Chapter 2

Synchronization between fractional order complex chaotic systems

2.1 Introduction

The fractional order dynamical systems are based on real variables, but the fractional order complex systems are in complex variables. Therefore it provides the best instrument to describe a variety of physical phenomena such as amplitudes of the electromagnetic field (Roldan et al. (1993)), thermal convection of liquid flow (Toronov and Derbov (1997)), detuned laser system (Ning and Haken (1990)). The fractional order complex dynamical systems are used to increase the content and security of transmitting information signals. This kind of system has played an important role in the dynamical systems viz., population inversion, polymer physics, etc. Therefore, it is a meaningful and interesting topic for scientists and researchers to study of dynamic behaviour, chaos control, synchronization of fractional order complex dynamical systems.

Lorenz (1963) was the first to study the 3D chaotic attractor in autonomous systems during the atmospheric study. This study has become a paradigm for the researcher working in nonlinear science. In 1982, the complex Lorenz system was first introduced by Flower et al., and its dynamical property was studied by Mahmoud et al. (2007a). Luo

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and Wang (2013b) studied the fractional order complex Lorenz system and its dynamical properties. The complex Lu system was proposed by Mahmoud et al. (2007c), and fractional order complex Lu system was studied by Jiang et al. (2014). Tigan and Opris (2008) proposed a 3D chaotic system called T system and later its dynamical behaviour was studied in details by Liu et al. (2014).

Many chaotic systems in practical applications have different structures. Thus the generalised study is always a useful tool for synchronization of the various chaotic systems. Active control method has widely been accepted as an efficient technique for the synchronization of non-identical chaotic systems, a feature for which it got an advantage over other synchronization methods. A generalised design of the active control strategy was developed by Ho and Hung (2002). It is treated as one of the most interesting control strategies for its simplicity but considered as an expensive strategy as it takes comparatively more time to synchronize as compared to few existing methods. The method is easy to design but cannot be adapted for unknown parameters, since systems' parameters fluctuate in an experimental situation due to internal and external features. Despite these facts, it is noticed that the active control method is efficient and convenient to synchronize or anti-synchronize since the Lyapunov exponents are not required for these calculations.

In the present chapter, a sincere attempt is taken to study the synchronization of fractional order complex chaotic systems using active control method. Computer simulation is carried out using Adams-Boshforth-Moulton method (Diethelm and Ford (2004), Diethelm et al. (2004)). The salient feature of the chapter is the study of the time of synchronization between complex chaotic systems for different particular cases as systems' pair approaches from integer order to fractional order.

2.2 Problem formulation and synchronization

Two fractional order complex chaotic systems are considered to formulate the problem. The master system is taken as

$$D^q x_m = B x_m + f(x_m) ,$$

where $x_m = x_{m_1} + i x_{m_2}$ is complex state variable. Separating into real and imaginary parts of above equation, we get

$$D^q x_{m_1} = B x_{m_1} + f_1(x_{m_1}) ,$$

$$D^q x_{m_2} = B x_{m_2} + f_2(x_{m_2}) , \tag{2.1}$$

The slave system is taken as

$$D^q y_s = C y_s + g(y_s) + u(t) .$$

where $y_s = y_{s_1} + i y_{s_2}$ is complex state variable,

$$D^q y_{s_1} = C y_{s_1} + g_1(y_{s_1}) + u_1(t) ,$$

$$D^q y_{s_2} = C y_{s_2} + g_2(y_{s_2}) + u_2(t) , \tag{2.2}$$

where $u(t) = u_1(t) + i u_2(t)$ is control function, $u_1(t) = [u_1, \ u_3, \cdots, u_{2n-1}]^T$ and $u_2(t) = [u_2, \ u_4, \cdots, u_{2n}]^T$.

Defining the error function between the state variables of the systems as

$$e_r = e_{r_1} + i e_{r_2} = y_s - x_m ,$$

the error system is obtained

$$D^q e_r = C e_r + (C - B)x_m + g(y_s) - f(x_m) + u(t)\ .$$

Using $e_{r_1} = y_{s_1} - x_{m_1}$ and $e_{r_2} = y_{s_2} - x_{m_2}$ and separating real and imaginary parts, we get

$$D^q e_{r_k} = C e_{r_k} + (C - B)x_{m_k} + g_k(y_{s_k}) - f_k(x_{m_k}) + u_k(t)\ , \quad k = 1\ , 2. \tag{2.3}$$

To stabilise the Equation (2.3), suppose (Agrawal et al. (2012), Zhou and Cheng (2008))

$$u_k(t) = (B - C)x_{m_k} - g_k(x_{m_k}) + f_k(x_{m_k}) - J[g_k(x_{m_k})]\, e_{r_k} + A e_{r_k}\ , \tag{2.4}$$

where $J[g_k(x_{m_k})]$ is the Jacobian matrix of $g_k(x_{m_k})$ and A is a controller gain matrix to be designed later.

