Chapter 5

Synchronization of time-delay chaotic systems with uncertainties and external disturbances

5.1 Introduction

The history of chaos theory is started in the late 19-th century with one of the most important mistakes done by Poincare (Bunimovich and Vela-Arevalo (2015)), which was later termed as nonlinear science. It has attracted a remarkable attention of the researchers working in the field of science and engineering during last few decades.

Uncertainty and external disturbance frequently occur in various systems due to unmodelled dynamics, measurement inaccuracy, structural variations in plates, modelling errors and measurement environment, receiver suffering in secure communication effect the precision of communication, etc. Due to this reason, the investigation of synchronizing of two identical/ non-identical chaotic systems has become an interesting and a challenging research area in the presence of uncertainties and external disturbances (Li et al. (2016), Liu et al. (2016), Ghosh et al. (2012), Ojo et al. (2015), Othman et al. (2016)). The ideas of uncertainty have extremely applied to various types of problems, viz., the vehicle scheduling problem, fuzzy clustering, the key problem, the shortest circuit path, reliability issues, investment risk analysis, storage, project review, sorting, the location problem, the assignment problem, update, data analysis, e-commerce,

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information processing, network optimization, project management, logistics and supply chain management, the decision support system, economic policy (Ansari and Das (2015), Yu and Cao (2007), Chen et al. (2012), Zhang et al. (2004), Aghababa and Akbari (2012)).

As the dynamical systems governed by delay differential equations (DDE) have an infinite dimensional state space, where the attractors of the solutions are high dimensional. Delay induces instabilities as and when described by DDEs, play a major role in modelling of natural phenomena and these are as important as the general differential equation models for many chaotic systems. Experiment results regarding time-delay show the design is effective and feasible, and it can be applied to many physical problems. The first study on synchronization of chaos in time-delay system had been reported by Pyragas (1998). There are many important applications of synchronization of time-delay chaotic systems in various fields such as physical systems (Lakshmanan and Murali (1996)), chemical systems (Han et al. (1995)), ecological systems (Blasius et al. (1999)), secure communication, modelling brain activity, system identification, and pattern recognition phenomena (Murali and Lakshmanan (2003), Cuomo and Oppenheim (1993)). Many researchers have addressed the phenomena of uncertainties and existence of uncertain parameters in time-delay chaotic systems (Wu et al. (2016), Lu et al. (2017), Zhang et al. (2015), Zhai et al. (2017)).

5.2 Problem statement

Let us consider an uncertain chaotic drive system as

$$\dot{X}_1(t) = (B + \Delta B)\, X_1(t) + f_1\big(X_1(t)\big) + d_1(t)\ , \tag{5.1}$$

and an uncertain chaotic response system as

$$\dot{X}_2(t) = (C + \Delta C)\, X_2(t) + f_2\big(X_2(t)\big) + d_2(t) + u(t)\ , \tag{5.2}$$

where $X_1 = [x_1, y_1, \cdots, z_1]^T \in R^n$ and $X_2 = [x_2, y_2, \cdots, z_2]^T \in R^n$ are the state vectors, B, $C \in R^{n\times n}$ are the constant matrices, f_1, $f_2 : R^n \to R^n$ are nonlinear functions, ΔB, $\Delta C \in R^{n\times n}$ are parametric uncertainties such that $\|\Delta B\| \le \delta_1$, $\|\Delta C\| \le \delta_2$ with δ_1 and δ_2 positive constants, and $d_1(t)$ and $d_2(t)$ are the external disturbances such that $\|d_1(t)\|$ $< \rho_1$, $\|d_2(t)\| < \rho_2$, where ρ_1, $\rho_2 > 0$ and $u(t) \in R^n$ is the active control function to be designed later. Defining the synchronization error state as $e(t) = X_2(t) - X_1(t)$, the error dynamics is obtained as

$$\dot{e}(t) = (C + \Delta C + \Delta B)\ e(t) + F\big(X_1(t), X_2(t)\big) + d_2(t) - d_1(t) + u(t)\ , \tag{5.3}$$

where $F\big(X_1(t), X_2(t)\big) = f_2(X_2(t)) - f_1(X_1(t)) + (C + \Delta C - B)\, X_1(t) - \Delta B\, X_2(t)$.

