Chapter 3

Synchronization between fractional order complex chaotic systems with uncertainty

3.1 Introduction

Synchronization of chaotic systems of coupled oscillators is one of the important issues in the frontier of nonlinear dynamics and complex systems. This study provides the understanding of the collective behaviours in many fields, such as the power grids, the flashing of fireflies, the rhythm of pacemaker cells of the heart, and even some social phenomena. Theoretically, the chaotic systems with uncertainty turn out to be paradigms for synchronization problem, which have inspired a wealth of works because of both their simplicity for mathematical treatment and their relevance to practice (Xu et al. (2016)). Examples of synchronization appear in various fields of science and engineering like coupled circuits in electronics, cardiac and respiratory systems or EEG signals in physiology, in different ecological systems, coupled laser systems in non-linear optics. The ideas can be applied to chaos secure communication for the sake of higher signal transmission efficiency and secure performance (Wang et al. (2016)). In such systems, many different synchronization states may arise as complete synchronization, combination synchronization, dual synchronization, dual combination synchronization, phase synchronization, cluster synchronization, lag synchronization, generalized synchronization, adaptive function projective combination synchronization, etc. (Naderi

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and Khaeiri (2016), Sun and Shen (2016), Mahmoud et al. (2016)). Several methods can be used to achieve such types of synchronizations viz., active control method, adaptive control method (Chen and Lu (2002a, 2002b)), OGY method (Ott et al. (1990)), sliding mode control method (Haeri et al. (2007)), linear and nonlinear feedback method, time-delay feedback approach, backstepping approach, etc.

Chaos synchronization using active control method was proposed by Bai and Lonngren (1997) for the integer order identical real chaotic systems. Researchers studied this technique further and applied to non-identical, fractional order, complex chaotic systems (Ho and Hung (2002), Agrawal et al. (2012), Srivastava et al. (2014)). The method is used by the author to achieve the synchronization of two general uncertain chaotic systems and to obtain a sufficient condition for the stability of the error dynamics to demonstrate effectiveness and feasibility of the considered technique.

Nevertheless, the works as mentioned earlier on the synchronization of complex chaotic systems mainly consider the complex chaotic systems without uncertain parameters. In other words, uncertainties frequently happen in various systems due to modelling errors, measurement inaccuracy, linear approximation, and so on. Uncertainties exist in operation research, management science, information science, system science, computer science and industrial engineering, and many other fields. So uncertainty is applied to solve many problems, viz., the vehicle scheduling problem, the key problem, the shortest circuit path, reliability issues, storage, sorting, the location problem, the assignment problem, update, fuzzy clustering, data analysis, information processing, network optimization, project review, project management, e-commerce, investment risk analysis, logistics and supply chain management, the decision support system, economic policy, etc. Many researchers have studied the problems of uncertainties and they have found that uncertain parameters exist in chaotic systems extensively. The fractional order chaotic system with uncertain parameters has already been studied in the real variable (Ran et al. (2014), Ma et al. (2013), Agrawal and Das (2014), Riahil et al. (2016), Hajipour and Aminabadi (2016), Li et al. (2015), Mathiyalagan et al. (2015), Wang and Qi (2016)), but for the case of the complex variable, it is first of its kind. In practical applications, chaotic systems often with uncertain parameters provide new challenges to the synchronization. Therefore, synchronization of the complex chaotic systems with uncertain parameters in fractional order systems is a significant research topic.

The fractional order chaotic system, a generalisation of the integer order chaotic system, is considered as a new alternative for which a considerable attention has been focused on developing techniques for modelling, synchronization and control of the family of generalised dynamical systems. Many researchers have made a lot of contributions in intelligent control approaches varied from the conventional to the advance (Soukkou et al. (2016)). In particular, Petras (2011) presented a survey of fractional dynamical systems, modelling, stability analysis and control. The fractional order complex chaotic systems are used to increase the content and security of transmitting information signals. This kind of systems has played an important role to discuss the chaotic systems in population inversion, polymer physics, etc. Therefore, it is a meaningful and exciting topic for scientists and researchers to study of dynamic behaviour, chaos control, chaos synchronization of fractional order complex chaotic systems (Xi et al. (2015)).

With the inspiration from the above discussions, the present chapter is constructed to obtain the synchronization of fractional order complex chaotic systems with uncertain parameters. Section 3.2 contains problem formulation of the fractional order drive-response systems in the presence of uncertainty, and demonstration of active control method is discussed. In section 3.3, systems' descriptions are given. Section 3.4 contains the synchronization of the considered systems which is achieved using active control method. Numerical simulation and results are carried out in section 3.5, which is followed by a conclusion given in section 3.6.

3.2 Problem formulation

Consider an uncertain fractional order complex chaotic system (Yadav et al. (2015)) as a drive system as

$$D^q X(t) = (A + \Delta A)\, X(t) + f_1\big(X(t)\big), \tag{3.1}$$

and another uncertain fractional order complex chaotic system as a response system as

$$D^q Y(t) = (B + \Delta B)\, Y(t) + f_2\big(Y(t)\big) + u(t)\ , \tag{3.2}$$

where $X = [x_1, x_2, \cdots, x_n]^T \in C^n$ and $Y = [y_1, y_2, \cdots, y_n]^T \in C^n$ are the state vectors, A, $B \in C^{n\times n}$ are the constant matrices, f_1, $f_2 : C^n \to C^n$ are nonlinear functions, ΔA, $\Delta B \in C^{n\times n}$ are parametric uncertainties such that $\|\Delta A\| \le \delta_1$, $\|\Delta B\| \le \delta_2$ with δ_1 and δ_2 positive constants, and $u(t) \in C^n$ is the active control function. Now, the aim is to design an appropriate active controller $u(t)$ such that the trajectories of the response system (3.2) asymptotically approach towards the drive system (3.1) and finally achieve synchronization. Defining the error state $e(t)$ as

$$e(t) = Y(t) - X(t)\ , \tag{3.3}$$

the error system is obtained as

$$D^q e(t) = (B + \Delta B + \Delta A)\ e(t) + F(X(t), Y(t)) + u(t)\ , \tag{3.4}$$

where $F(X(t), Y(t)) = f_2(Y(t)) - f_1(X(t)) + (B + \Delta B - A)\ X(t) - \Delta A Y(t)$ ·

To stabilise error system (3.4), an appropriate control function $u(t)$ has to be chosen using active control method.

