Chapter 4

Comparative study of synchronization methods of fractional order chaotic systems

4.1 Introduction

The fractional calculus which was in earlier stage considered as mathematical curiosity now becomes the object for the extensive development of fractional order differential equations for its applications in various areas of science and engineering. Geometric and physical interpretations of fractional differentiation and fractional integration can be found in the monograph of Podlubny's (Podlubny (1999)).

Chaos synchronization is an important topic in the nonlinear science. Generally speaking, the synchronization phenomenon has the feature that the trajectories of two systems (drive and response systems) are identical, despite starting from different initial conditions. However, slight changes in the initial conditions may lead to completely different trajectories. Therefore, how to control two chaotic systems to be synchronized have received lots of interest in past two decades to the researchers working in the field of chaos theory.

Active control method has widely been accepted as an efficient technique for the synchronization of non-identical chaotic systems, a feature for which it has got advantage

The contents of this chapter have been published in ***Nonlinear Engineering***.
http://dx.doi.org/10.1515/nleng-2016-0023

over other synchronization methods. After giving a generalised design of the method, this approach had been treated as one of the most interesting control strategies for its simplicity. It is seen from literature that Vincent and Laoye (2007) used active control based synchronization scheme for controlling directed transport arising from co-existing attractors in non-equilibrium physics.

Backstepping design and active control are both recognised as powerful design methods for chaos synchronization. Backstepping design can guarantee global stability, tracking and transient performance for a broad class of strict-feedback nonlinear systems (Zhang et al. (2005), Kokotovic (1992), Krstic et al. (1995)). This method has been employed for controlling and synchronizing many chaotic systems as well as hyperchaotic systems. Some of the method is widely used for synchronization of chaotic systems for the advantage that it needs only one controller to realise synchronization between chaotic systems and finally there are no derivatives in the controller (Tan et al. (2003)).

In the present chapter, active control method and backstepping approach are used to synchronize the fractional order chaotic systems. The fractional order Chen and Qi systems are taken to synchronize using both the methods. In 2008, both the methods were used by Vincent during synchronization of identical integer order systems already considered by A.M. Harb and M.A. Zohdy (Vincent (2008)). This work has inspired the authors to extend it in fractional order system. The main feature of the article is a comparative study of the time of synchronization through numerical simulation and graphical presentation.

4.2 Systems' descriptions

4.2.1 Fractional order Chen system

The fractional order Chen system (Li and Peng (2004)) of order q is given by

$$D^q x_1 = a_1(x_2 - x_1) \,,$$

$$D^q x_2 = (a_3 - a_1)x_1 - x_1 x_3 + a_3 x_2 \,,$$

$$D^q x_3 = x_1 x_2 - a_2 x_3 \,, \qquad 0 < q < 1 \,, \tag{4.1}$$

where a_1 , a_2 , a_3 are the parameters of the system (4.1). The phase portrait of the fractional order Chen system in $x_1 - x_2 - x_3$ space is depicted in Figure 4.1 at $q = 0.96$ for the values of parameters $a_1 = 35$, $a_2 = 3$, $a_3 = 28$ and the initial condition $[10,\, 25,\, 36]^T$.

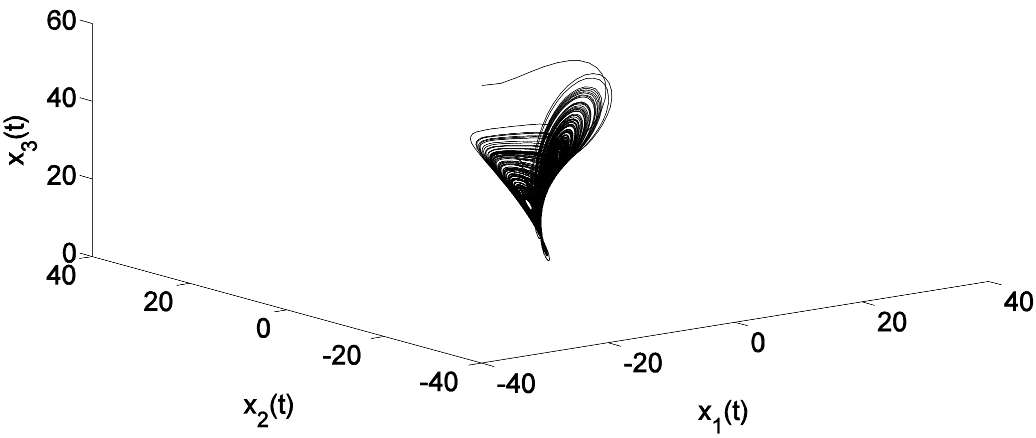

Figure 4.1: Phase portrait of the fractional order Chen system in $x_1 - x_2 - x_3$ space at the order $q = 0.96$.

4.2.2 Fractional order Qi system

The fractional order Qi system (Wang et al. (2010)) of order q is given by

$$D^q y_1 = b_1(y_2 - y_1) + y_2 y_3 \ ,$$

$$D^q y_2 = b_3 y_1 - y_2 - y_1 y_3 \ ,$$

$$D^q y_3 = -b_2 y_3 + y_1 y_2 \ , \tag{4.2}$$

where b_1, b_2 and b_3 are the parameters of the system. The phase portrait of system (4.2) at $q = 0.96$ is depicted through the Figure 4.2 for the values of the parameters $b_1 = 35$, $b_2 = \frac{8}{3}$, $b_3 = 80$ and initial condition $[3, 2, 1]^T$.