To stabilise error system (5.3), active control method is employed to choose appropriate control function $u(t)$.

5.3 Systems' descriptions

5.3.1 Time-delay advanced Lorenz system

The time-delay advanced Lorenz system (Zhang et al. (2009)) is given by

$$\dot{x}_1(t) = a\,(y_1(t) - x_1(t)) + p\,x_1(t-\tau)\ ,$$

$$\dot{y}_1(t) = -x_1(t)\,z_1(t) + b\,x_1(t) + c\,y_1(t)\ ,$$

$$\dot{z}_1(t) = x_1^2(t) - d\,z_1(t)\ , \tag{5.4}$$

which shows chaotic behaviour for the control parameters' values $a = 20$, $b = 14$, $c = 10.6$, $d = 2.8$, $p = 3$ and $\tau = 0.001$ and initial condition $[x_1(t)\ ,\ y_1(t)\ ,\ z_1(t)]$ = $[-20\ ,\ 8\ ,\ 20]^T$, where $-\tau \le t \le 0$. The phase portrait of system (5.4) is shown by the Figure 5.1 (a).

The time-delay advanced Lorenz system (5.4) in the presence of uncertain terms and external disturbances is described as

$$\dot{x}_1(t) = a\,(y_1(t) - x_1(t)) + p\,x_1(t-\tau) + 0.13\,x_1(t) + 0.02\,z_1(t) + 10\sin 10t\ ,$$

$$\dot{y}_1(t) = -x_1(t)\,z_1(t) + b\,x_1(t) + c\,y_1(t) + 0.08\,x_1(t) - 20\cos 10t\ ,$$

$$\dot{z}_1(t) = x_1^2(t) - d\,z_1(t) + 0.01\,y_1(t) + 0.8\,z_1(t) + 15\sin 10t\ , \tag{5.5}$$

where the uncertain term $\Delta B = \begin{bmatrix} 0.13 & 0 & 0.02 \\ 0.08 & 0 & 0 \\ 0 & 0.01 & 0.8 \end{bmatrix}$, and disturbance term $d_1(t) = \begin{bmatrix} 10\sin 10t \\ -20\cos 10t \\ 15\sin 10t \end{bmatrix}$. Figure 5.1 (b) shows the phase portrait of the system (5.5) in $x_1 - y_1 - z_1$ space.

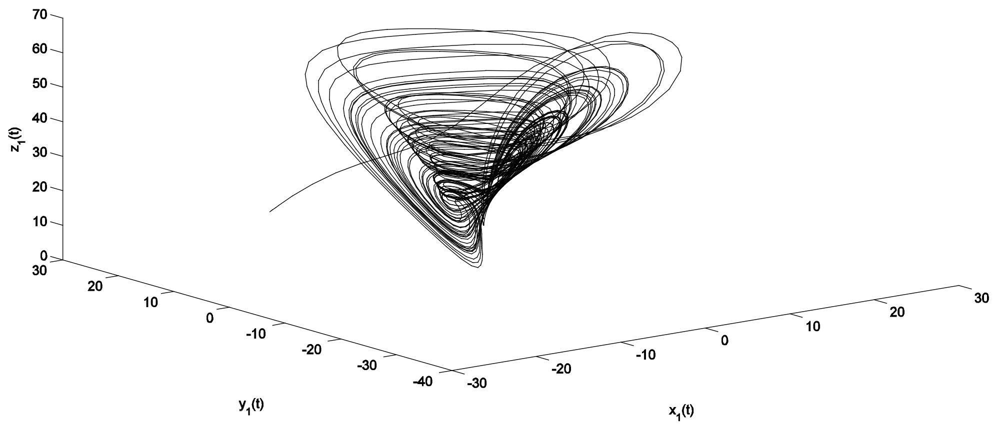

Figure 5.1 (a)