With the aid of Equation (2.4), the Equation (2.3) becomes

$$D^q e_{r_k} = B e_{r_k} + g_k(y_{s_k}) - g_k(x_{m_k}) - J[g_k(x_{m_k})]\, e_{r_k} + A e_{r_k} \tag{2.5}$$

Using Taylor series formula,

$$g_k(y_{s_k}) = g_k(x_{m_k}) + J[g_k(x_{m_k})]e_{r_k} + o(e_{r_k})\ , \tag{2.6}$$

where $J[g_k(x_{m_k})] = \left[\frac{\partial}{\partial x} g_k(x)\right]_{x = x_{m_k}}$ and $o(e_{r_k})$ represents the higher order terms of the expression such that $\lim\limits_{e_{r_k} \to 0} \frac{o(e_{r_k})}{|e_{r_k}|} = 0$, under the assumption that the error e_{r_k} is sufficiently small, one can neglect $o(e_{r_k})$ and then Equation (2.5) reduces to

$$D^q e_{r_k} = (C + A)e_{r_k}\ . \tag{2.7}$$

Definitely $e_{r_k} = y_{s_k} - x_{m_k} = 0$ is one fixed point of the error Equation (2.7). The Jacobian matrix of the above equation at this fixed point is $(C + A)$. As shown by Matignon (1996) during analysis of the stability criterion for fractional differential equation, the fixed point becomes asymptotically stable if the argument of the eigenvalues λ_i of matrix

$(C+A)$ satisfy $\left|\arg(\lambda_i(C+A))\right| > 0.5\pi q$. Therefore $e_{r_k} \to 0$ as time $t \to \infty$, indicating that the systems (2.1) and (2.2) are synchronized.

2.3 Systems' descriptions

2.3.1 The fractional order complex Lorenz system

The complex Lorenz system was described as

$$\dot{x}_1' = a_1(x_2' - x_1') \ ,$$

$$\dot{x}_2' = a_2 x_1' - x_2' - x_1' x_3' \ ,$$

$$\dot{x}_3' = \frac{1}{2}(\overline{x}_1' x_2' + x_1' \overline{x}_2') - a_3 x_3' \ , \qquad (2.8)$$

where $x' = [x_1', \, x_2', \, x_3']^T$ is the state variable vector, $x_1' = x_1 + i\,x_2$ and $x_2' = x_3 + i\,x_4$ are complex variables while $x_3' = x_5$ is real variable and $i = \sqrt{-1}$.

The fractional order complex Lorenz system is given by

$$D^q x_1' = a_1(x_2' - x_1') \ ,$$

$$D^q x_2' = a_2 x_1' - x_2' - x_1' \, x_3' \ ,$$

$$D^q x_3' = \frac{1}{2}(\overline{x}_1' \, x_2' + x_1' \, \overline{x}_2') - a_3 x_3' \ , \qquad (2.9)$$

where a_1 is the Prandtl number, a_2 is the Rayleigh number and a_3 is the size of the region approximated by the system. When the parameters' values are $a_1 = 10$, $a_2 = 180$, $a_3 = 1$ and initial condition $x'(0) = [2+3i, \, 5+6i, \ \ 9]^T$ at order $q = 0.99$, the system (2.9) possesses the chaotic attractor described by Figure 1.

Since the Caputo's fractional order differential operator is linear, the system (2.9) can be written as