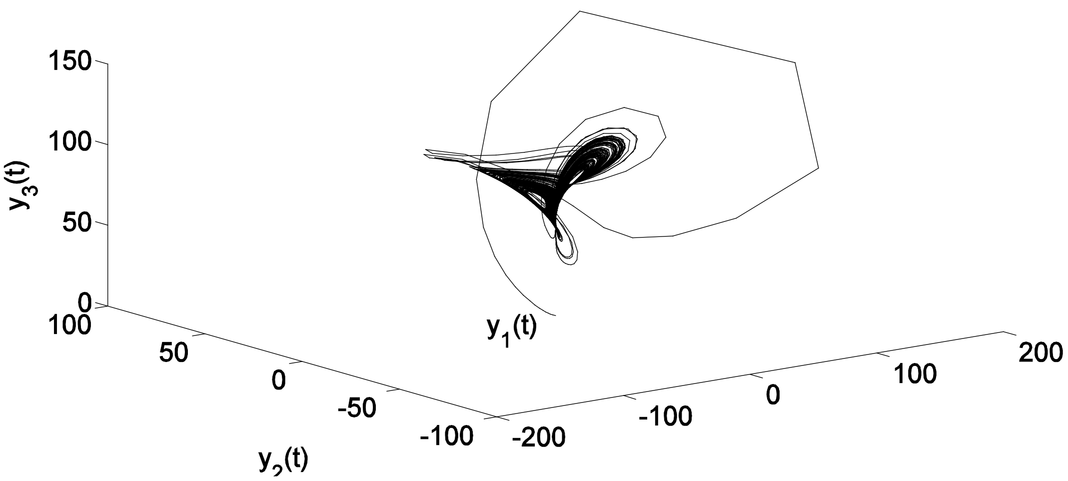

Figure 4.2 Phase portrait of the fractional order Qi system in $y_1 - y_2 - y_3$ space for the order of derivative $q = 0.96$.

4.3 Synchronization of fractional order Chen and Qi systems using active control method

In this section, the aim is to achieve synchronization between fractional order Chen system and Qi system by using active control method. Assume that the Chen system (4.1) drives the Qi system (4.2). The response system is re-written as

$$D^q y_1 = b_1(y_2 - y_1) + y_2 y_3 + u_1(t) ,$$

$$D^q y_2 = b_3 y_1 - y_2 - y_1 y_3 + u_2(t) ,$$

$$D^q y_3 = -b_2 y_3 + y_1 y_2 + u_3(t) , \tag{4.3}$$

where $u_1(t)$, $u_2(t)$ and $u_3(t)$ are control functions.

Defining the error $e(t) = [e_1(t), e_2(t), e_3(t)]^T$ as

$$e_i(t) = y_i(t) - x_i(t) , \qquad i = 1, 2, 3 . \tag{4.4}$$

The error system becomes

$$D^q e_1 = b_1(e_2 - e_1) + (b_1 - a_1)(x_2 - x_1) + y_2 y_3 + u_1 ,$$

$$D^q e_2 = b_3 e_1 - e_2 + (b_3 - a_3 + a_1)x_1 + x_1 x_3 - y_1 y_3 - (1 + a_3)x_2 + u_2 ,$$

$$D^q e_3 = -b_2 e_3 + (a_2 - b_2)x_3 + y_1 y_2 - x_1 x_2 + u_3 . \tag{4.5}$$

To achieve the aim, the active control function is re-defined as

$$u_1 = V_1 - (b_1 - a_1)(x_2 - x_1) - y_2 y_3 ,$$

$$u_2 = V_2 - (b_3 - a_3 + a_1)x_1 - x_1 x_3 + y_1 y_3 + (1 + a_3)x_2 ,$$

$$u_3 = V_3 - (a_2 - b_2)x_3 - y_1 y_2 + x_1 x_2 .$$

Using above equations in system (4.5), the error system becomes

$$D^q e_1 = b_1(e_2 - e_1) + V_1,$$

$$D^q e_2 = b_3 e_1 - e_2 + V_2,$$

$$D^q e_3 = -b_2 e_3 + V_3. \tag{4.6}$$

To control the linear error system (4.6) with control inputs $V_1(t)$, $V_2(t)$ and $V_3(t)$ as functions of the error states $e_i(t)$, $i = 1, 2, 3,$ our aim is to find the feedback control function in such a way that $e_i(t) \to 0$ as $t \to \infty$, $i = 1, 2, 3,$ so that the systems (4.1) and (4.3) are globally synchronized. To achieve this, there are many choices for the control functions $V_1(t)$, $V_2(t)$ and $V_3(t)$. Let us choose

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = A \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

where A is a 3×3 constant matrix. In order to make the closed loop system stable, the elements of the matrix A is chosen in such a way that the error system must have all the eigenvalues with negative real parts. Let the matrix A is chosen in the form

$$A = \begin{bmatrix} b_1 - 1 & -b_1 & 0 \\ -b_3 & 0 & 0 \\ 0 & 0 & b_2 - 1 \end{bmatrix}.$$

In this particular choice, the closed loop system has the eigenvalues -1, -1 and -1. This choice leads to a stable system, and thus the synchronization between fractional order Chen and Qi systems is achieved.