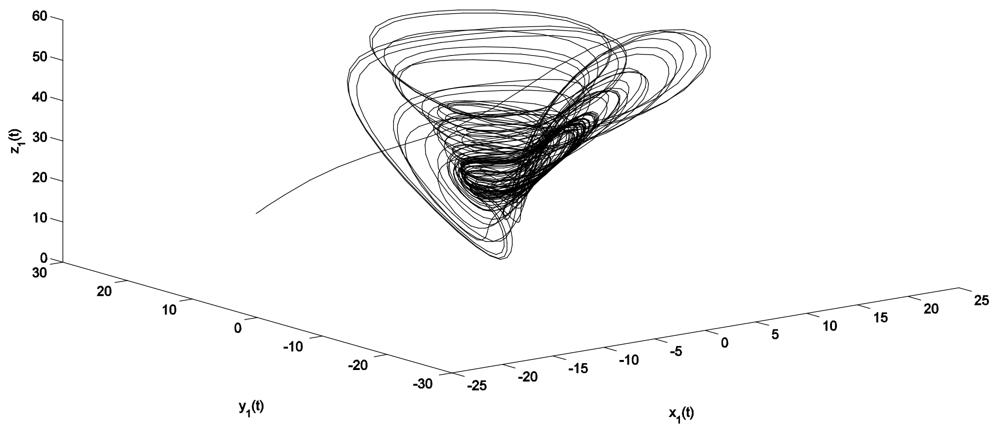

Figure 5.1 (b)

Figure 5.1: Phase portraits of **(a)** time-delay advanced Lorenz system, and **(b)** time-delay advanced Lorenz system with uncertainties and disturbances.

5.3.2 Double time-delay Rossler system

The double time-delay Rossler system (Ghosh et al. (2008)) with uncertainties and external disturbances is described as

$$\dot{x}_2(t) = -y_2(t) - z_2(t) + a_1\, x_2(t-\tau_1) + a_2\, x_2(t-\tau_2) + 0.01 x_2(t) + 0.5\sin 5t\ ,$$

$$\dot{y}_2(t) = x_2(t) + b_1\, y_2(t) + 0.05 z_2(t) + 0.4\cos 5t\ ,$$

$$\dot{z}_2(t) = b_2 + x_2(t)\, z_2(t) - c_1\, z_2(t) - 0.02\, y_2(t) + 0.1 z_2(t) - 0.1\sin 5t\ , \tag{5.6}$$

where uncertain term $\Delta C = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0 & 0.05 \\ 0 & -0.02 & 0.1 \end{bmatrix}$, and disturbance term

$d_2(t) = \begin{bmatrix} 0.5\sin 5t \\ 0.4\cos 5t \\ -0.1\sin 5t \end{bmatrix}$. The phase portraits of the system (5.6) without and with ΔC and $d_2(t)$ are shown in Figure 5.2 (a) and Figure 5.2 (b) respectively for the parameters $a_1 = 0.2$, $a_2 = 0.5$, $b_1 = b_2 = 0.2$, $c = 5.7$, $\tau_1 = 1.0$, $\tau_2 = 2.0$ and the initial condition $[x_2(t)\ ,\ y_2(t)\ ,\ z_2(t)] = [0.5\ ,\ 1\ ,\ 1.5]^T$.

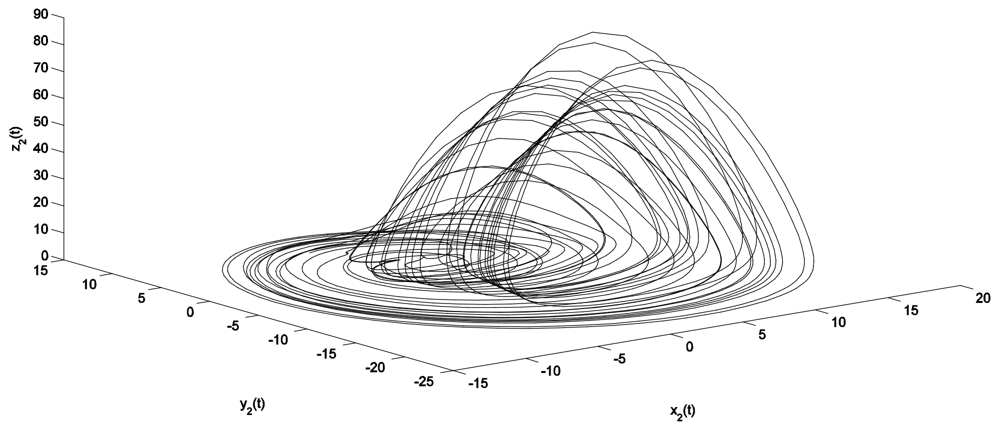

Figure 5.2 (a)