3.2.1 Active control method

According to the active control method, the control input $u(t)$ is defined to eliminate nonlinear term $F(X(t), Y(t))$ in error system (3.4). Defining the control function as

$$u(t) = V(t) - F(X(t), Y(t))\ , \tag{3.5}$$

the error system (3.4) becomes,

$$D^q e(t) = (B + \Delta B + \Delta A)\ e(t) + V(t)\ , \tag{3.6}$$

where the error system (3.6) is a linear system with a control function $V(t)$ as a function of the error vector $e(t)$. The controller function $V(t)$ is chosen in such a way that the system (3.6) becomes stable. There is not a unique choice for such functions. One can choose

$$V(t) = C\, e(t)\ , \tag{3.7}$$

where C is an $n \times n$ constant matrix. Then error system (3.6) becomes

$$D^q e(t) = (B + C + \Delta B + \Delta A)\ e(t)\ . \tag{3.8}$$

Matrix C is chosen so that the eigenvalues of the matrix $B + C + \Delta B + \Delta A$ are satisfied for $\arg(\lambda) > q\pi/2$. This choice will lead to $\lim_{t\to\infty} \|e(t)\| = 0$ and hence the chaos synchronization between considered uncertain fractional order complex chaotic systems is achieved (Matignon (1996)).

3.3 Systems' descriptions

3.3.1 The fractional order complex Lorenz system with uncertainty

The fractional order complex Lorenz system (Luo and Wang (2013b)) is given by

$$D^q x_1' = a_1(x_2' - x_1') \ ,$$

$$D^q x_2' = a_2 x_1' - x_2' - x_1' x_3' \ ,$$

$$D^q x_3' = \frac{1}{2}(\bar{x}_1' x_2' + x_1' \bar{x}_2') - a_3 x_3' \ , \tag{3.9}$$

where $x' = [x_1', x_2', x_3']^T$ is the state variable vector, $x_1' = x_1 + i\,x_2$ and $x_2' = x_3 + i\,x_4$ are complex variables while $x_3' = x_5$ is real variable and a_1, a_2, a_3 are parameters. The fractional order complex Lorenz system (3.9) with uncertain term is described as

$$D^q x_1' = a_1(x_2' - x_1') + 0.2x_2' - 0.05x_3' \ ,$$

$$D^q x_2' = a_2 x_1' - x_2' - x_1' x_3',+0.3x_1' \ ,$$

$$D^q x_3' = \frac{1}{2}(\bar{x}_1' x_2' + x_1' \bar{x}_2') - a_3 x_3' - 0.5x_3' \ . \tag{3.10}$$

Separating into real and imaginary parts, the system (3.10) becomes

$$D^q x_1 = a_1(x_3 - x_1) + 0.2x_3 - 0.05x_5 \ ,$$

$$D^q x_2 = a_1(x_4 - x_2) + 0.2x_4 \ ,$$

$$D^q x_3 = a_2 x_1 - x_3 - x_1 x_5 + 0.3x_1 \ ,$$

$$D^q x_4 = a_2 x_2 - x_4 - x_2 x_5 + 0.3x_2 \ ,$$

$$D^q x_5 = x_1 x_3 + x_2 x_4 - a_3 x_5 - 0.5x_5 \ , \tag{3.11}$$

where the uncertain parameter is given by

$$\Delta A = \begin{bmatrix} 0 & 0 & 0.2 & 0 & -0.05 \\ 0 & 0 & 0 & 0.2 & 0 \\ 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.5 \end{bmatrix}.$$

Taking the values of the parameters as $a_1 = 10$, $a_2 = 180$, $a_3 = 1$ and initial condition $x(0) = [2,\ 3,\ 5,\ 6,\ 9]^T$ at the fractional derivative $q = 0.95$, the system (3.11) possesses the chaotic attractor given in Figure 3.1.

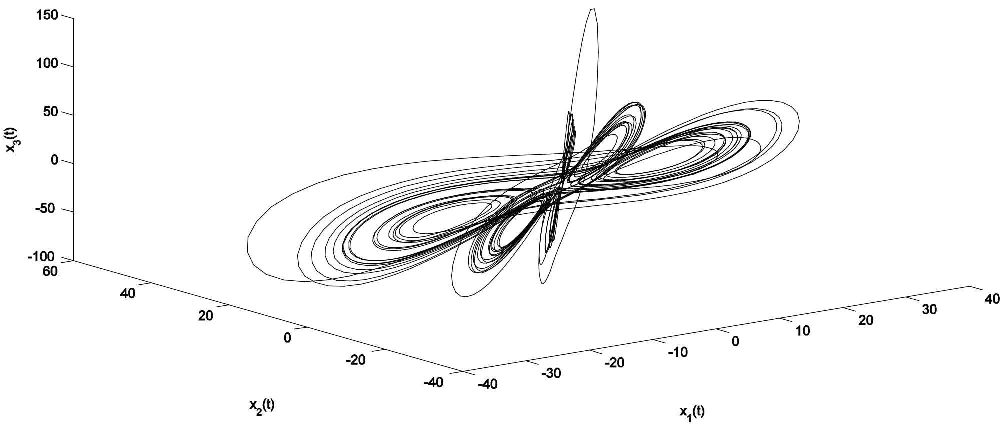


Figure 3.1 (a)

