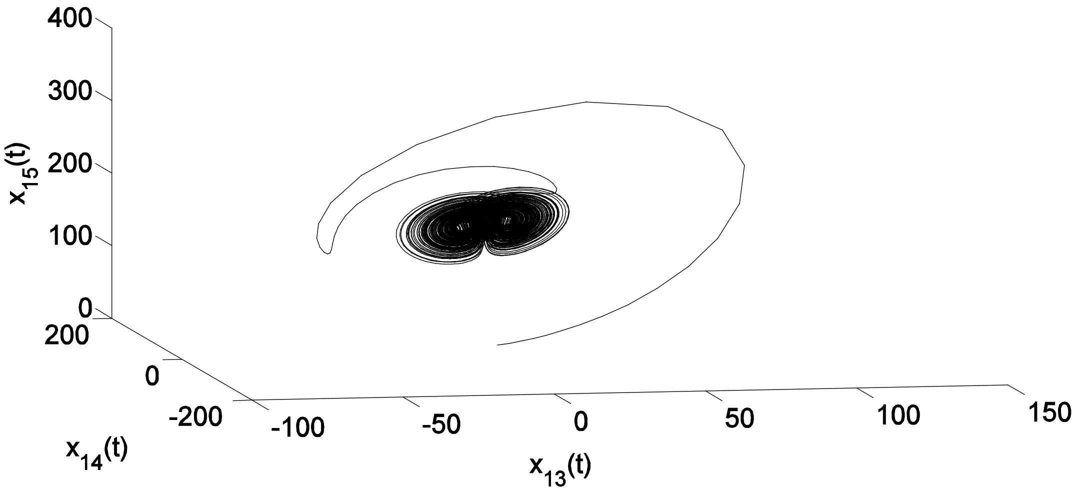

Figure 7.1 (f)

Figure 7.1: Phase portraits of the complex Lorenz system for the order of derivative $q = 0.95$ in **(a)** $x_{11} - x_{12} - x_{13}$ space, **(b)** $x_{11} - x_{12} - x_{14}$ space, **(c)** $x_{11} - x_{12} - x_{15}$ space, **(d)** $x_{12} - x_{13} - x_{14}$ space, **(e)** $x_{12} - x_{13} - x_{15}$ space, **(f)** $x_{13} - x_{14} - x_{15}$ space.

7.4.2 The fractional order complex T system

The fractional order complex T system (Liu et al. (2014)) is given by

$$D^q u_1 = a_{21}(u_2 - u_1) ,$$

$$D^q u_2 = (a_{22} - a_{21})u_1 - a_{21}u_1 u_3 ,$$

$$D^q u_3 = \frac{1}{2}(\overline{u_1}\, u_2 + u_1\, \overline{u_2}) - a_{23}u_3 , \qquad 0 < q < 1, \tag{7.15}$$

where $u = [u_1, \; u_2, \; u_3]^T$ is state vector variable with $u_1 = x_{21} + i\,x_{22}$ and $u_2 = x_{23} + i\,x_{24}$ are complex variables and $u_3 = x_{25}$ is real variable; a_{21}, a_{22}, a_{23} are parameters.

System (7.15) can be written as

$$D^q x_{21} = a_{21}(x_{23} - x_{21}) \ ,$$

$$D^q x_{22} = a_{21}(x_{24} - x_{22}) \ ,$$

$$D^q x_{23} = (a_{22} - a_{21})x_{21} - a_{21}x_{21}x_{25} \ ,$$

$$D^q x_{24} = (a_{22} - a_{21})x_{22} - a_{21}x_{22}x_{25} \ ,$$

$$D^q x_{25} = x_{21}x_{23} + x_{22}x_{24} - a_{23}x_{25} \ . \qquad (7.16)$$

For parameters' values $a_{21} = 2.1$, $a_{22} = 30$, $a_{23} = 0.6$ and the initial condition $u(0) = [8+7i, \quad 5+6i, \quad 10]^T$, the above system exhibits chaotic behaviour at $q = 0.94$, which is described in Figure 7.2 through various state space plots.

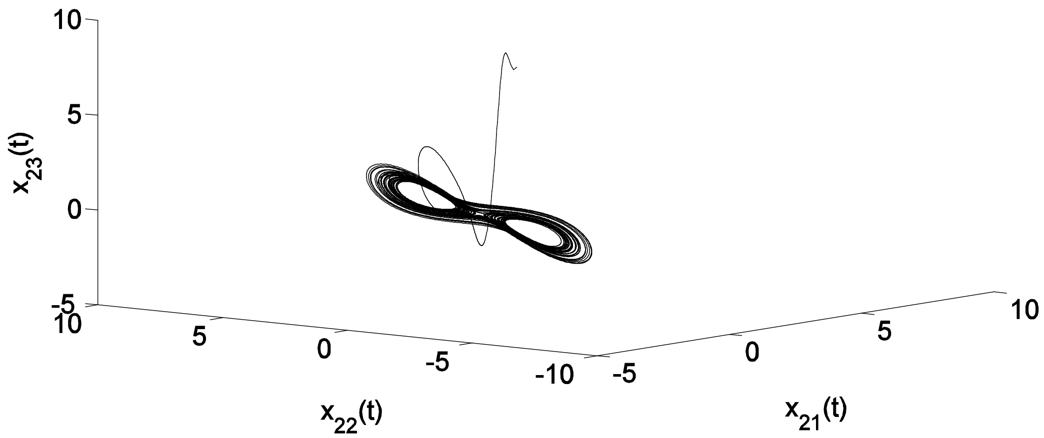

Figure 7.2 (a)