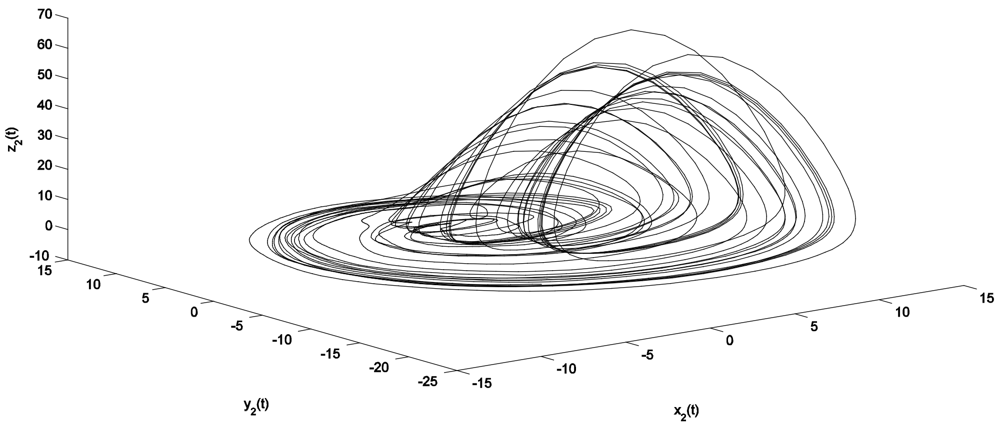

Figure 5.2 (b)

Figure 5.2: Phase portraits of **(a)** double time-delay Rossler system; **(b)** double time-delay Rossler system with uncertainties and external disturbances.

5.4 Synchronization using active control method

In this section, synchronization is achieved using active control method. Let us assume that the system (5.5) drives the system (5.6). Then the response system is re-written as

$$\dot{x}_2(t) = -y_2(t) - z_2(t) + a_1\, x_2(t-\tau_1) + a_2\, x_2(t-\tau_2) + 0.01 x_2(t) + 0.5\sin 5t + u_1(t)\ ,$$

$$\dot{y}_2(t) = x_2(t) + b_1\, y_2(t) + 0.05 z_2(t) + 0.4\cos 5t + u_2(t)\ ,$$

$$\dot{z}_2(t) = b_2 + x_2(t)\, z_2(t) - c_1\, z_2(t) - 0.02\, y_2(t) + 0.1 z_2(t) - 0.1\sin 5t + u_3(t)\ , \qquad (5.7)$$

where $u_1(t),\ u_2(t)$ and $u_3(t)$ are control functions.

Comparing system (5.1) with system (5.5), it is obtained constant matrix $B = \begin{bmatrix} -a & a & 0 \\ b & c & 0 \\ 0 & 0 & -d \end{bmatrix}$ and nonlinear function $f_1 = \begin{bmatrix} p\, x_1(t-\tau) \\ -x_1(t)\, z_1(t) \\ x_1^2(t) \end{bmatrix}$. Similarly, system (5.2) with system (5.7), $C = \begin{bmatrix} 0 & -1 & -1 \\ 1 & b_1 & 0 \\ 0 & 0 & -c_1 \end{bmatrix}$ and nonlinear function

$f_2 = \begin{bmatrix} a_1\, x_2(t-\tau_1) + a_2\, x_2(t-\tau_2) \\ 0 \\ b_2 + x_2(t)\, z_2(t) \end{bmatrix}$.