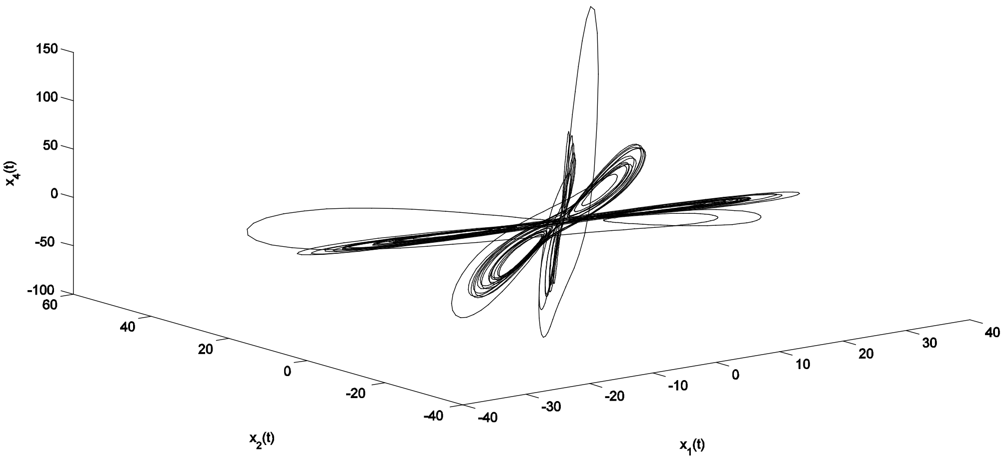
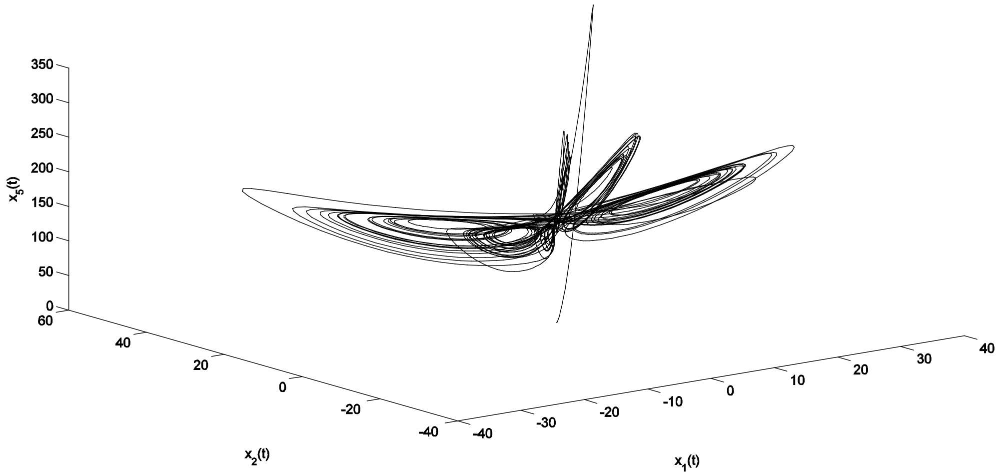


Figure 3.1 (b)



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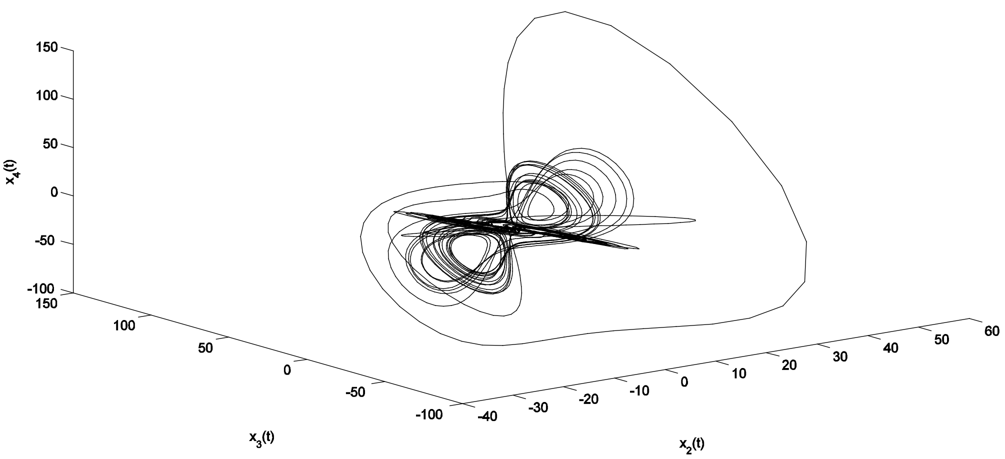

Figure 3.1 (d)

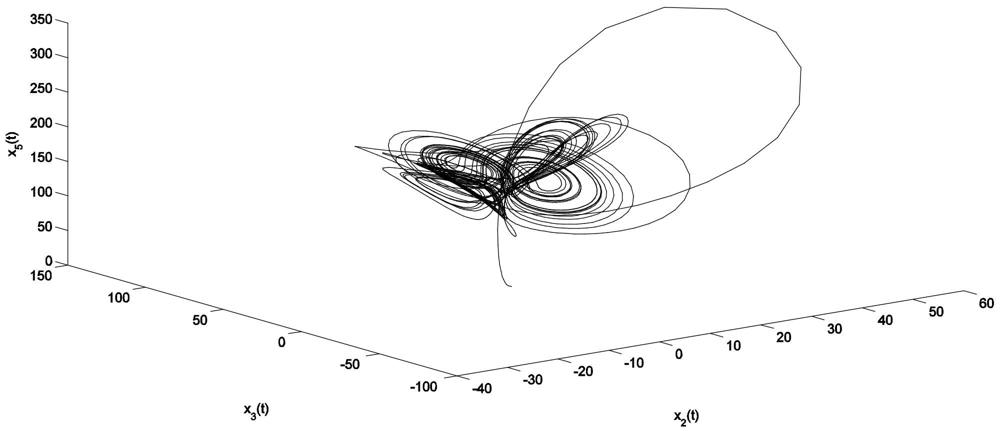

Figure 3.1 (e)

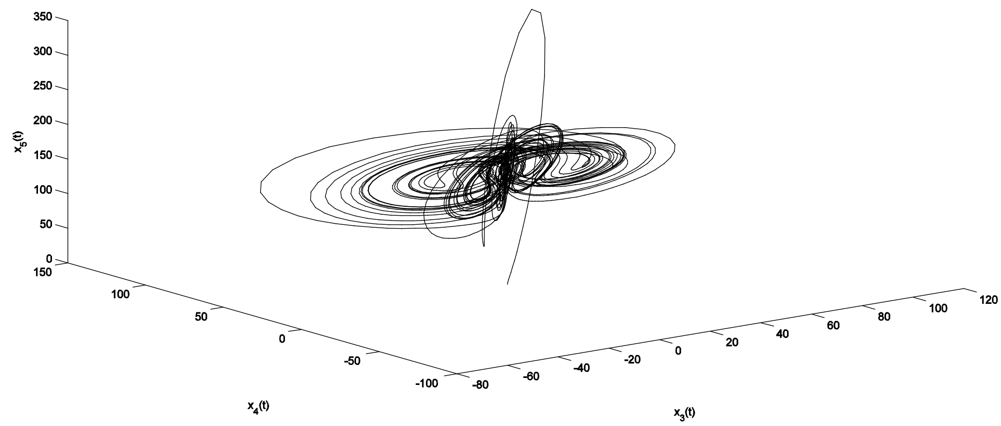

Figure 3.1 (f)