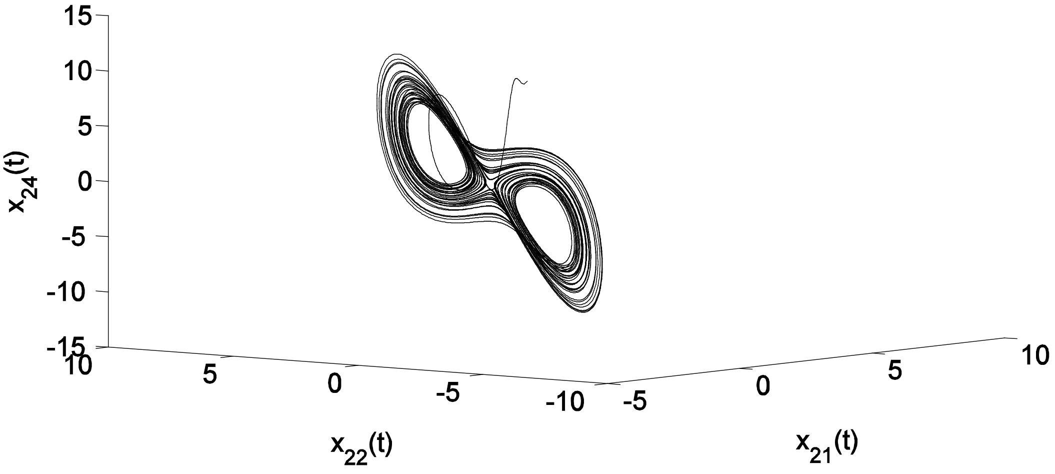

Figure 7.2 (b)

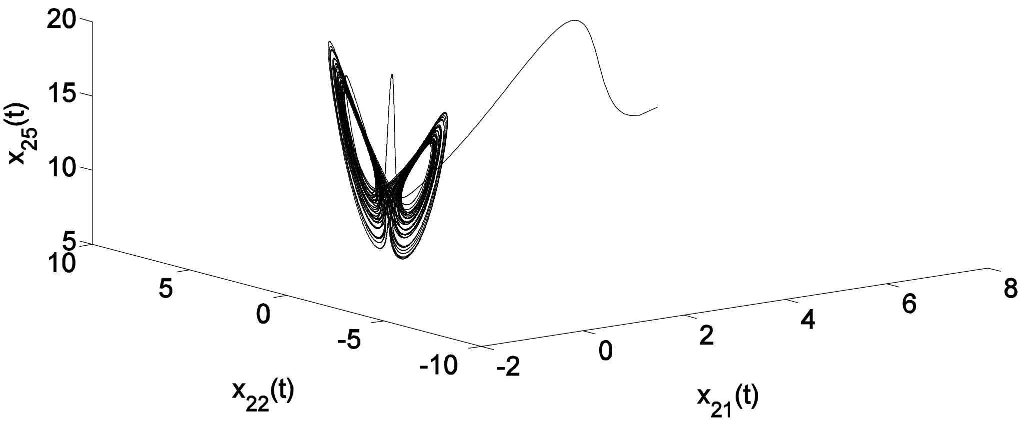

Figure 7.2 (c)

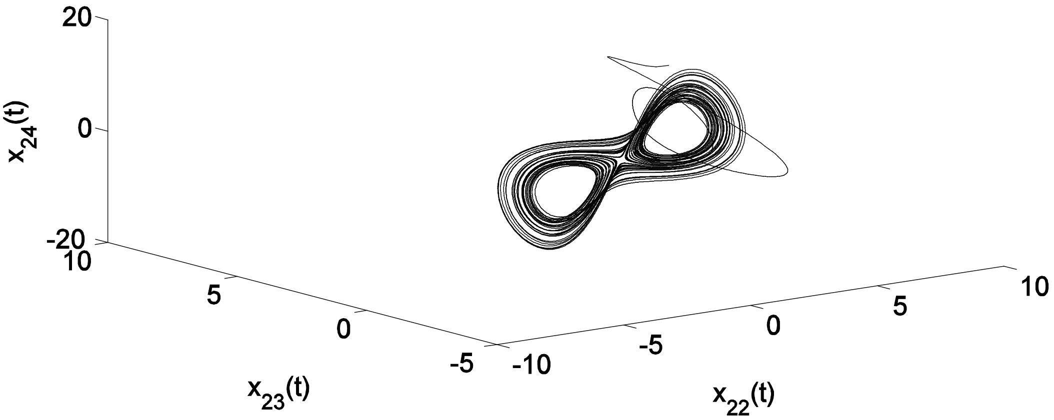

Figure 7.2 (d)

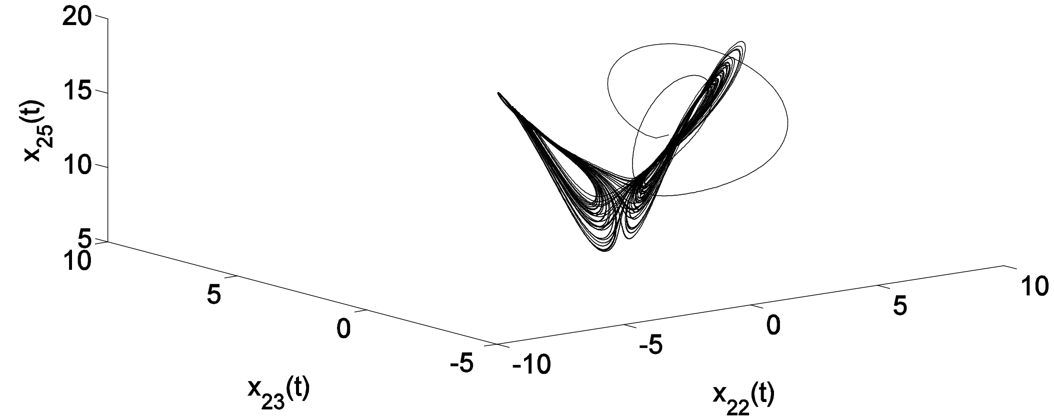

Figure 7.2 (e)