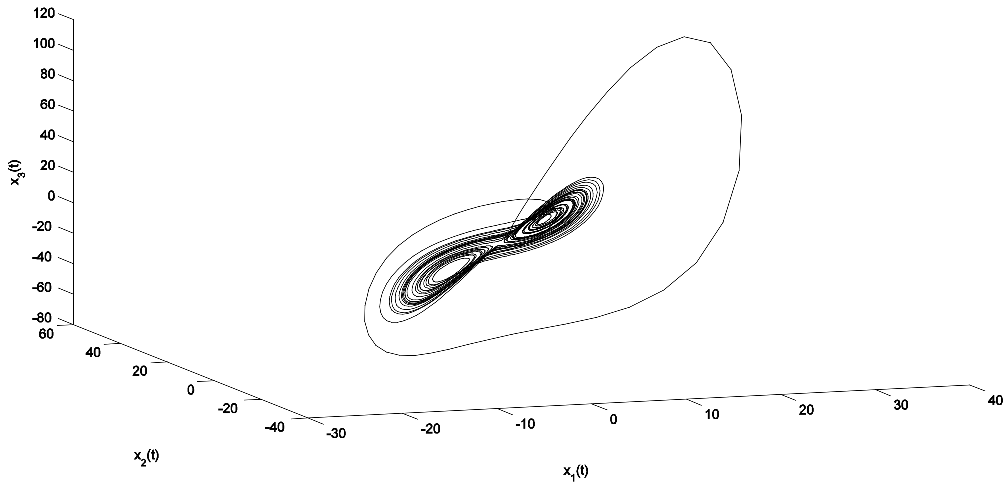
$$D^q x_1 = a_1(x_3 - x_1) ,$$

$$D^q x_2 = a_1(x_4 - x_2) ,$$

$$D^q x_3 = a_2 x_1 - x_3 - x_1 x_5 ,$$

$$D^q x_4 = a_2 x_2 - x_4 - x_2 x_5 ,$$

$$D^q x_5 = x_1 x_3 + x_2 x_4 - a_3 x_5 . \tag{2.10}$$



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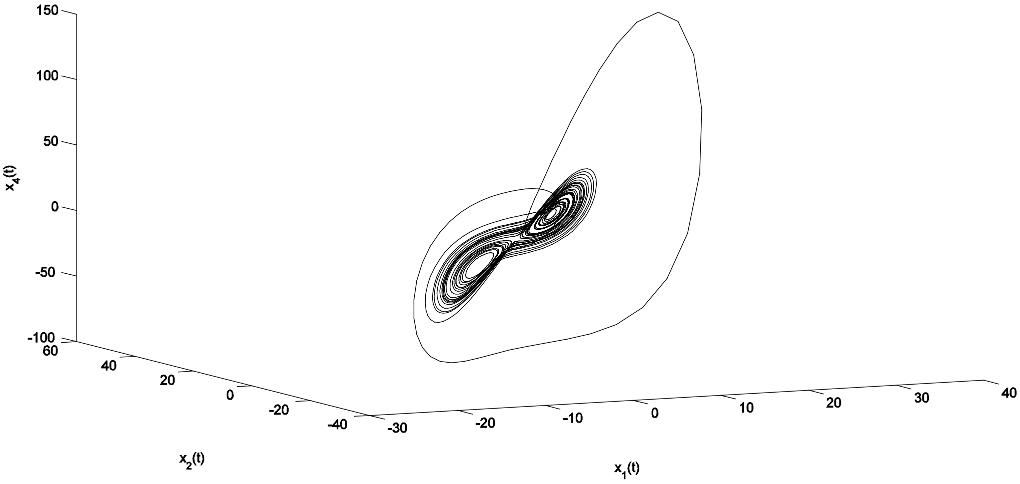

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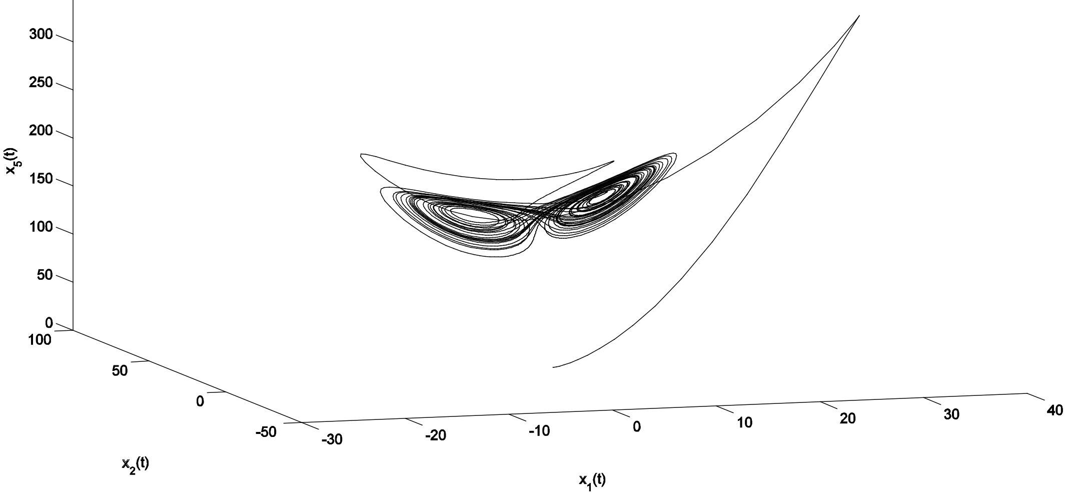

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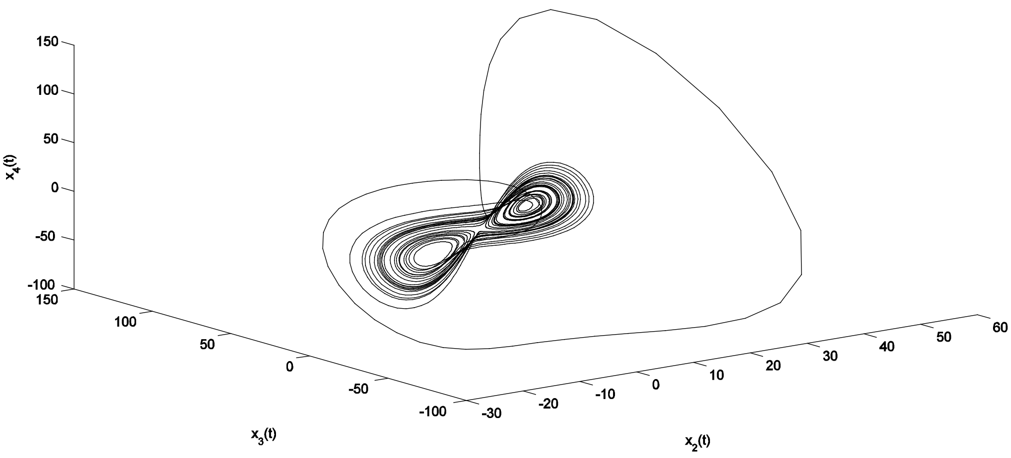