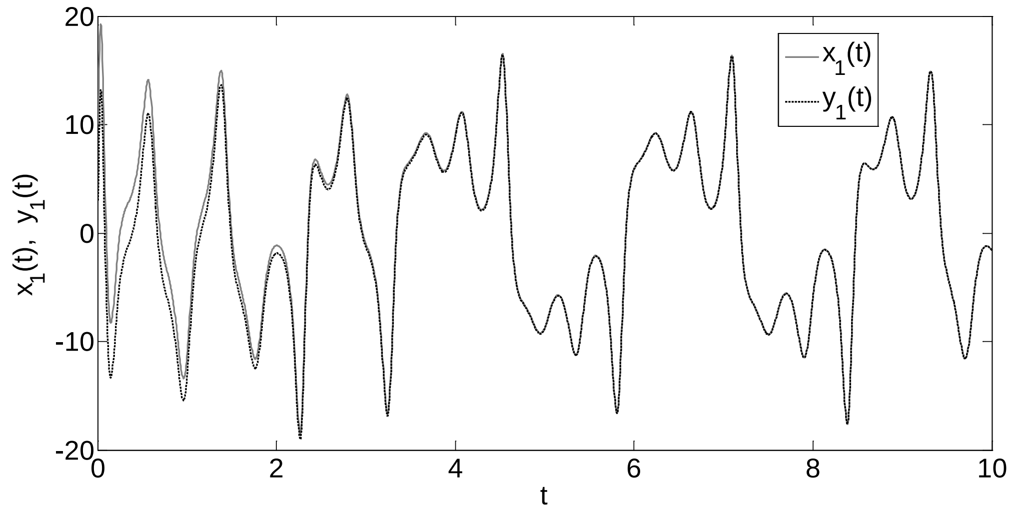

Figure 4.3 (a)

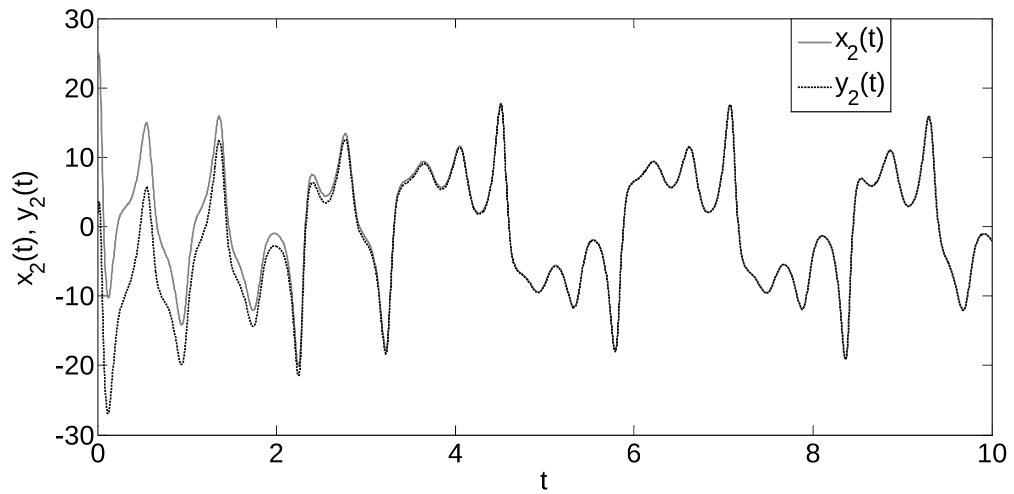

Figure 4.3 (b)

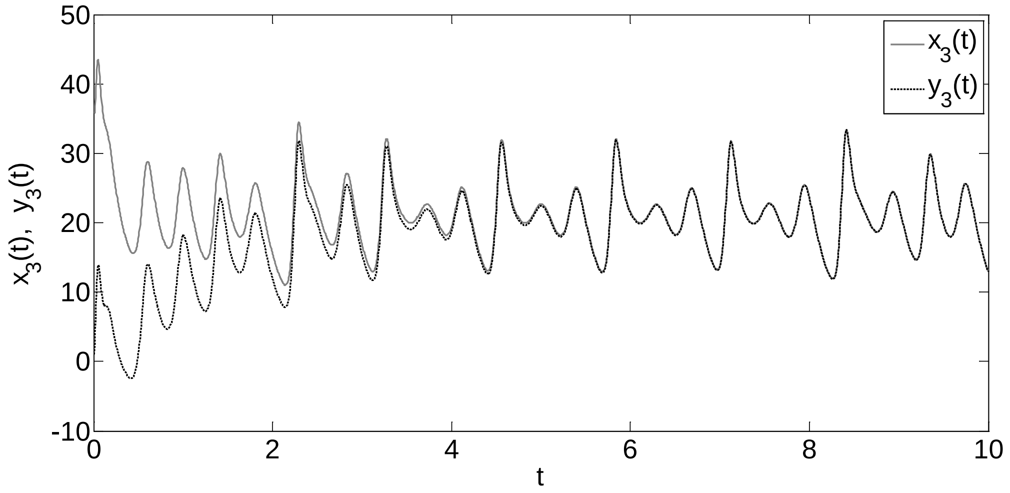

Figure 4.3 (c)

Figure 4.3: State trajectories of drive system (4.1) and response system (4.3) for fractional order $q = 0.96$ using active control method between: **(a)** x_1 and y_1; **(b)** x_2 and y_2; **(c)** x_3 and y_3.