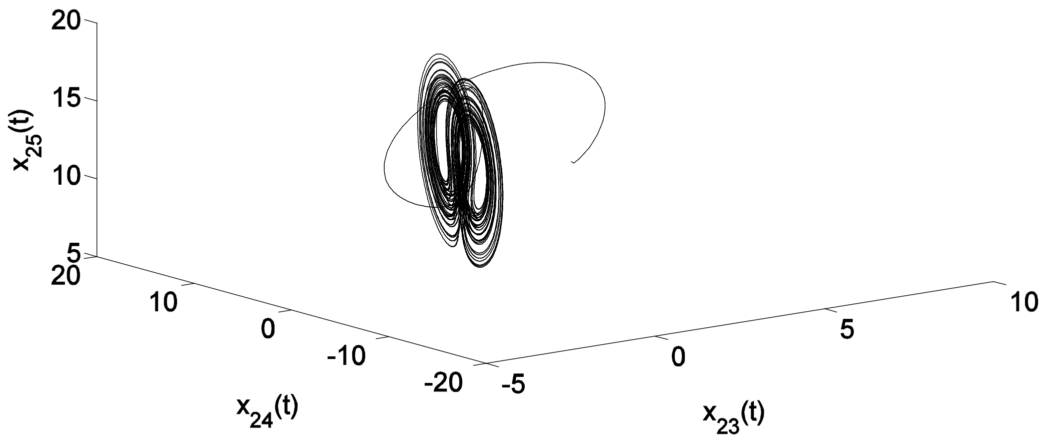

Figure 7.2 (f)

Figure 7.2: Phase portraits of the complex T system for the order of derivative $q = 0.94$ in **(a)** $x_{21} - x_{22} - x_{23}$ space, **(b)** $x_{21} - x_{22} - x_{24}$ space, **(c)** $x_{21} - x_{22} - x_{25}$ space, **(d)** $x_{22} - x_{23} - x_{24}$ space, **(e)** $x_{22} - x_{23} - x_{25}$ space, **(f)** $x_{23} - x_{24} - x_{25}$ space.

7.5 Illustration of the scheme

In this section, the effectiveness of the proposed scheme is realised through consideration of fractional order complex Lorenz system and fractional order complex T system. Let us consider the systems (7.14) and (7.16) as first two master systems. The other two master systems are considered as

$$D^q y_{11} = b_{11}(y_{13} - y_{11}) \,,$$

$$D^q y_{12} = b_{11}(y_{14} - y_{12}) \,,$$

$$D^q y_{13} = b_{12} y_{11} - y_{13} - y_{11} y_{15} \,,$$

$$D^q y_{14} = b_{12} y_{12} - y_{14} - y_{12} y_{15} \ ,$$

$$D^q y_{15} = y_{11} y_{13} + y_{12} y_{14} - b_{13} y_{15} \ , \qquad (7.17)$$

and

$$D^q y_{21} = b_{21}(y_{23} - y_{21}) \ ,$$

$$D^q y_{22} = b_{21}(y_{24} - y_{22}) \ ,$$

$$D^q y_{23} = (b_{22} - b_{21}) y_{21} - b_{21} y_{21} y_{25} \ ,$$

$$D^q y_{24} = (b_{22} - b_{21}) y_{22} - b_{21} y_{22} y_{25} \ ,$$

$$D^q y_{25} = y_{21} y_{23} + y_{22} y_{24} - b_{23} y_{25} \ . \qquad (7.18)$$

The corresponding two slave systems can be written as

$$D^q z_{11} = c_{11}(z_{13} - z_{11}) + u_{11} \ ,$$

$$D^q z_{12} = c_{11}(z_{14} - z_{12}) + u_{12} \ ,$$

$$D^q z_{13} = c_{12} z_{11} - z_{13} - z_{11} z_{15} + u_{13} \ ,$$

$$D^q z_{14} = c_{12} z_{12} - z_{14} - z_{12} z_{15} + u_{14} \ ,$$

$$D^q z_{15} = z_{11} z_{13} + z_{12} z_{14} - c_{13} z_{15} + u_{15} \ , \qquad (7.19)$$

and

$$D^q z_{21} = c_{21}(z_{23} - z_{21}) + u_{21} \ ,$$

$$D^q z_{22} = c_{21}(z_{24} - z_{22}) + u_{22} \ ,$$

$$D^q z_{23} = (c_{22} - c_{21}) z_{21} - c_{21} z_{21} z_{25} + u_{23} \ ,$$

$$D^q z_{24} = (c_{22} - c_{21}) z_{22} - c_{21} z_{22} z_{25} + u_{24} \ ,$$

$$D^q z_{25} = z_{21} z_{23} + z_{22} z_{24} - c_{23} z_{25} + u_{25} \ , \qquad (7.20)$$

when u_{1j} and u_{2j} $(j=1,2,3,4,5)$ are control functions.

Taking $A=B=C=P=Q=R=I$ and $k_1=k_2=1$, it is obtained the control functions as

$$\begin{aligned}
u_{11} &= -c_{11}(z_{13}-z_{11})+a_{11}(x_{13}-x_{11})+b_{11}(y_{13}-y_{11})+e_{11} \ , \\
u_{12} &= -c_{11}(z_{14}-z_{12})+a_{11}(x_{14}-x_{12})+b_{11}(y_{14}-y_{12})+e_{12} \ , \\
u_{13} &= -c_{12}z_{11}+z_{13}+z_{11}z_{15}+a_{12}x_{11}-x_{13}-x_{11}x_{15}+b_{12}y_{11}-y_{13}-y_{11}y_{15}+e_{13} \ , \\
u_{14} &= -c_{12}z_{12}+z_{14}+z_{12}z_{15}+a_{12}x_{12}-x_{14}-x_{12}x_{15}+b_{12}y_{12}-y_{14}-y_{12}y_{15}+e_{14} \ , \\
u_{15} &= -z_{11}z_{13}-z_{12}z_{14}+c_{13}z_{15}+x_{11}x_{13}+x_{12}x_{14}-a_{13}x_{15}+y_{11}y_{13}+y_{12}y_{14} \\
&\quad -b_{13}y_{15}+e_{15} \ ,
\end{aligned} \tag{7.21}$$