Figure 3.1: Phase portraits of the fractional order complex Lorenz system with uncertain parameters for the order of the derivative $q = 0.99$ in **(a)** $x_1 - x_2 - x_3$ space; **(b)** $x_1 - x_2 - x_4$ space; **(c)** $x_1 - x_2 - x_5$ space; **(d)** $x_2 - x_3 - x_4$ space; **(e)** $x_2 - x_3 - x_5$ space; **(f)** $x_3 - x_4 - x_5$ space.

3.3.2 The fractional order complex T system with uncertainty

The fractional order complex T system (Liu et al. (2014)) is given by

$$D^q y_1' = b_1(y_2' - y_1') \ ,$$

$$D^q y_2' = (b_2 - b_1)y_1' - b_1 \, y_1' \, y_3' \ ,$$

$$D^q y_3' = \frac{1}{2}(\bar{y}_1' \, y_2' + y_1' \, \bar{y}_2') - b_3 \, y_3' \ , \tag{3.12}$$

where $y' = [y_1', \, y_2', \, y_3']^T$ is the state variable vector of the system, $y_1' = y_1 + i\, y_2$ and $y_2' = y_3 + i\, y_4$ are complex variables, $y_3' = y_5$ is real variable and $b_1, \, b_2, \, b_3$ are parameters. The fractional order complex T system (3.7) with uncertain parameters is given as

$$D^q y_1' = b_1(y_2' - y_1') - 0.81 y_1' \ ,$$

$$D^q y_2' = (b_2 - b_1)y_1' - b_1 \, y_1' \, y_3' + 0.3 y_2' \ ,$$

$$D^q y_3' = \frac{1}{2}(\bar{y}_1' \, y_2' + y_1' \, \bar{y}_2') - b_3 y_3' - 0.4 y_3' \ . \tag{3.13}$$

Separating complex variables into real and imaginary parts, the system (3.13) is reduced to

$$D^q y_1 = b_1(y_3 - y_1) - 0.81 y_1 \ ,$$

$$D^q y_2 = b_1(y_4 - y_2) - 0.81 y_2 \ ,$$

$$D^q y_3 = (b_2 - b_1)y_1 - b_1 y_1 y_5 + 0.3 y_3 \ ,$$

$$D^q y_4 = (b_2 - b_1)y_2 - b_1 y_2 y_5 + 0.3 y_4 \ ,$$

$$D^q y_5 = y_1 y_3 + y_2 y_4 - b_3 y_5 - 0.4 y_5 \ , \tag{3.14}$$

where the uncertain parameter is given by

$$\Delta B = \begin{bmatrix} -0.81 & 0 & 0 & 0 & 0 \\ 0 & -0.81 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & -0.4 \end{bmatrix}.$$

The system (3.14) possesses chaotic attractors which are described through Figure 3.2 for the values of the parameters $b_1 = 2.1$, $b_2 = 30$, $b_3 = 0.6$, initial condition $y(0) = [8, 7, 6, 8, 7]^T$ at $q = 0.94$.

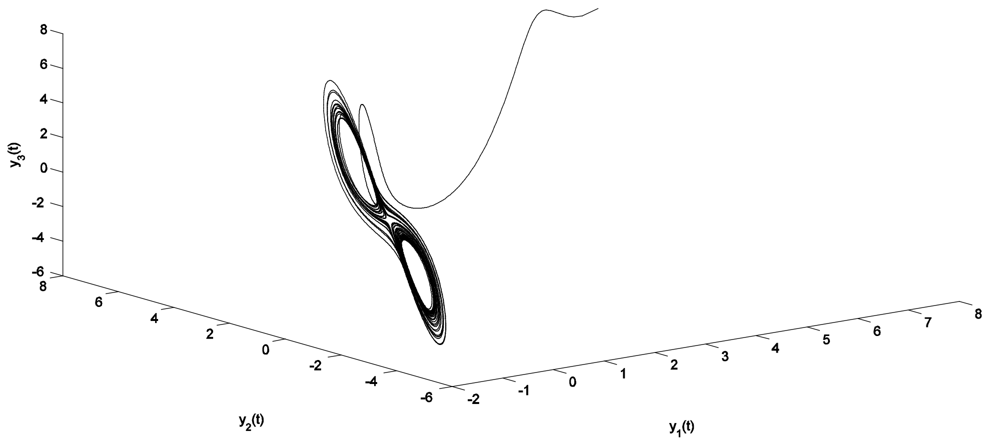

Figure 3.2 (a)

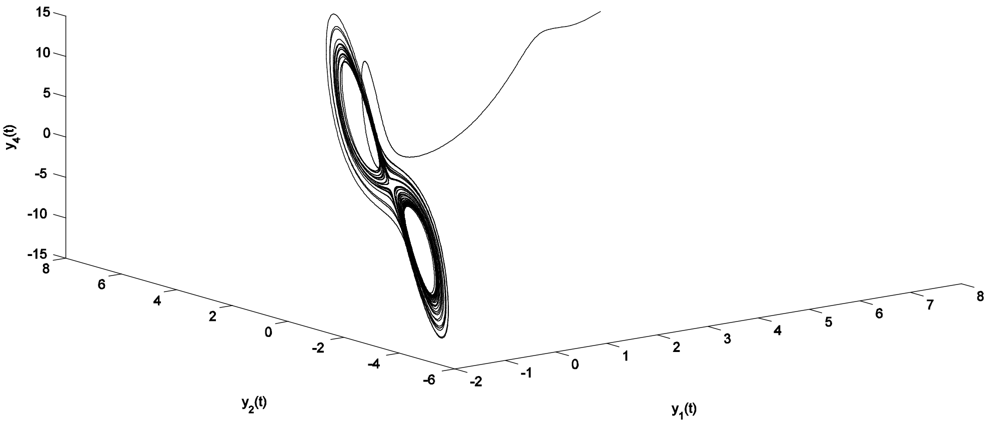

Figure 3.2 (b)