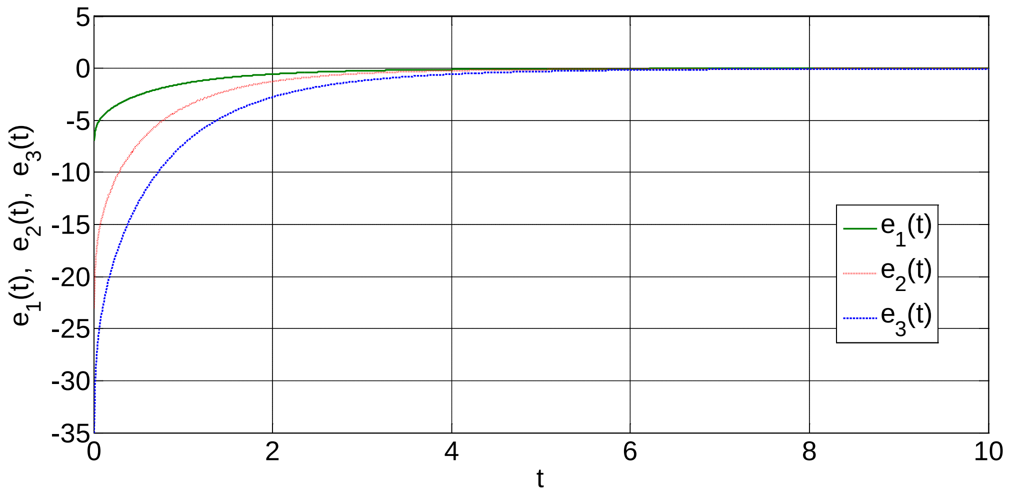

Figure 4.4 (a)

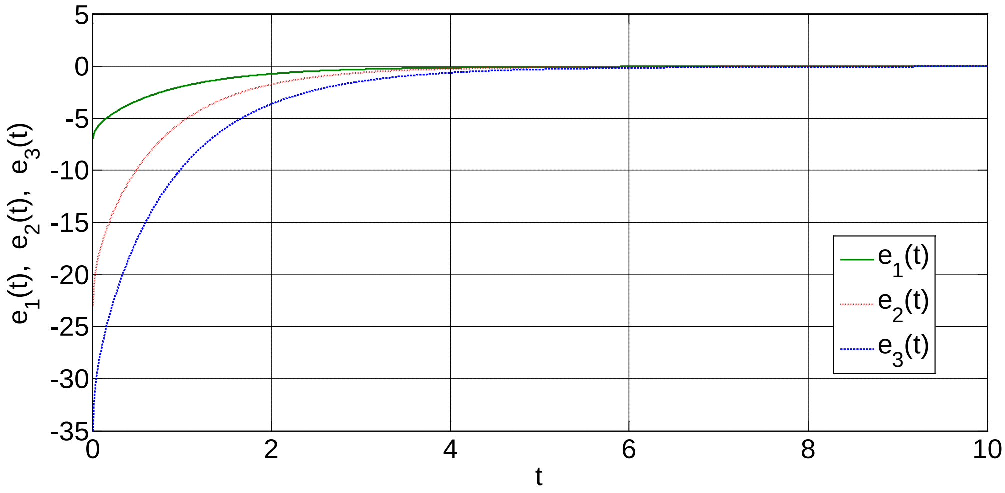

Figure 4.4 (b)

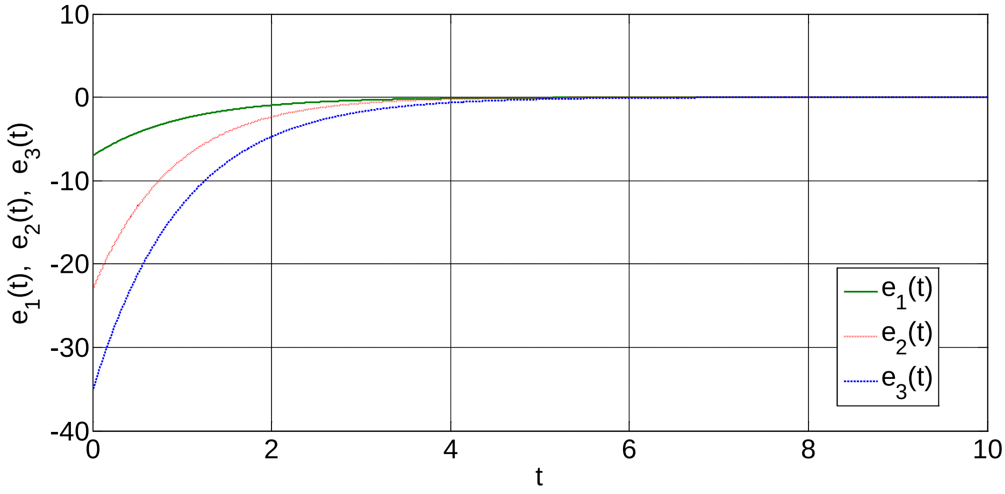

Figure 4.4 (c)

Figure: 4.4 The evolution of the error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$ using active control method at: **(a)** $q = 0.92$; **(b)** $q = 0.96$; **(c)** $q = 1$.