and

$$\begin{aligned}
u_{21} &= -c_{21}(z_{23}-z_{21})+a_{21}(x_{23}-x_{21})+b_{21}(y_{23}-y_{21})+e_{21} \ , \\
u_{22} &= -c_{21}(z_{24}-z_{22})+a_{21}(x_{24}-x_{22})+b_{21}(y_{24}-y_{22})+e_{22} \ , \\
u_{23} &= -(c_{22}-c_{21})z_{21}+c_{21}z_{21}z_{25}+(a_{22}-a_{21})x_{21}-a_{21}x_{21}x_{25}+(b_{22}-b_{21})y_{21} \\
&\quad -b_{21}y_{21}y_{25}+e_{23} \ , \\
u_{24} &= -(c_{22}-c_{21})z_{22}+c_{21}z_{22}z_{25}+(a_{22}-a_{21})x_{22}-a_{21}x_{22}x_{25}+(b_{22}-b_{21})y_{22} \\
&\quad -b_{21}y_{22}y_{25}+e_{24} \ , \\
u_{25} &= -z_{21}z_{23}-z_{22}z_{24}+c_{23}z_{25}+x_{21}x_{23}+x_{22}x_{24}-a_{23}x_{25}+y_{21}y_{23}+y_{22}y_{24} \\
&\quad -b_{23}y_{25}+e_{25} \ .
\end{aligned} \tag{7.22}$$

The above control functions are to be designed in such a manner that the considered systems (7.14), (7.16)-(7.20) will be stabilised through the convergence of errors obtained using Theorem 7.4 as time approaches to infinity.

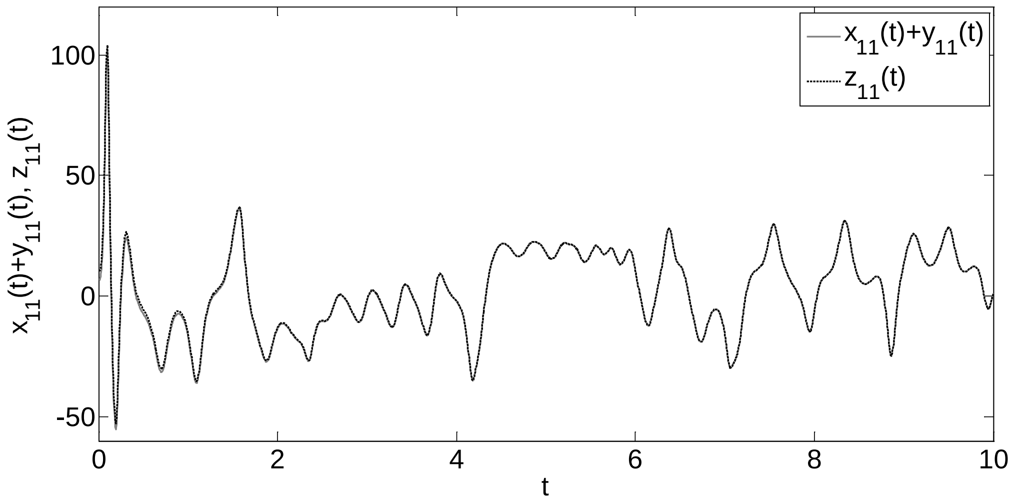

Figure 7.3 (a)

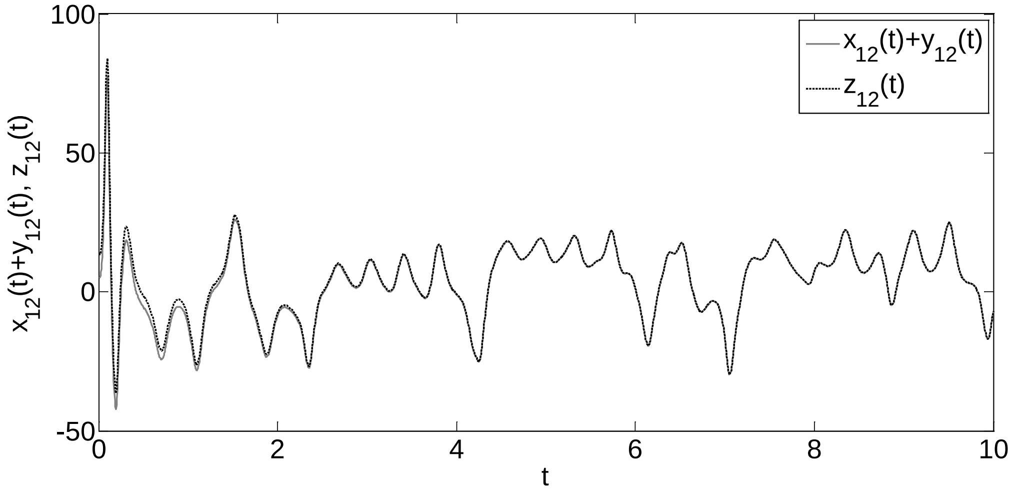

Figure 7.3 (b)

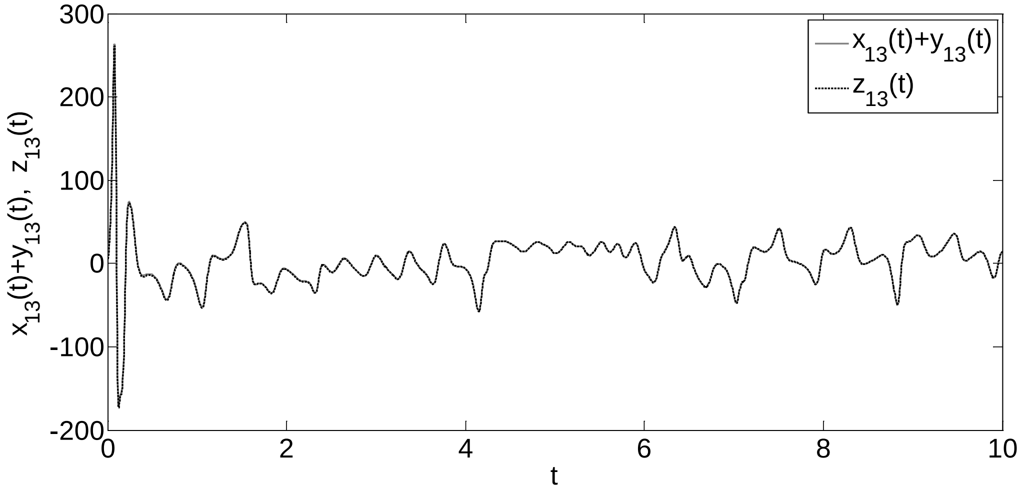

Figure 7.3 (c)