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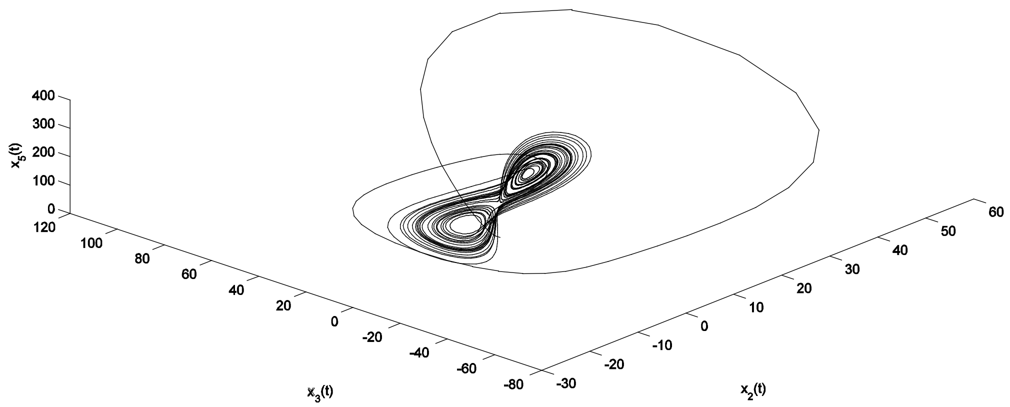

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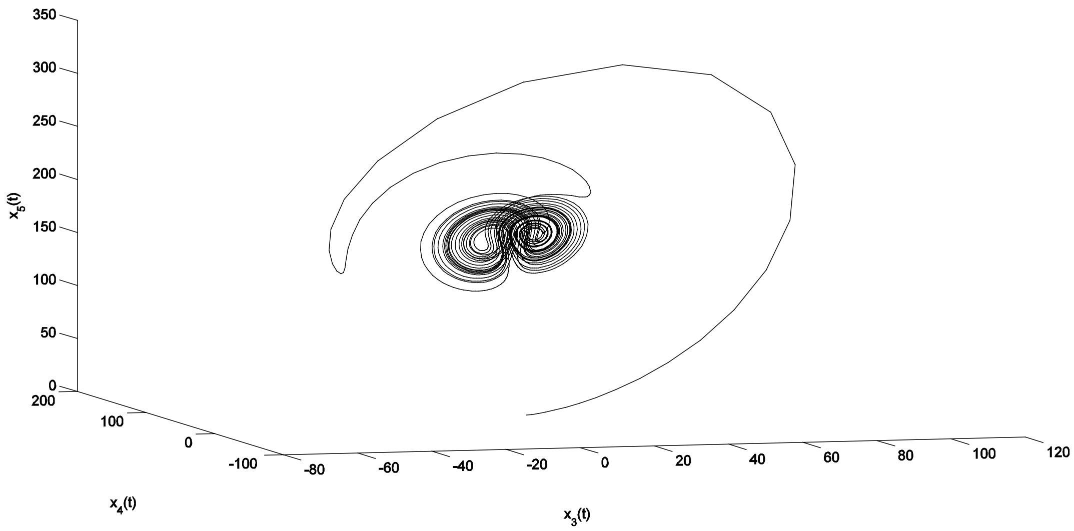


Figure 2.1 (f)

Figure 2.1: Phase portraits of the complex Lorenz system for the order of derivative $q = 0.99$ in **(a)** $x_1 - x_2 - x_3$ space; **(b)** $x_1 - x_2 - x_4$ space; **(c)** $x_1 - x_2 - x_5$ space; **(d)** $x_2 - x_3 - x_4$ space; **(e)** $x_2 - x_3 - x_5$ space; **(f)** $x_3 - x_4 - x_5$ space.

2.3.2 The fractional order complex Lu system

The complex Lu system is given by

$$\dot{y}'_1 = b_1(y'_2 - y'_1) ,$$

$$\dot{y}'_2 = b_2 y'_2 - y'_1 y'_3 ,$$

$$\dot{y}'_3 = \frac{1}{2}(\bar{y}'_1 y'_2 + y'_1 \bar{y}'_2) - b_3 y'_3 , \tag{2.11}$$

where $y' = [y'_1,\ y'_2,\ y'_3]^T$ is the state vector variable, $y'_1 = y_1 + i\,y_2$ and $y'_2 = y_3 + i\,y_4$ are complex variables while $y'_3 = y_5$ is real variable.