4.4 Synchronization of fractional order Chen and Qi systems using backstepping approach

In order to synchronize the fractional order Chen and Qi systems using backstepping approach, the fractional order Chen system (4.1) is taken as drive system and fractional order Qi system (4.3) considered as response system (4.3).

Defining error states as $e_1 = y_1 - x_1$, $e_2 = y_2 - x_2$, $e_3 = y_3 - x_3$, the dynamical error system reduces to

$$D^q e_1 = b_1(e_2 - e_1) + e_2 e_3 + e_2 x_3 + e_3 x_2 + \phi_1 + u_1 ,$$

$$D^q e_2 = b_3 e_1 - e_2 - e_1 e_3 - e_1 x_3 - e_3 x_1 + \phi_2 + u_2 ,$$

$$D^q e_3 = -b_2 e_3 + e_1 e_2 + e_1 x_2 + e_2 x_1 + \phi_3 + u_3 , \tag{4.7}$$

where

$$\phi_1 = x_2 x_3 + (b_1 - a_1)(x_2 - x_1) ,$$

$$\phi_2 = b_3 x_1 - x_2 - (a_3 - a_1)x_1 - a_3 x_2 ,$$

$$\phi_3 = (a_2 - b_2)x_3 .$$

System (4.7) can be considered as control problem where the system is to be controlled by the control functions $u_1(t)$, $u_2(t)$ and $u_3(t)$ which are the functions of error vectors $e_1(t)$, $e_2(t)$ and $e_3(t)$.

If the error states $e_1(t)$, $e_2(t)$ and $e_3(t)$ converge to zero as time t becomes large, then the systems (4.1) and (4.3) are said to be synchronized. Our next aim is to design control function properly using backstepping approach towards synchronization of chaotic systems.

Theorem 4.1: If the control function are chosen as

$$u_1 = -\phi_1 - w_3 x_2 \ ,$$

$$u_2 = -\phi_2 - b_3 w_1 - b_1 w_1 \ ,$$

$$u_3 = -\phi_3 - w_1 x_2 - w_1 w_2 \ ,$$

where $w_1 = e_1$, $w_2 = e_2$, $w_3 = e_3$, then the drive system (4.1) will be synchronized with the response system (4.3).

Proof: To achieve the results let us use backstepping approach which has three steps.

Step-I: Defining $w_1 = e_1$, it is obtained

$$D^q w_1 = b_1(e_2 - w_1) + e_2 e_3 + e_2 x_3 + e_3 x_2 + \phi_1 + u_1 \ , \tag{4.8}$$

where $e_2 = \alpha_1(w_1)$ is regarded as an virtual controller. For the design of $\alpha_1(w_1)$ to stabilize $w_1-\text{subsystem}$, choose the Lyapunov function as

$$v_1 = \frac{1}{2} w_1^2 \ .$$

The $q-$th order derivative of v_1 is

$$D^q v_1 = \frac{1}{2}\frac{d^q w_1^2}{dt^q} \le w_1 \frac{d^q w_1}{dt^q}$$

Using Lemma 1.10, it is obtained

$$D^q v_1 \le w_1[b_1(\alpha_1(w_1) - w_1) + \alpha_1(w_1)e_3 + \alpha_1(w_1)x_3 + e_3 x_2 + \phi_1 + u_1] \ .$$

Let us choose $u_1 = -\phi_1 - e_3 x_2$ and $\alpha_1(w_1) = 0$, then $D^q v_1 \le -b_1 w_1^2 < 0$ is negative definite. This implies that the $w_1 -$subsystem (4.8) is asymptotically stable.

Defining the error variable between e_2 and the estimative virtual controller $\alpha_1(w_1)$ as $w_2 = e_2 - \alpha_1(w_1)$, the $(w_1, w_2)-$subsystem is obtained as

$$D^q w_1 = b_1(w_2 - w_1) + w_2 e_3 + w_2 x_3 \ ,$$

$$D^q w_2 = b_3 w_1 - w_2 - w_1 e_3 - w_1 x_3 - e_3 x_1 + \phi_2 + u_2 \ , \tag{4.9}$$

where $e_3 = \alpha_2(w_1, w_2)$ is a virtual controller.