Now expressing error functions as $e_1(t) = x_2(t) - x_1(t)$, $e_2(t) = y_2(t) - y_1(t)$ and $e_3(t) = z_2(t) - z_1(t)$, the error systems is obtained as

$$\dot{e}_1(t) = 0.14\, e_1(t) - e_2(t) - 0.98\, e_3(t) - p\, x_1(t-\tau) + a_1\, x_2(t-\tau_1) + a_2\, x_2(t-\tau_2)$$
$$+ (0.01 + a) x_1(t) - (a+1) y_1(t) - z_1(t) - 0.13\, x_2(t) - 0.02\, z_2(t) + 0.5 \sin 5t$$
$$-10 \sin 10t + u_1(t) \ ,$$

$$\dot{e}_2(t) = 1.08\, e_1(t) + b_1\, e_2(t) + 0.05\, e_3(t) + (1-b)\, x_1(t) + (b_1 - c)\, y_1(t) + 0.05\, z_1(t)$$
$$-0.08\, x_2(t) + x_1(t)\, z_1(t) + 0.4 \cos 5t + 20 \cos 10t + u_2(t) \ ,$$

$$\dot{e}_3(t) = -0.01 e_2(t) + (0.9 - c_1) e_3(t) + b_2 - 0.02\, y_1(t) + (d - c_1 - 0.1)\, z_1(t)$$
$$-0.01\, y_2(t) - 0.8\, z_2(t) - x_1^2(t) + x_2(t)\, z_2(t) - 0.1 \sin 5t$$
$$-15 \sin 10t + u_3(t) \ . \tag{5.8}$$

Defining the active control functions as

$$u_1(t) = p\, x_1(t-\tau) - a_1\, x_2(t-\tau_1) - a_2\, x_2(t-\tau_2) - (0.01 + a) x_1(t) + (a+1) y_1(t)$$
$$+ z_1(t) + 0.13\, x_2(t) + 0.02\, z_2(t) - 0.5 \sin 5t + 10 \sin 10t + V_1(t) \ ,$$

$$u_2(t) = -(1-b)\,x_1(t) - (b_1 - c)\,y_1(t) - 0.05\,z_1(t) + 0.08\,x_2(t) - x_1(t)\,z_1(t)$$
$$- 0.4\cos 5t - 20\cos 10t + V_2(t)\ ,$$

$$u_3(t) = -b_2 + 0.02\,y_1(t) - (d - c_1 - 0.1)\,z_1(t) + 0.01\,y_2 + 0.8\,z_2 + x_1^2(t)$$
$$- x_2(t)\,z_2(t) + 0.1\sin 5t + 15\sin 10t + V_3(t)\ , \tag{5.9}$$

the error system (5.8) is reduced to

$$\dot{e}_1(t) = 0.14\,e_1(t) - e_2(t) - 0.98\,e_3(t) + V_1(t)\ ,$$
$$\dot{e}_2(t) = 1.08\,e_1(t) + b_1\,e_2(t) + 0.05\,e_3(t) + V_2(t)\ ,$$
$$\dot{e}_3(t) = -0.01\,e_2(t) + (0.9 - c_1)\,e_3(t) + V_3(t)\ . \tag{5.10}$$

The error system (5.10) is to be controlled as a linear system with new control inputs $V_1(t), V_2(t)$ and $V_3(t)$ as functions of error states. Defining the appropriate feedback control $V_i(t)$ in order to stabilise the system (5.10) so that $e_i(t) \to 0$ as $t \to \infty$ $i = 1,\ 2,\ 3$. This implies that system (5.5) is globally synchronised with system (5.7). This can be achieved by choosing a 3×3 matrix A such that

$$\begin{bmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \end{bmatrix} = A \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}.$$

There is no unique choice for matrix A. Let the matrix A is chosen in the form

$$A = \begin{bmatrix} -1.14 & 1 & 0.98 \\ -1.08 & -1-b_1 & -0.05 \\ 0 & 0.01 & -1.9+c_1 \end{bmatrix}.$$

Thus the synchronization between time-delay advanced Lorenz and double time-delay Rossler systems with uncertain terms and external disturbances is achieved.