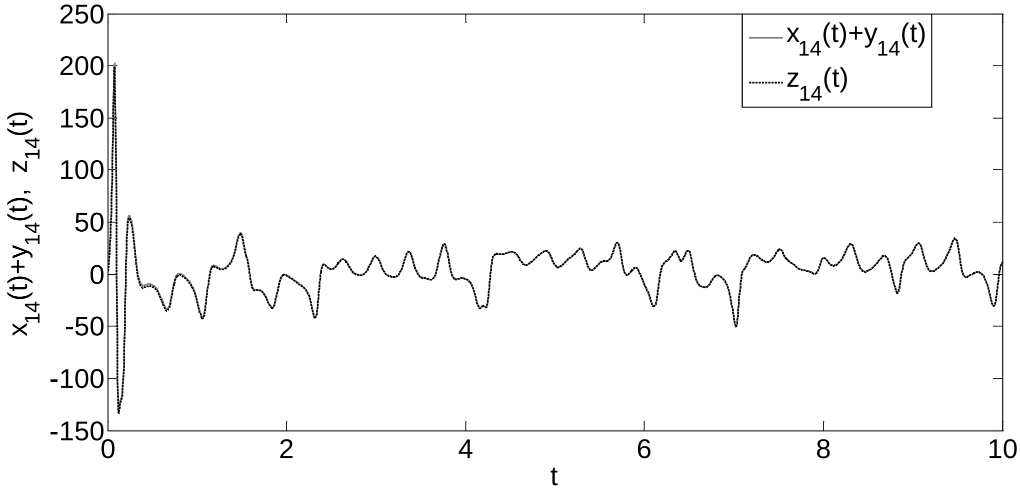

Figure 7.3 (d)

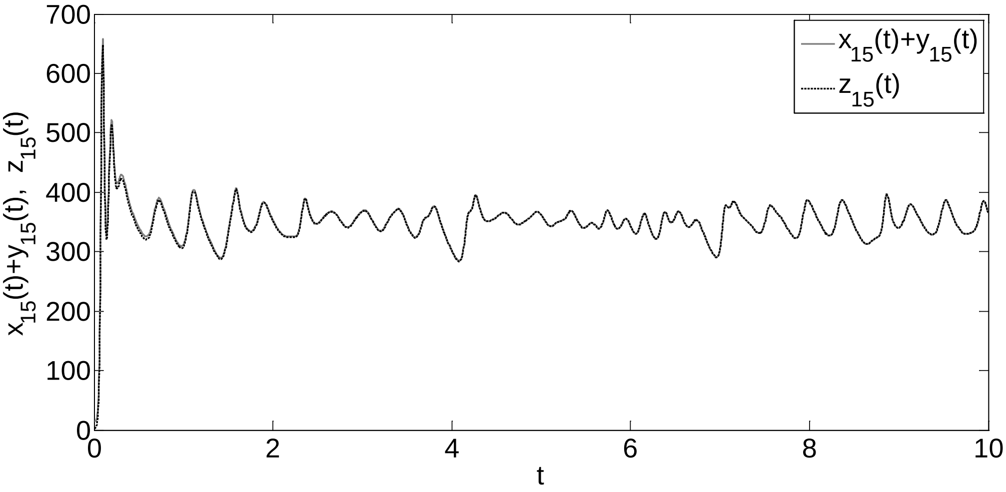

Figure 7.3 (e)

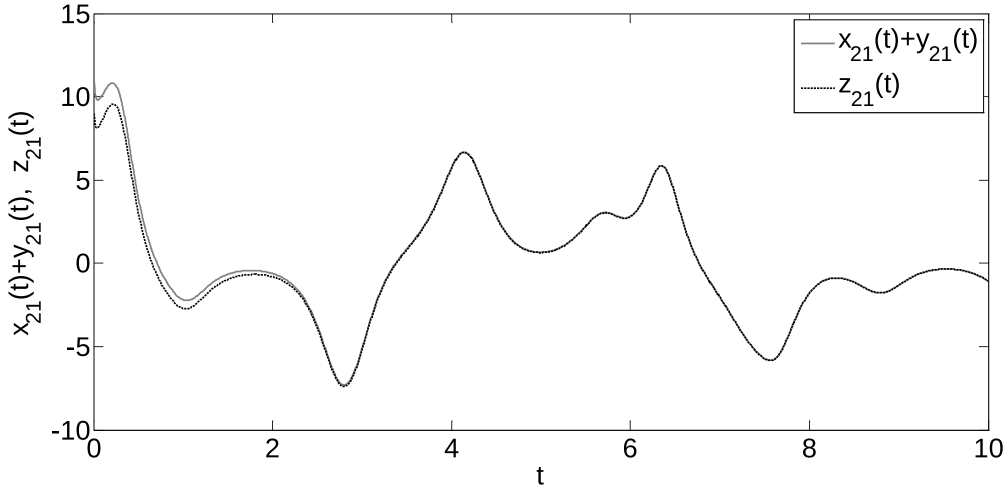

Figure 7.3 (f)