The fractional order complex Lu system is

$$D^q y'_1 = b_1(y'_2 - y'_1) ,$$

$$D^q y'_2 = b_2 y'_2 - y'_1 y'_3 ,$$

$$D^q y'_3 = \frac{1}{2}(\bar{y}'_1\, y'_2 + y'_1\, \bar{y}'_2) - b_3 y'_3 . \tag{2.12}$$

Figure 2.2 shows the chaotic attractor of the system at $q = 0.96$ for the value of parameters $b_1 = 42$, $b_2 = 22$ and $b_3 = 5$, and initial condition $y'(0)$ = $[-1+9i\ , 8-5i\ , 1]^T$.

Separating into real and imaginary parts of the system (2.12) gives rise to

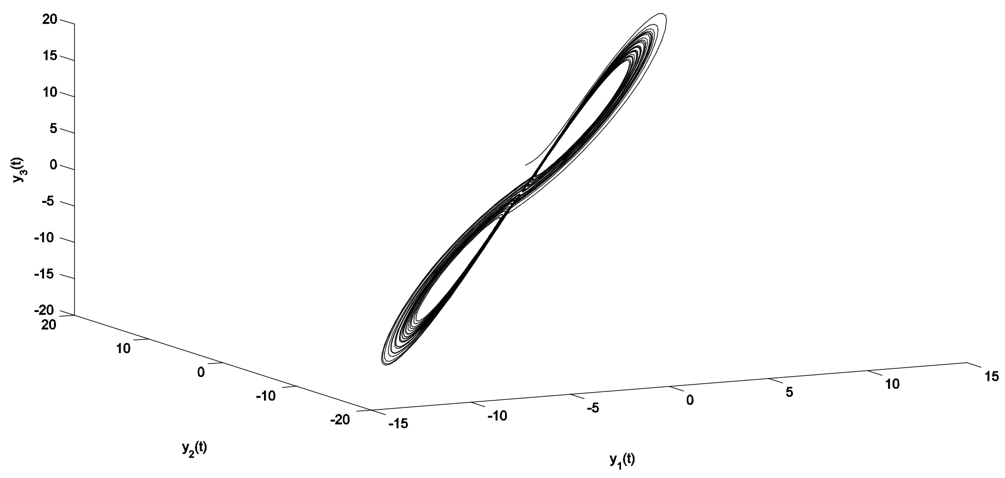
$$D^q y_1 = b_1(y_3 - y_1) ,$$

$$D^q y_2 = b_1(y_4 - y_2) ,$$

$$D^q y_3 = b_2 y_3 - y_1 y_5 ,$$

$$D^q y_4 = b_2 y_4 - y_2 y_5 ,$$

$$D^q y_5 = y_1 y_3 + y_2 y_4 - b_3 y_5 . \tag{2.13}$$



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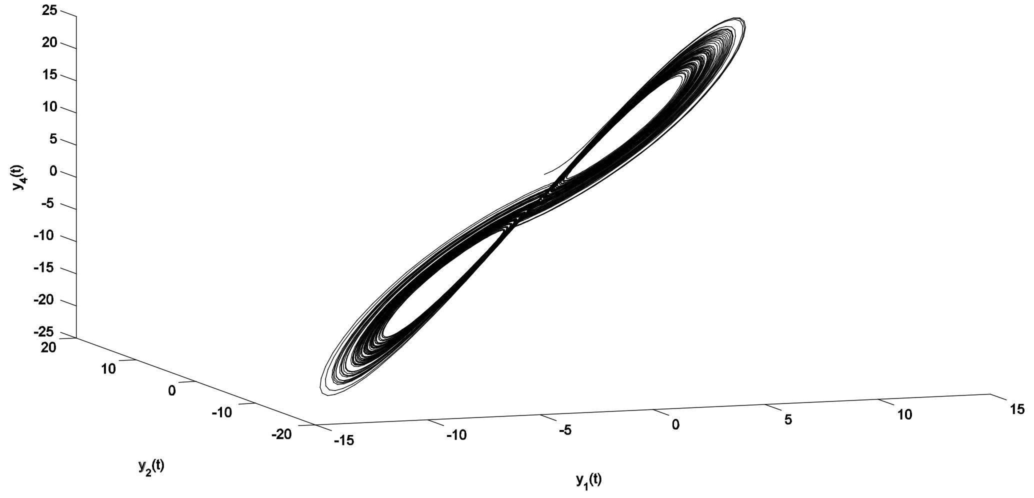