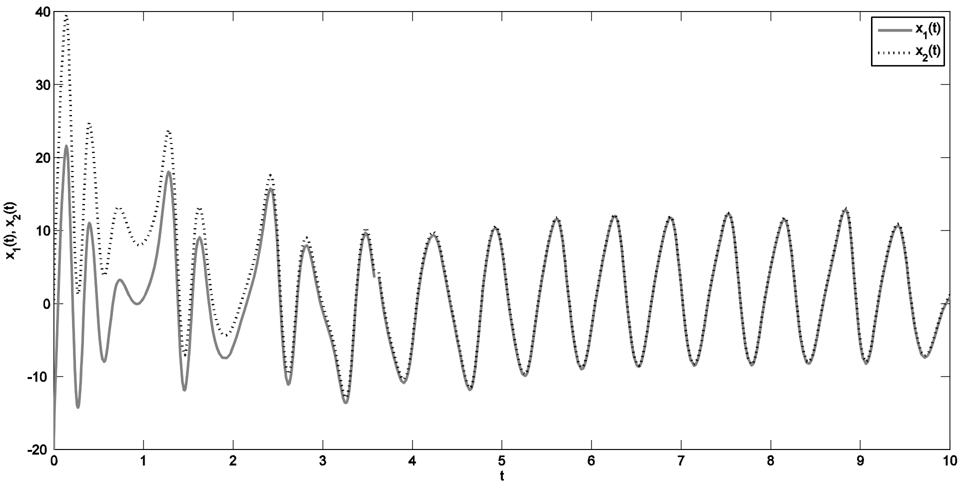

Figure 5.3 (a)

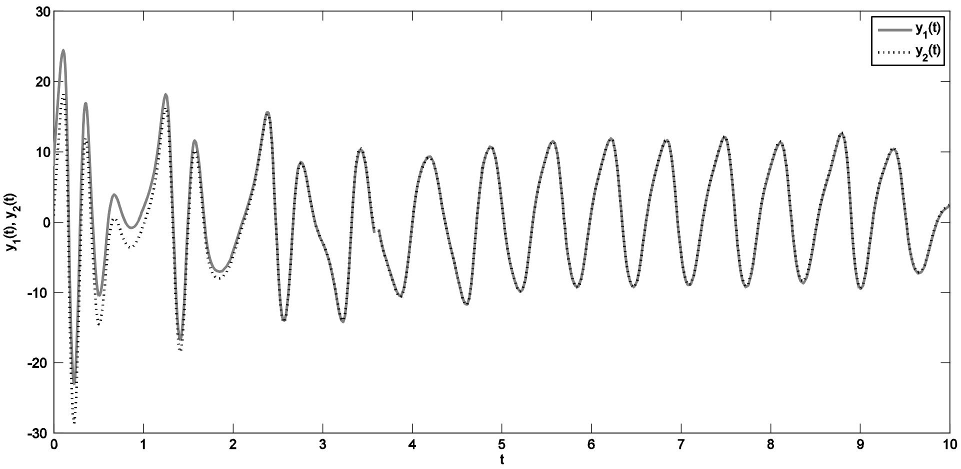

Figure 5.3 (b)