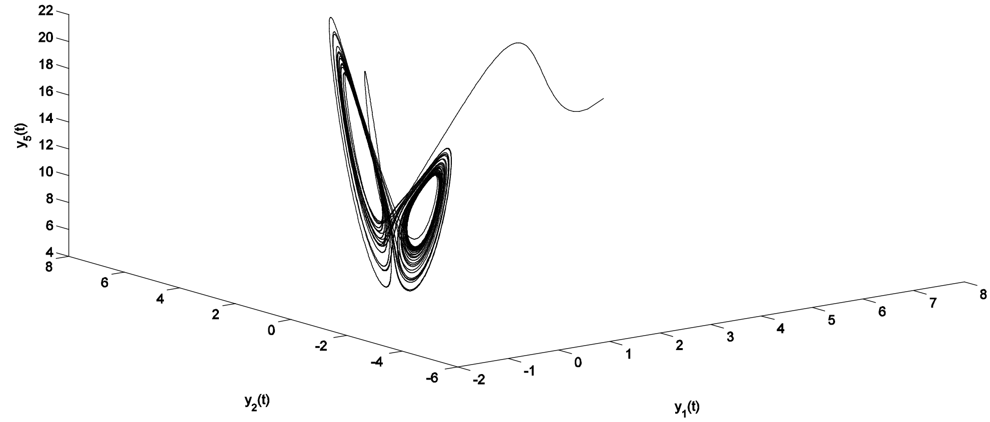

Figure 3.2 (c)

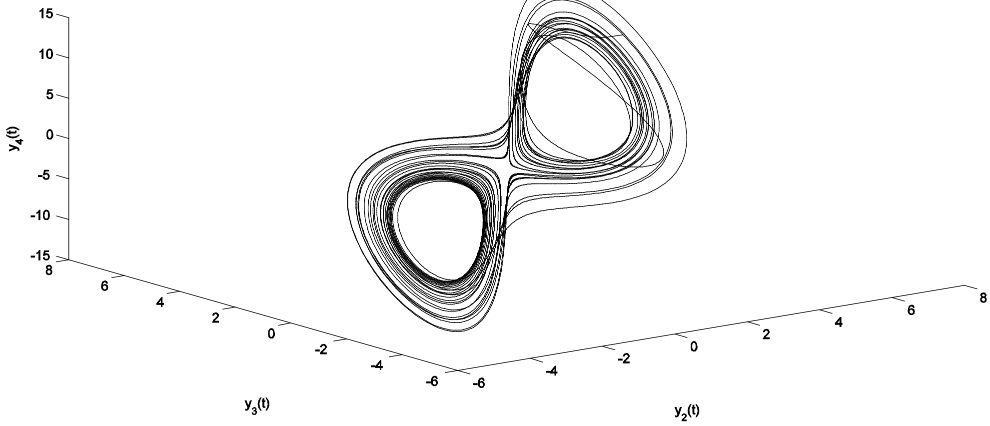

Figure 3.2 (d)

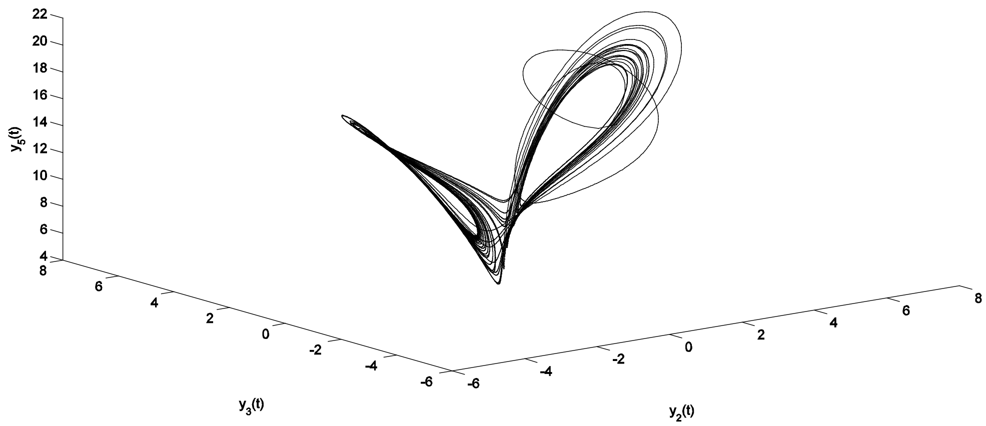

Figure 3.2 (e)

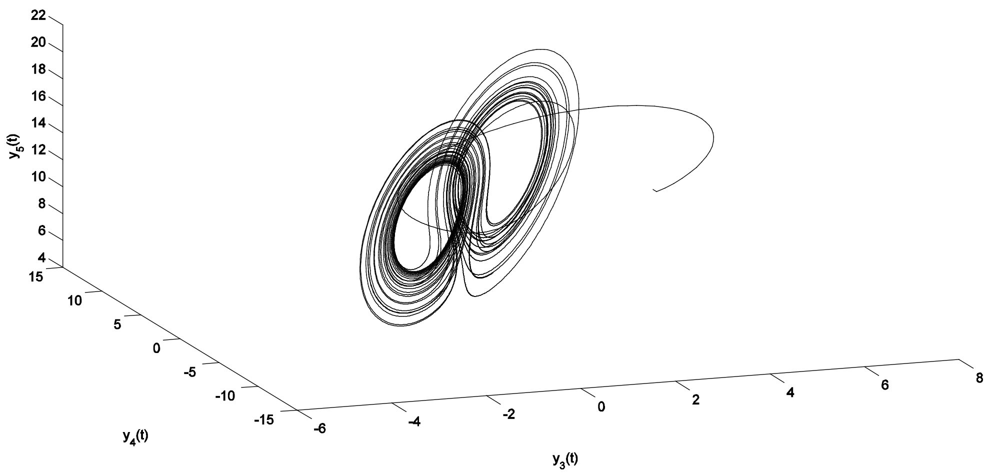

Figure 3.2 (f)

Figure 3.2: Phase portraits of the fractional order complex T system with uncertain parameters for the order of the derivative $q = 0.94$ in **(a)** $y_1 - y_2 - y_3$ space; **(b)** $y_1 - y_2 - y_4$ space; **(c)** $y_1 - y_2 - y_5$ space; **(d)** $y_2 - y_3 - y_4$ space; **(e)** $y_2 - y_3 - y_5$ space; **(f)** $y_3 - y_4 - y_5$ space.