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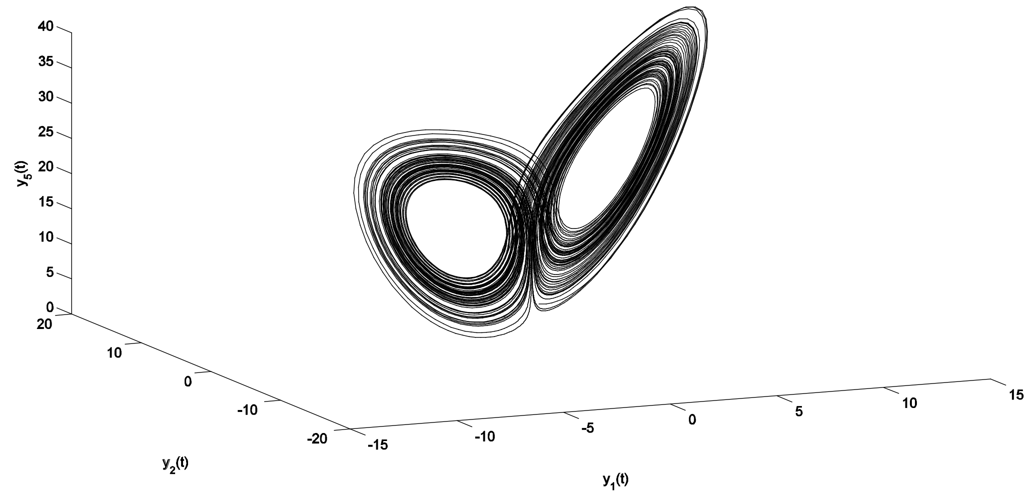

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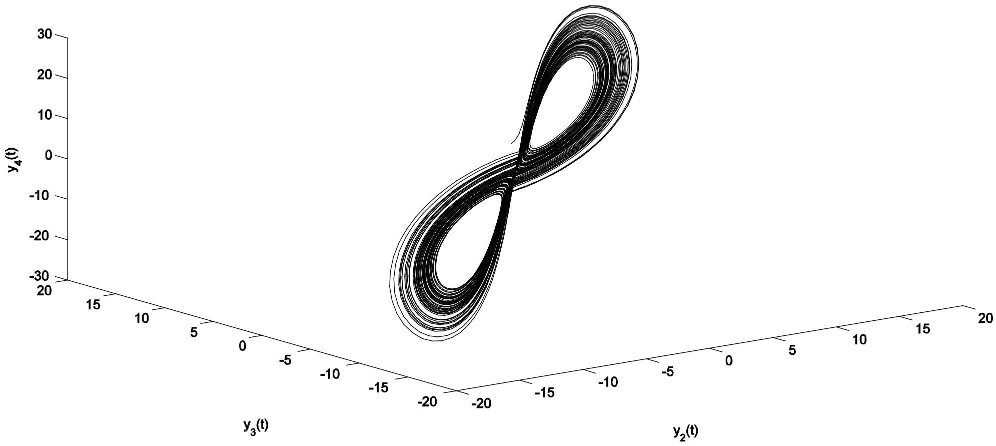

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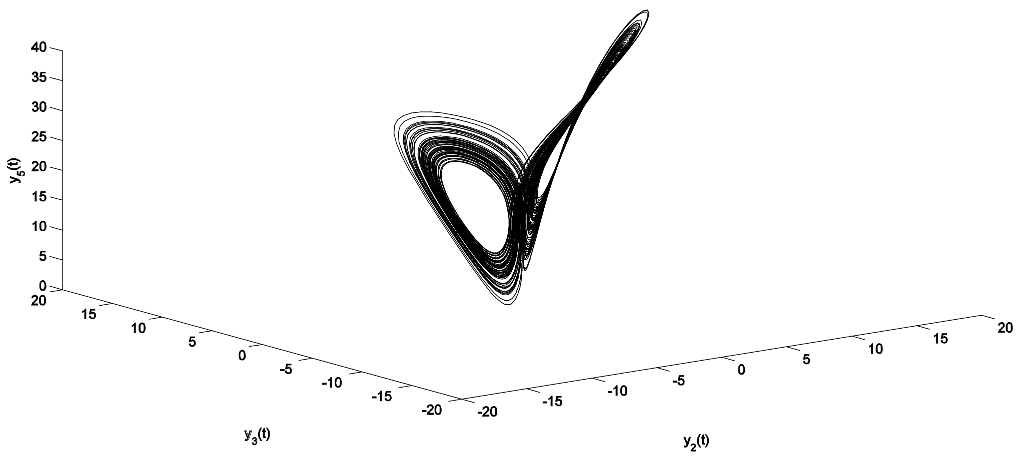

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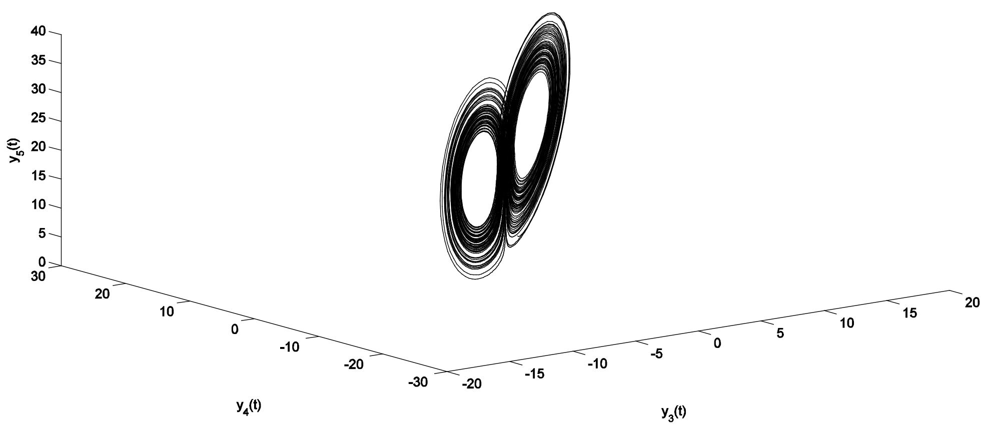