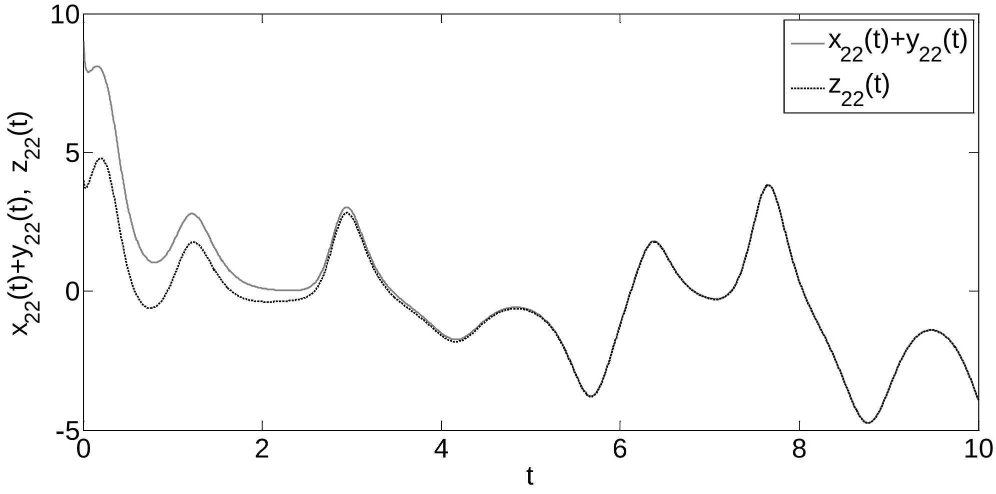

Figure 7.3 (g)

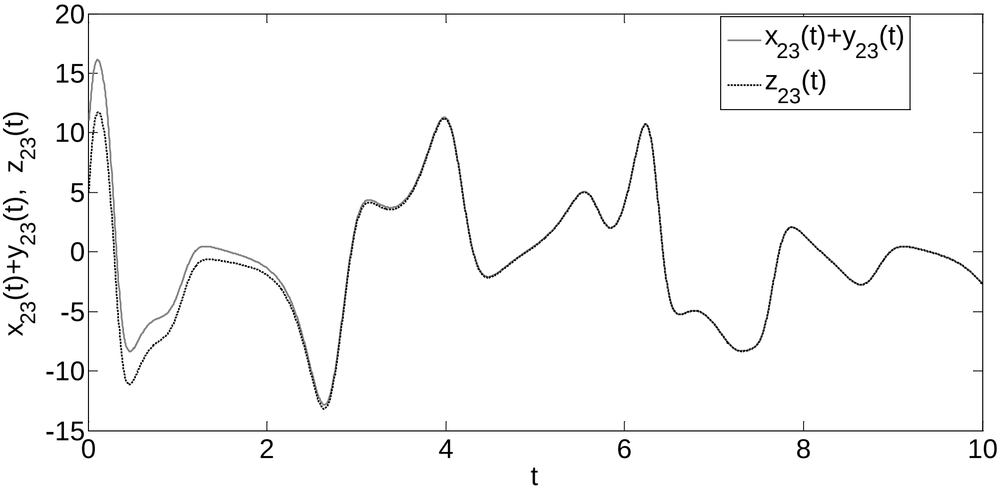

Figure 7.3 (h)

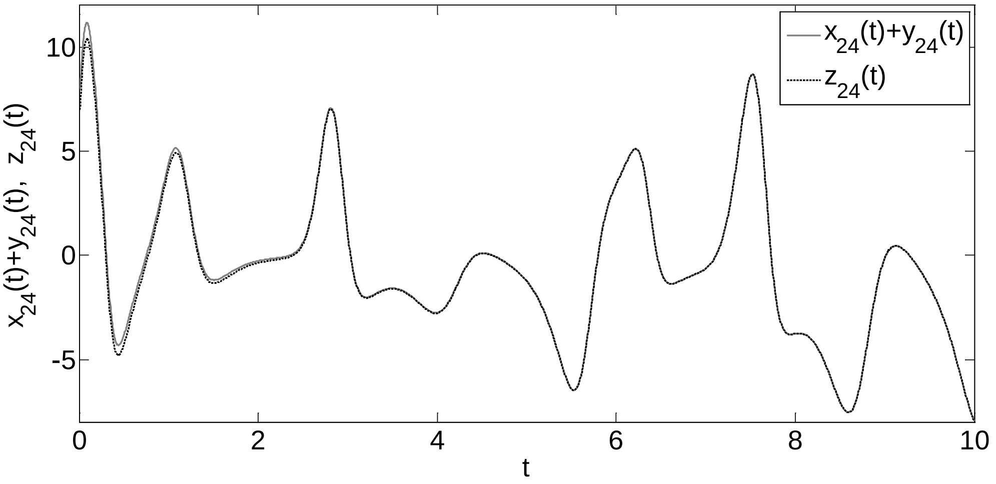

Figure 7.3 (i)

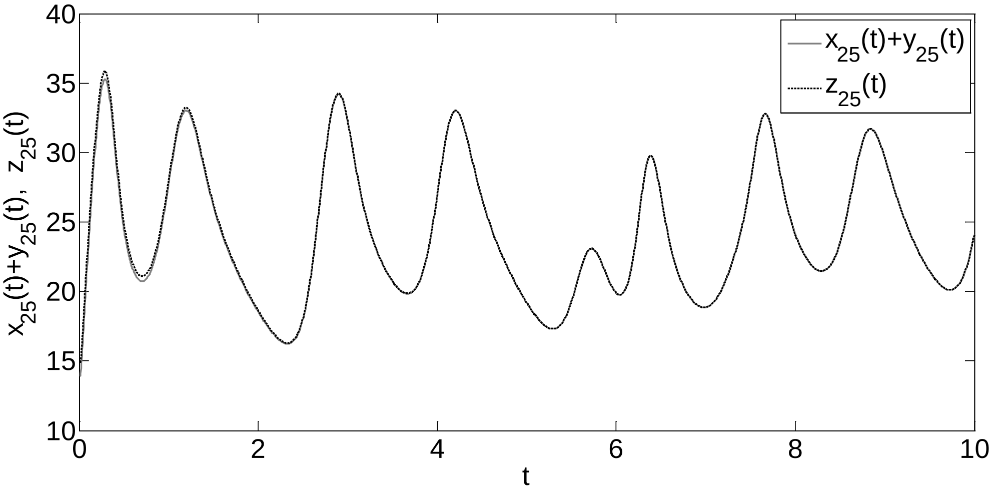

Figure 7.3 (j)