3.4 Synchronization of Lorenz and T systems via active control method

Let us define the system (3.11) as a drive system and system (3.14) as response system. Then response system with control function $u(t) = [u_1(t), u_2(t), u_3(t), u_4(t), u_5(t)]^T$ is given by

$$D^q y_1 = b_1(y_3 - y_1) - 0.81y_1 + u_1 \ ,$$

$$D^q y_2 = b_1(y_4 - y_2) - 0.81y_2 + u_2 \ ,$$

$$D^q y_3 = (b_2 - b_1)y_1 - b_1 y_1 y_5 + 0.3y_3 + u_3 \ ,$$

$$D^q y_4 = (b_2 - b_1)y_2 - b_1 y_2 y_5 + 0.3y_4 + u_4 \ ,$$

$$D^q y_5 = y_1 y_3 + y_2 y_4 - b_3 y_5 - 0.4y_5 + u_5 \ . \tag{3.15}$$

Considering the error functions as $e_i(t) = y_i(t) - x_i(t),\ i = 1, 2, 3, 4, 5$ and choosing the control functions as

$$u_1 = V_1 - (a_1 - b_1 - 0.81)x_1 - (b_1 - a_1)x_3 + 0.2\,y_3 - 0.05\,y_5 \ ,$$

$$u_2 = V_2 - (a_1 - b_1 - 0.81)x_2 - (b_1 - a_1)x_4 + 0.2\,y_4 \ ,$$

$$u_3 = V_3 - (b_2 - b_1 - a_2)x_1 - 1.3\,x_3 + b_1\,y_1\,y_5 - x_1\,x_5 + 0.3\,y_1 \ ,$$

$$u_4 = V_4 - (b_2 - b_1 - a_2)x_2 - 1.3\,x_4 + b_1\,y_2\,y_5 - x_2\,x_5 + 0.3\,y_2 \ ,$$

$$u_5 = V_5 - (a_3 - b_3 - 0.4)x_5 - y_1 y_3 - y_2 y_4 + x_1 x_3 + x_2 x_4 - 0.5\,y_5 \ , \tag{3.16}$$

the error system is reduced to

$$D^q e_1 = -(b_1 + 0.81)\,e_1 + (b_1 + 0.2)e_3 - 0.05\,e_5 + V_1 \ ,$$

$$D^q e_2 = -(b_1 + 0.81)\,e_2 + (b_1 + 0.2)e_4 + V_2 \ ,$$

$$D^q e_3 = (b_2 - b_1 + 0.3)e_1 + 0.3e_3 + V_3 \ ,$$

$$D^q e_4 = (b_2 - b_1 + 0.3)e_2 + 0.3e_4 + V_4 \ ,$$

$$D^q e_5 = -(b_3 + 0.9)e_5 + V_5 \ , \qquad (3.17)$$

where $V(t) = [V_1(t), V_2(t), V_3(t), V_4(t), V_5(t)]^T$ and $e(t) = [e_1(t), e_2(t), e_3(t), e_4(t), e_5(t)]^T$.

Let the matrix C is chosen in the form

$$C = \begin{bmatrix} b_1 - 0.19 & 0 & -b_1 - 0.2 & 0 & 0.05 \\ 0 & b_1 - 0.19 & 0 & -b_1 - 0.2 & 0 \\ b_1 - b_2 - 0.3 & 0 & -1.3 & 0 & 0 \\ 0 & b_1 - b_2 - 0.3 & 0 & -1.3 & 0 \\ 0 & 0 & 0 & 0 & b_3 - 0.1 \end{bmatrix}.$$

In this particular choice, the error system (3.17) becomes

$$D^q e_i = -e_i \ , \ \ i = 1,\ 2,\ 3,\ 4, 5\ . \qquad (3.18)$$

The closed loop system (3.18) has the eigenvalues -1, -1, -1, -1 and -1. Hence the condition $|\arg(\lambda_i)| > q\pi/2$ is satisfied for $q < 2$. Since in my proposed approach the values q are $q \le 1$, then the error states $e_i(t)$, $i = 1, 2, 3, 4, 5$ converge to zero as time t approaches to infinity and thus the synchronization between considered systems is achieved.

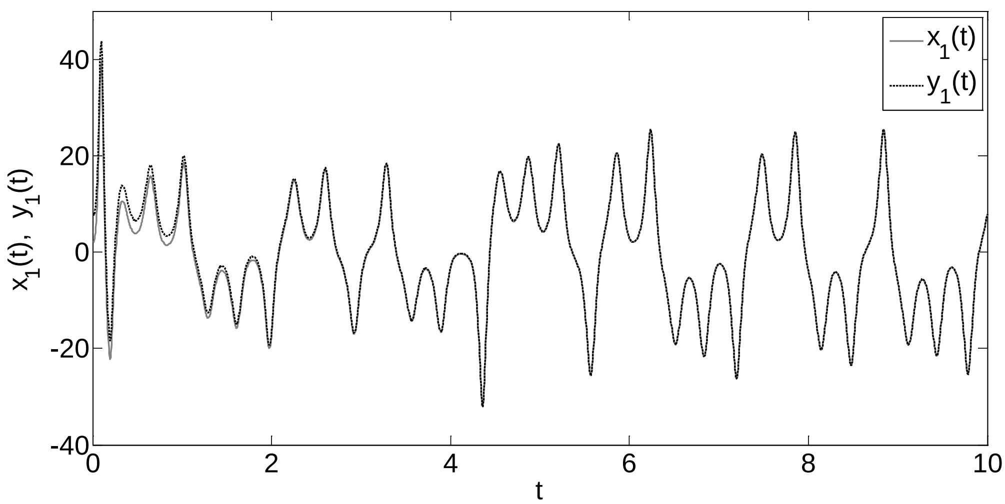

Figure 3.3 (a)