Figure 2.2 (f)

Figure 2.2: Phase portraits of the complex Lu system for the order of derivative $q = 0.96$ in **(a)** $y_1 - y_2 - y_3$ space; **(b)** $y_1 - y_2 - y_4$ space; **(c)** $y_1 - y_2 - y_5$ space; **(d)** $y_2 - y_3 - y_4$ space; **(e)** $y_2 - y_3 - y_5$ space; **(f)** $y_3 - y_4 - y_5$ space.

2.3.3 The fractional order complex T system

The complex T system is given by

$$\dot{z}_1' = c_1(z_2' - z_1') \ ,$$

$$\dot{z}_2' = (c_2 - c_1)z_1' - c_1\, z_1'\, z_3' \ ,$$

$$\dot{z}_3' = \frac{1}{2}(\bar{z}_1'\, z_2' + z_1'\, \bar{z}_2') - c_3\, z_3' \ , \tag{2.14}$$

where $z' = [z_1',\ z_2',\ z_3']^T$ is the state vector variable, $z_1' = z_1 + iz_2$ and $z_2' = z_3 + iz_4$ are complex variables, $z_3' = z_5$ is real variable and $c_1,\ c_2,\ c_3$ are parameters with $c_1 \neq 0$. This system possesses a chaotic attractor as shown in Figure 2.3, when the parameters are taken as $c_1 = 2.1$, $c_2 = 30$, $c_3 = 0.6$ and initial condition $z'(0) = [8+7i,\ 5+6i,\ 10]^T$ at $q = 0.94$.

The fractional order complex T system is given by

$$D^q z_1' = c_1(z_2' - z_1') \ ,$$

$$D^q z_2' = (c_2 - c_1)z_1' - c_1\, z_1'\, z_3' \ ,$$

$$D^q z_3' = \frac{1}{2}(\bar{z}_1'\, z_2' + z_1'\, \bar{z}_2') - c_3\, z_3' \ , \tag{2.15}$$

where $z' = [z_1',\ z_2',\ z_3']^T$ is the vector of state variables. System (2.15) can be written as

$$D^q z_1 = c_1(z_3 - z_1) \ ,$$

$$D^q z_2 = c_1(z_4 - z_2) \ ,$$

$$D^q z_3 = (c_2 - c_1)z_1 - c_1 z_1 z_5 \ ,$$

$$D^q z_4 = (c_2 - c_1)z_2 - c_1 z_2 z_5 \ ,$$

$$D^q z_5 = z_1 z_3 + z_2 z_4 - c_3 z_5 \ . \tag{2.16}$$

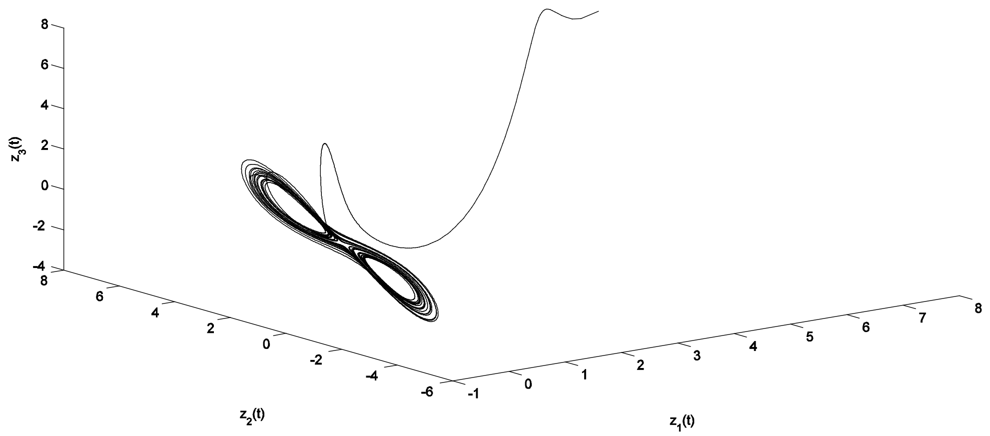
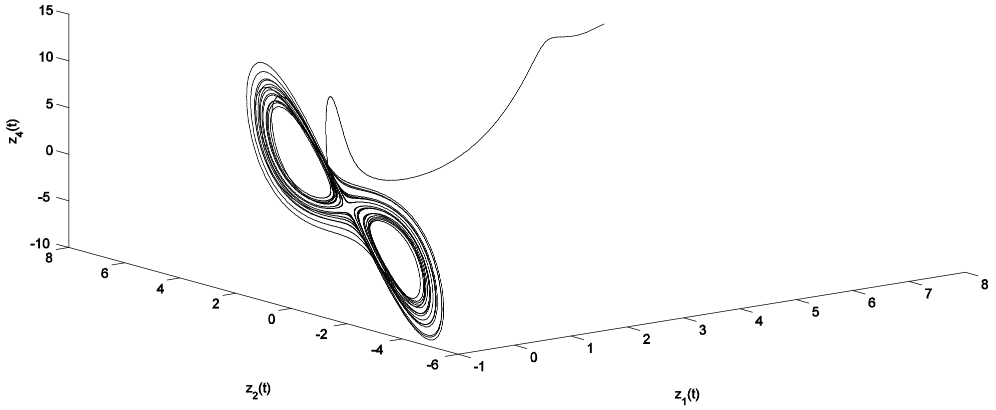