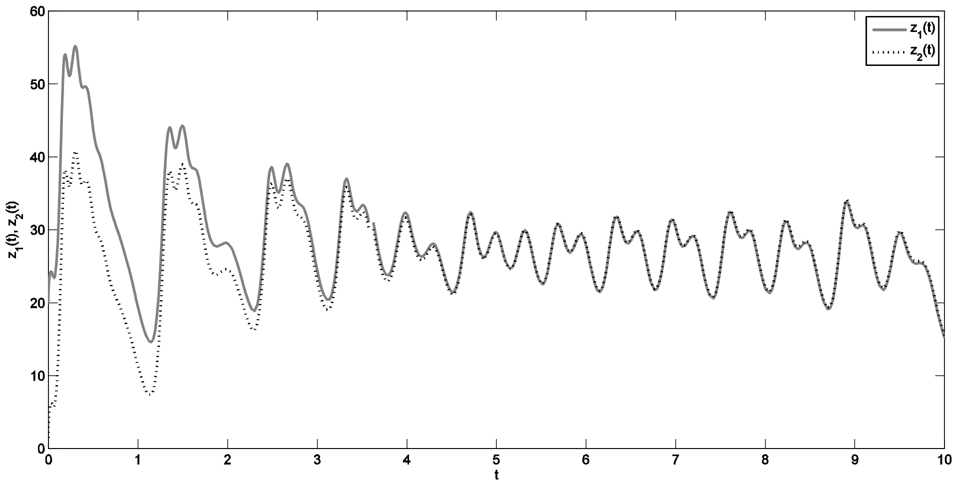

Figure 5.3 (c)

Figure 5.3: State trajectories of drive system (5.5) and response system (5.8) between: **(a)** $x_1(t)$ and $x_2(t)$, **(b)** $y_1(t)$ and $y_2(t)$, **(c)** $z_1(t)$ and $z_2(t)$.

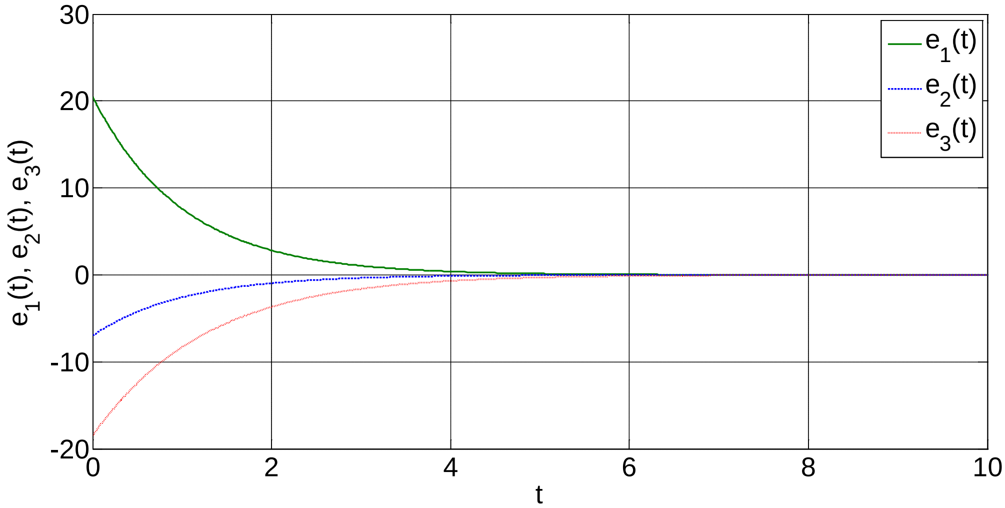

Figure 5.4 The evolution of the error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$.

5.5 Simulation results

In this section, numerical simulation results are given to verify the effectiveness of the active control method during synchronization of time-delay advanced Lorenz system and double time-delay Rossler system with parametric uncertainties and external disturbances. Fourth order Runge-Kutta method is used to solve the delay differential equations with time step size 0.001. The initial conditions are taken as before, and thus the initial error is $[e_1(t), e_2(t), e_3(t)] = [20.5,\ -7,\ -18.5]^T$. The state trajectories of drive and response systems are presented in Figure 5.3. It is seen from Figure 5.4 that the time-delay advanced Lorenz and double time-delay Rossler systems are synchronized with error functions converge to zero as time becomes large.

5.6 Concluding remarks

Three important goals have been achieved in the present study. First one is successfully demonstration of the synchronization between time-delay Lorenz and Rossler systems in the presence of uncertainties and external disturbances using active control method. Second one is the graphical presentation of numerical results with error states tending to zero as time goes to infinity, which clearly exhibits that the active control method is very much effective and reliable for synchronization of time-delay chaotic systems even with the uncertain terms and external disturbances. Third and the last achievement is the problem statement for synchronization of two time-delay chaotic systems in the presence of uncertainties and external disturbances in a very systematic way, which will surely attract the remarkable attention by the researchers working in the applications oriented field of physical systems in nonlinear sciences.

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