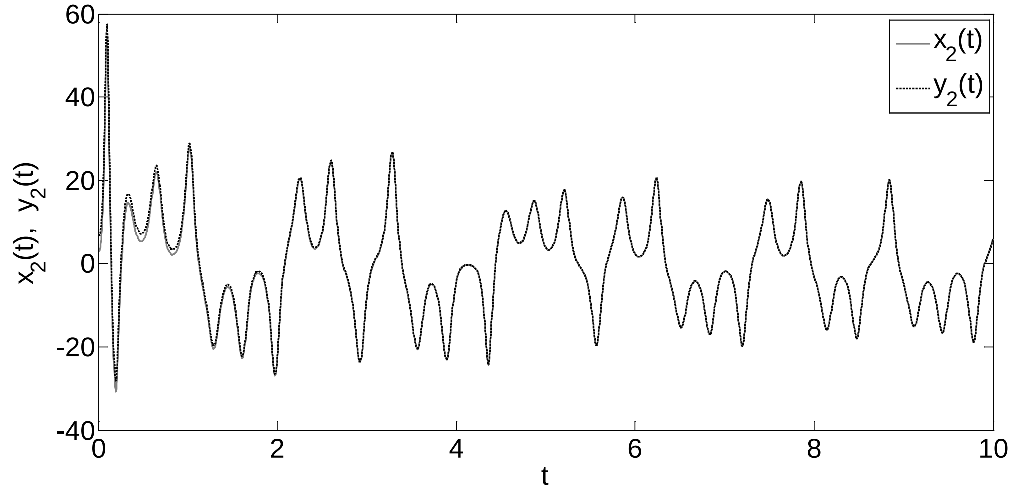

Figure 3.3 (b)

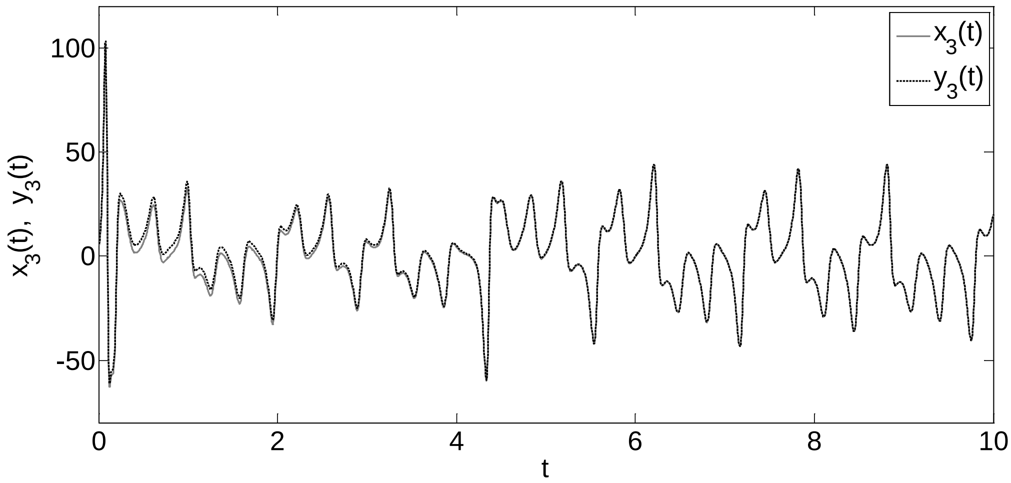

Figure 3.3 (c)

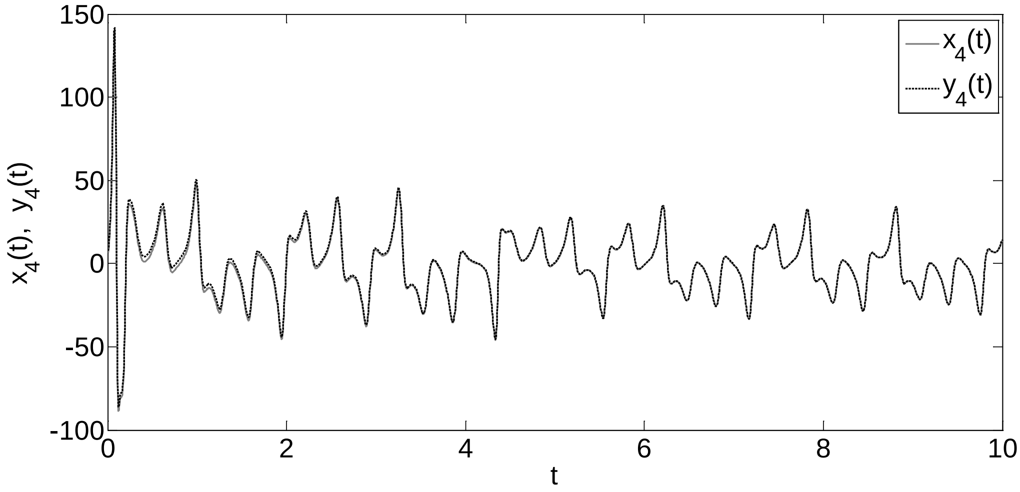

Figure 3.3 (d)

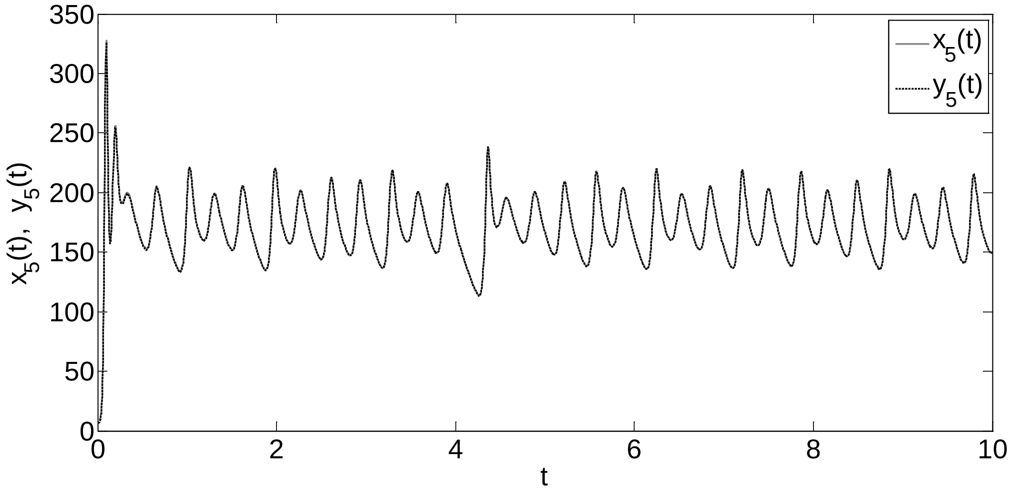

Figure 3.3 (e)