Figure 7.3 Dual combination synchronization of complex chaotic systems (7.14), (7.16)-(7.20) at $q = 0.95$ between: **(a)** $x_{11}(t) + y_{11}(t)$ and $z_{11}(t)$; **(b)** $x_{12}(t) + y_{12}(t)$ and $z_{12}(t)$; **(c)** $x_{13}(t) + y_{13}(t)$ and $z_{13}(t)$; **(d)** $x_{14}(t) + y_{14}(t)$ and $z_{14}(t)$; **(e)** $x_{15}(t) + y_{15}(t)$ and $z_{15}(t)$; **(f)** $x_{21}(t) + y_{21}(t)$ and $z_{21}(t)$; **(g)** $x_{22}(t) + y_{22}(t)$ and $z_{22}(t)$; **(h)** $x_{23}(t) + y_{23}(t)$ and $z_{23}(t)$; **(i)** $x_{24}(t) + y_{24}(t)$ and $z_{24}(t)$; **(j)** $x_{25}(t) + y_{25}(t)$ and $z_{25}(t)$

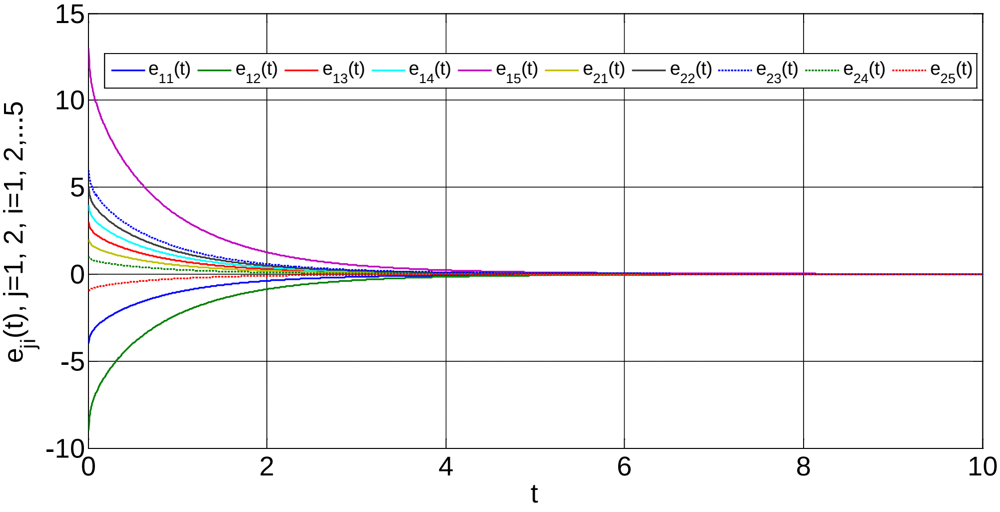

Figure 7.4 (a)

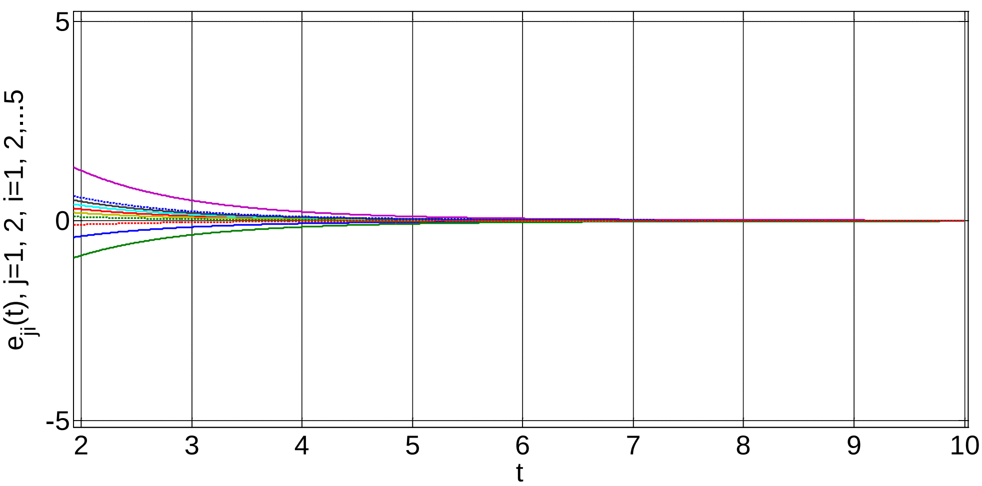

Figure 7.4 (b)

Figure 7.4 The evaluation of error functions $e_{ij}(t),\ i=1,2;\ j=1,2,\cdots,5$ at $q=0.95$.

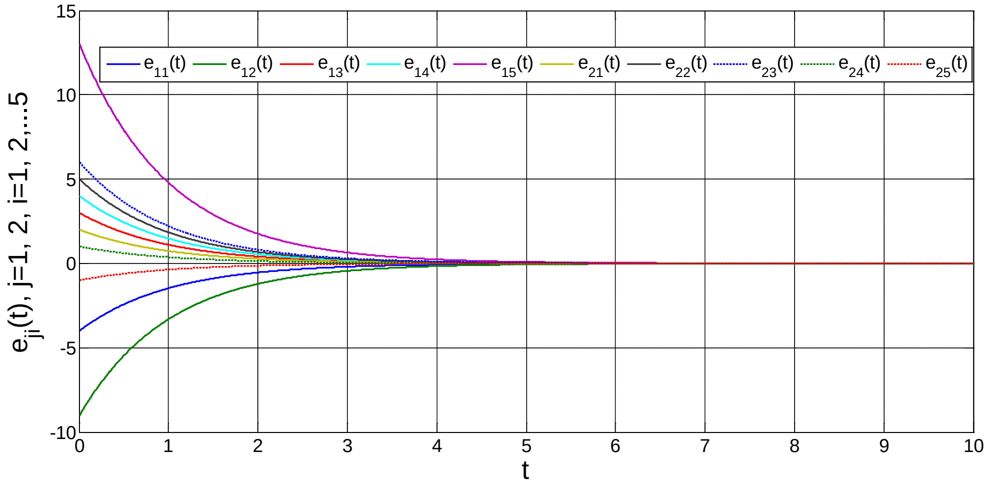

Figure 7.5 (a)