Step-II: In this step, to stabilize $(w_1, w_2)-$subsystem (4.9), the Lyapunov function v_2 is defined as

$$v_2 = v_1 + \frac{1}{2} w_2^2 = \frac{1}{2} w_1^2 + \frac{1}{2} w_2^2 \ .$$

The $q-$th order derivative of v_2 w.r. to t is

$$D^q v_2 = \frac{1}{2}\frac{d^q w_1^2}{dt^q} + \frac{1}{2}\frac{d^q w_2^2}{dt^q} \le w_1 \frac{d^q w_1}{dt^q} + w_2 \frac{d^q w_2}{dt^q}$$

Again using Lemma 1.10, above expression becomes

$$D^q v_2 \le -b_1 w_1^2 - w_2^2 + w_1[b_1 w_2 + w_2 \alpha_2(w_1, w_2) + w_2 x_3]$$

$$+ w_2[b_3 w_1 - w_1 \alpha_2(w_1, w_2) - w_1 x_3 - \alpha_2(w_1, w_2) x_1 + \phi_2 + u_2(t)] \, .$$

Lat us choose $\alpha_2(w_1, w_2) = 0$ and $u_2(t) = -\phi_2 - b_3 w_1 - b_1 w_1$, then $D^q v_2 \le -b_1 w_1^2 - w_2^2 < 0$ is negative define which shows that $(w_1, w_2)-$subsystem (4.9) is asymptotically stable.

Similarly considering $w_3 = e_3 - \alpha_2(w_1, w_2)$, the $(w_1, w_2, w_3)-$system becomes

$$D^q w_1 = b_1(w_2 - w_1) + w_2 w_3 + w_2 x_3 \ ,$$

$$D^q w_2 = -w_2 - w_1 w_3 - w_1 x_3 - w_3 x_1 - b_1 w_1 \ ,$$

$$D^q w_3 = -b_2 w_3 + w_1 w_2 + w_1 x_2 + w_2 x_1 + \phi_3 + u_3 \ . \tag{4.10}$$

Step-III: In order to stabilise $(w_1, w_2, w_3)-$ system, let us define the Lyapunov function v_3 as

$$v_3 = v_2 + \frac{1}{2} w_3^2 = \frac{1}{2} w_1^2 + \frac{1}{2} w_2^2 + \frac{1}{2} w_3^2 \ .$$

The $q-$th order fractional order derivative of v_3 w.r. to t is

$$D^q v_3 = \frac{1}{2}\frac{d^q w_1^2}{dt^q} + \frac{1}{2}\frac{d^q w_2^2}{dt^q} + \frac{1}{2}\frac{d^q w_3^2}{dt^q}$$

Once again using Lemma 1.10, above equation becomes

$$D^q v_3 \le w_1 \frac{d^q w_1}{dt^q} + w_2 \frac{d^q w_2}{dt^q} + w_3 \frac{d^q w_3}{dt^q}$$

$$D^q v_3 \le -b_1 w_1^2 - w_2^2 - b_2 w_3^2 - w_2 w_3 x_1 + w_3[w_1 w_2 + w_1 x_2 + w_2 x_1 + \phi_3 + u_3] \, .$$

Now choosing $u_3(t) = -\phi_3 - w_1 x_2 - w_1 w_2$, we get $D^q v_3 \le -b_1 w_1^2 - w_2^2 - b_2 w_3^2 < 0$, which implies that $(w_1, w_2, w_3)-$ system is asymptotically stable. Thus for $w_1 = e_1$, $w_2 = e_2 - \alpha_1(w_1) = e_2$, $w_3 = e_3 - \alpha_2(w_1, w_2) = e_3$, the state errors e_1, e_2 and e_3 converge to zero after a finite period of time, confirm the synchronization between fractional order Chen and Qi systems.

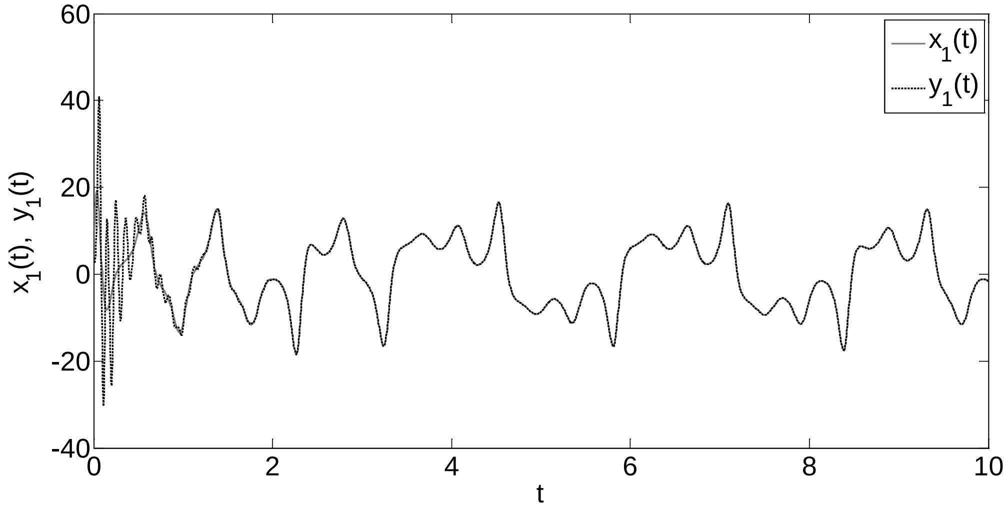

Figure 4.5 (a)