Figure 3.3: Plots of state trajectories of drive system (3.11) and response system (3.15) for fractional order $q = 0.95$ between: **(a)** $x_1(t)$ and $y_1(t)$; **(b)** $x_2(t)$ and $y_2(t)$; **(c)** $x_3(t)$ and $y_3(t)$; **(d)** $x_4(t)$ and $y_4(t)$; **(e)** $x_5(t)$ and $y_5(t)$.

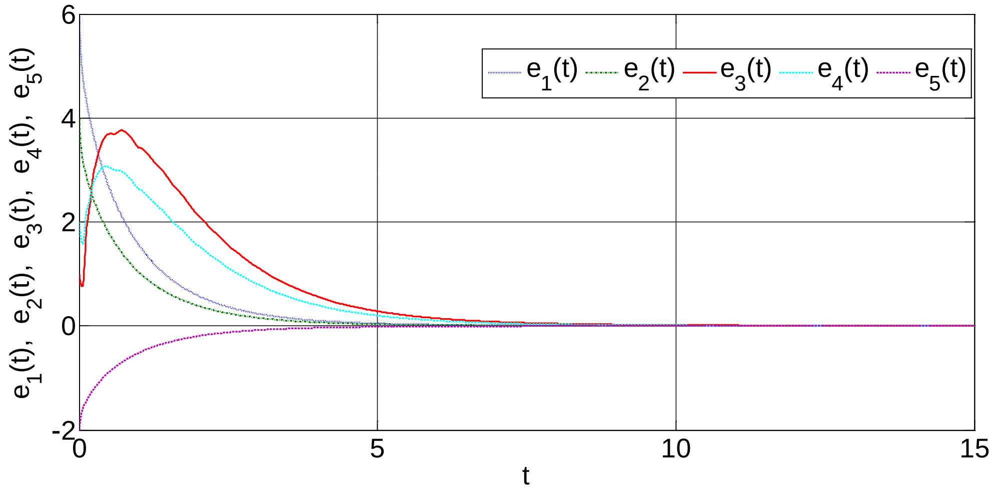

Figure 3.4 (a)

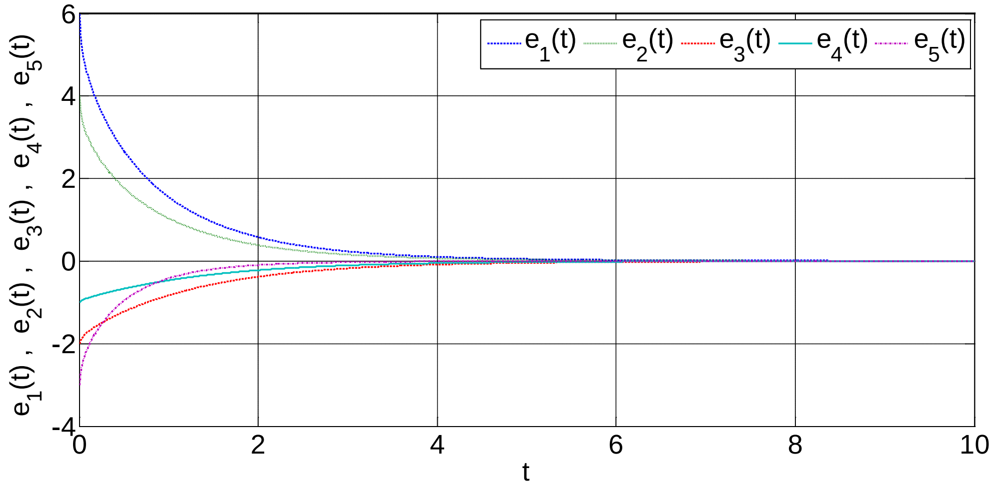

Figure 3.4 (b)

Figure 3.4: Plots of error functions between of drive system (3.11) and response system (3.15) at fractional order $q = 0.95$; **(a)** with uncertain term; and **(b)** without uncertain term.

3.5 Numerical simulation and results

Numerical simulation is given to visualise the synchronization between considered systems and to verify the effectiveness of the proposed method. Adams-Bashforth-Moulton method is used to solve the fractional order differential equations with time step size 0.001. The initial conditions and parameters of the systems are taken as before, and thus the initial error is $[e_1(t),\, e_2(t),\, e_3(t)\, e_4(t),\, e_5(t)]^T = [6,\, 4,\, 1,\, 2,\, -2]^T$. The graphical results of state trajectories of drive system (3.11) and response system (3.15) are presented in Figure 3.3 for the fractional order $q = 0.95$. The study of synchronization for the same chaotic systems have already been studied in chapter 2. Thus the study with uncertain parameters is new contribution. The graphical presentation of the synchronization through error analysis is depicted in Figure 3.4 (a) at $q = 0.99$. The salient feature of the present contribution is requirement of less time for synchronization of considered chaotic systems without uncertain terms [Figure 3.4 (b)] in comparison with the presence of the terms [Figure 3.4 (a)].

3.6 Conclusion

The main contribution of the work is the study of the synchronization of fractional order complex chaotic systems with uncertain parameters using active control method. Based on stability theory the synchronization between fractional order complex Lorenz system and complex T system with uncertainty through proper design of control functions is the first achievement of the present study. The second one is the proper choice of the uncertain parameters for which the chaotic nature of the considered complex systems is maintained. The third one is the exhibition of reliability and effectiveness of the considered method during synchronization of fractional order complex systems, through graphical presentations of the simulation results, even in the presence of uncertain parameters. The most significant point of the study is in finding the effect on time of synchronization with and without the presence of uncertain terms in fractional order system.

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