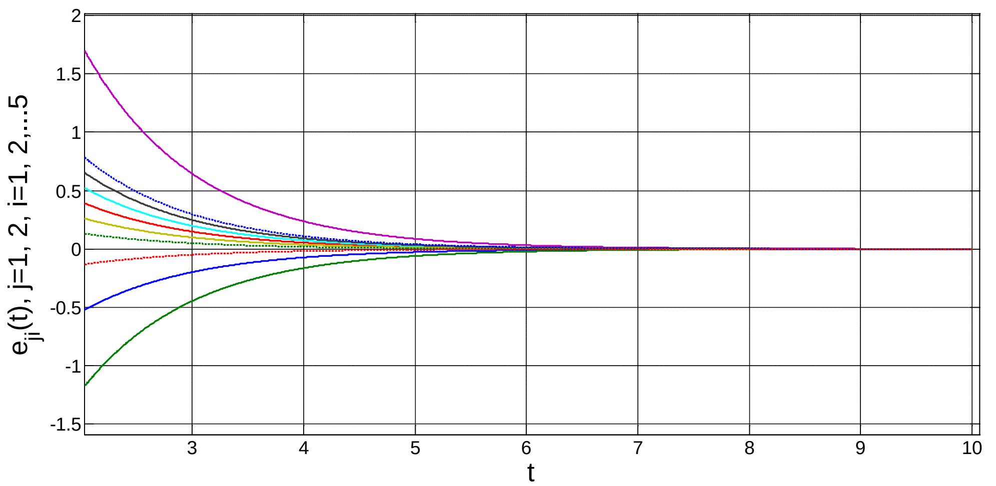

Figure 7.5 (b)

Figure 7.5 The evaluation of error functions $e_{ij}(t),\ i=1,2;\ j=1,2,\cdots,5$ at $q=1$.

7.6 Simulation results and discussion

In this section, to verify and demonstrate the feasibility of dual combination synchronization of fractional order complex chaotic systems, it is obtained the simulation results of the considered fractional order Lorenz and T systems in complex space at the fractional order $q = 0.95$. During simulation, the values of the parameters remain unchanged. The initial conditions are taken as $[x_{11}(t), x_{12}(t), x_{13}(t), x_{14}(t), x_{15}(t)]^T$ = $[2, 3, 5, 6, 9]^T$; $[x_{21}(t),\ x_{22}(t),\ x_{23}(t),\ x_{24}(t),\ x_{25}(t)]^T = [8, 7, 5, 6, 10]^T$; $[y_{11}(t),\ y_{12}(t),\ y_{13}(t),\ y_{14}(t),\ y_{15}(t)]^T = [5, 2, 3, 1, 6]^T$; $[y_{21}(t), y_{22}(t), y_{23}(t), y_{24}(t), y_{25}(t)]^T$ = $[3, 2, 6, 2, 4]^T$; $[z_{11}(t),\ z_{12}(t),\ z_{13}(t),\ z_{14}(t),\ z_{15}(t)]^T = [11, 14, 5, 3, 2]^T$ and $[z_{21}(t),\ z_{22}(t),\ z_{23}(t),\ z_{24}(t),\ z_{25}(t)]^T = [9, 4, 5, 7, 15]^T$; and hence the initial error is $[e_{11}(t),\ e_{12}(t),\ e_{13}(t),\ e_{14}(t),\ e_{15}(t),\ e_{21}(t),\ e_{22}(t),\ e_{23}(t),\ e_{24}(t),\ e_{25}(t)]^T = [-4, -9, 3, 4, 13, 2, 5, 6, 1, -1]^T$. Figures 7.3 (a)-(e) display the time response of the states $x_{1j}(t) + y_{1j}(t)$ and $z_{1j}(t)$ of the master systems (7.14), (7.16), (7.17), (7.18) and the slave systems (7.19), (7.20) with controllers (7.21), (7.22) where $j = 1(1)5$. Similarly Figures 7.3 (f)-(j) depict the time response of the states $x_{2j}(t) + y_{2j}(t)$ and $z_{2j}(t)$ of master systems and slave systems respectively for $j = 1(1)5$. Figure 7.4 (a) and Figure 7.5 (a) show that the error vectors asymptotically converge to zero as time becomes large which implies that dual combination synchronizations among the considered six chaotic systems are achieved at $q = 0.95$ and $q = 1$ respectively. For better illustration and understanding, the time differences for two cases are shown through Figures 7.4 (b) and 7.5 (b). The figures clearly exhibit that it takes less time for synchronization when the order of the derivative approaches from standard order to the fractional order system.

7.7 Conclusion

The primary purpose of the present chapter is to propose a novel scheme for the dual combination synchronization of fractional order complex chaotic systems. Another goal of this study is the stability analysis using Lyapunov stability theory with the proposed scheme used successfully for synchronization of four master and two slave systems. The author concludes that the scheme is very much useful for synchronization of a number of chaotic systems in fractional order as well as standard order cases. The author believes that the proposed scheme will play an important role in practical applications and it will attract the attention of the researchers working in the field of dynamical systems in both standard order and fractional order. The meaningful outcome of this study is the demonstration that less time is required for synchronization of the considered chaotic systems as the time derivative approaches towards fractional order from standard order.

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