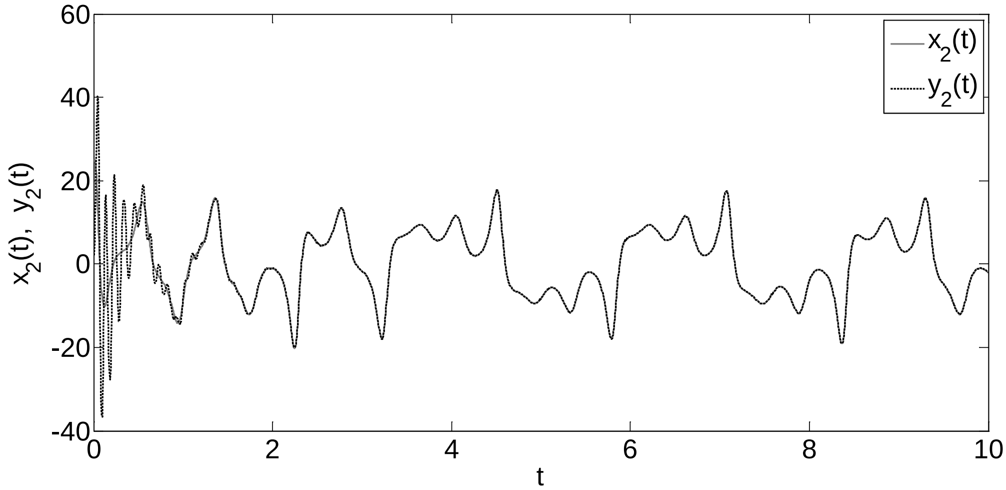

Figure 4.5 (b)

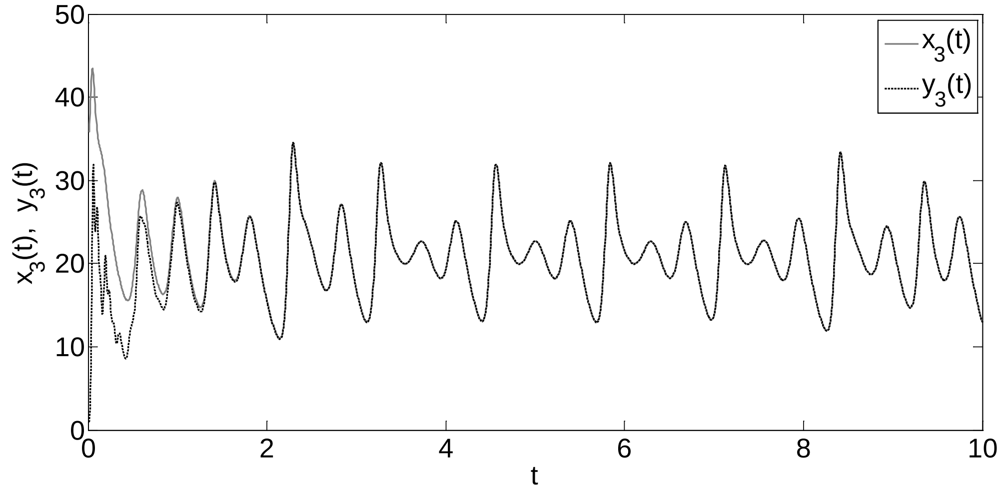

Figure 4.5 (c)

Figure 4.5: State trajectories of drive system (1) and response system (3) for fractional order $q = 0.96$ between: **(a)** x_1 and y_1 ; **(b)** x_2 and y_2 ; **(c)** x_3 and y_3 using backstepping method.

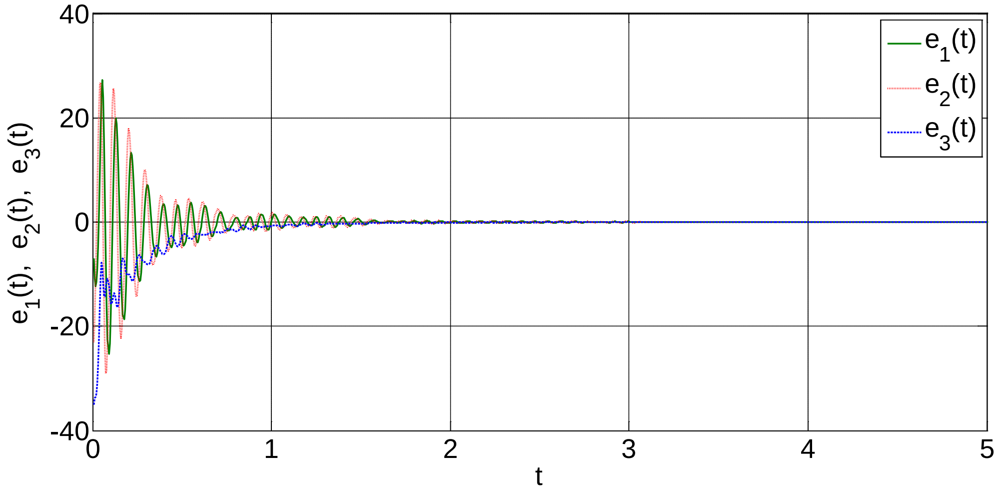

Figure 4.6 (a)

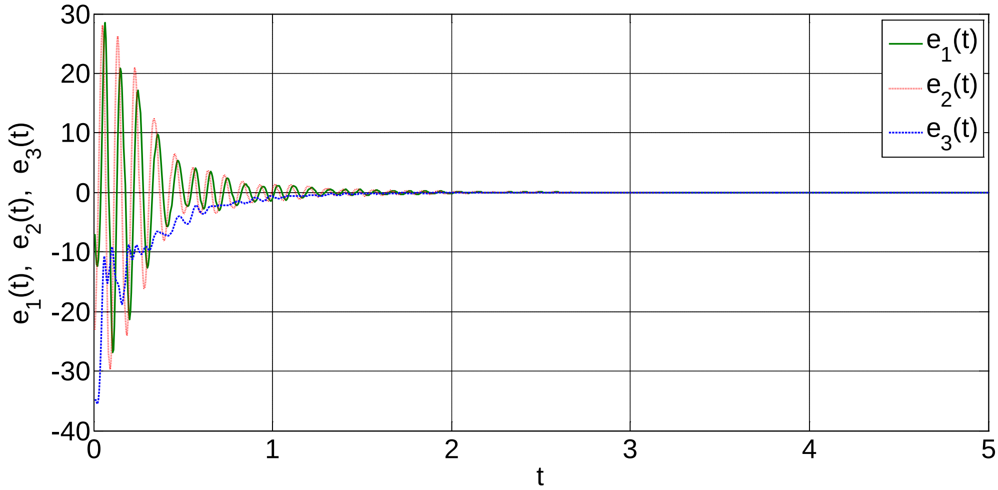

Figure 4.6 (b)

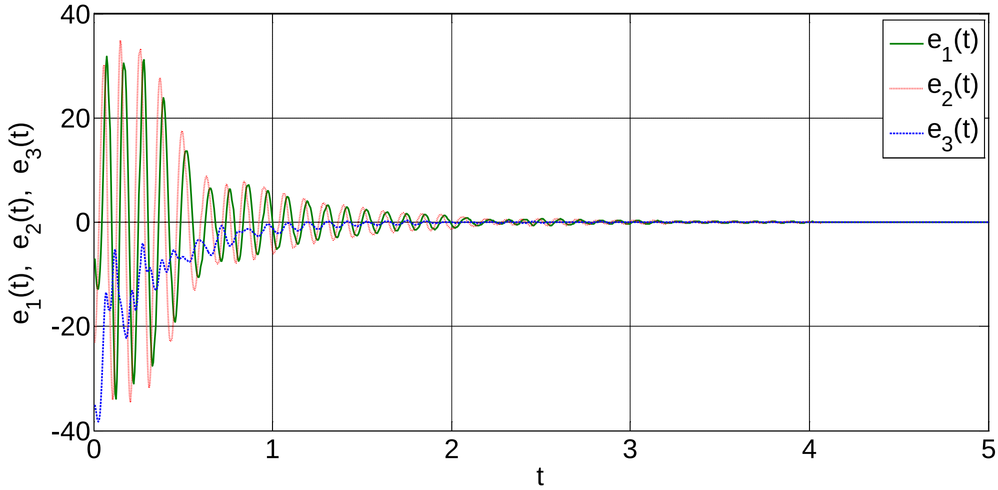

Figure 4.6 (c)

Figure 4.6: The evolution of the error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$ using backstepping method at: **(a)** $q = 0.92$; **(b)** $q = 0.96$; **(c)** $q = 1$.

4.5 Numerical simulation and results

In this section, the earlier considered values of the parameters are taken for both the systems. The initial conditions of drive and response systems are taken as $[x_1(0), x_2(0), x_3(0)]^T = [10, 25, 36]^T$ and $[y_1(0), y_2(0), y_3(0)]^T = [3, 2, 1]^T$ respectively. Hence the initial conditions of error system is $[e_1(0), e_2(0), e_3(0)] = [-7, -23, -35]^T$. During synchronization of the systems, the time step size is taken as 0.005. Synchronization in $x_1 - y_1$ space, $x_2 - y_2$ space and $x_3 - y_3$ space are depicted through Figure 4.3 and Figure 4.5 at $q = 0.96$ for active control method and backstepping approach respectively. The error functions are depicted in Figure 4.4 and Figure 4.6 for active control method and backstepping approach respectively at $q = 0.92, 0.96, 1$. It is clear from the figures that in both the cases it takes less time to synchronize as the systems' pair approaches from standard order to fractional order. Also, it is found from the figures that it takes less time for synchronization for the backstepping method compared to the active control method.

4.6 Conclusion

The theme of the present chapter is to investigate the synchronization between two non-identical fractional order chaotic systems using active control method and backstepping method. Based on stability analysis, the required synchronization of the chaotic systems viz., Chen and Qi systems has been achieved. The components of error systems tending to zero as time progresses is an attempt through proper choices of control functions. This approach helps to get the time required for synchronization. The novelty of this chapter is finding that less time is required for synchronization, when the system' pair approaches to fractional order from the standard order, upon application of both the methods exhibited through graphical presentations. Additionally, a comparison of time requirement for both standard and fractional order derivatives applying the active control method and backstepping method has been made.

*****