Figure 2.3 (a)



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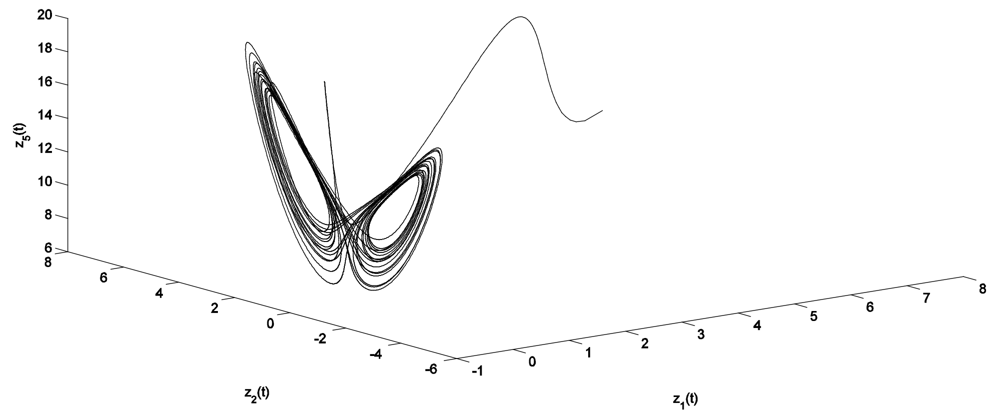

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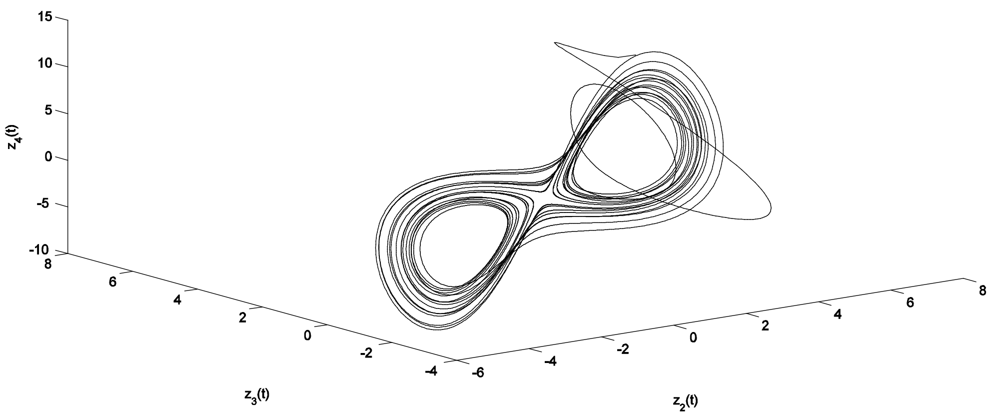

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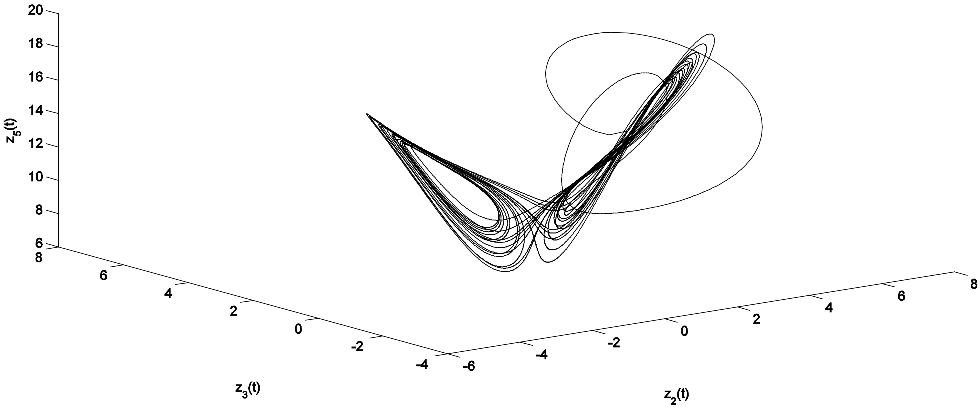

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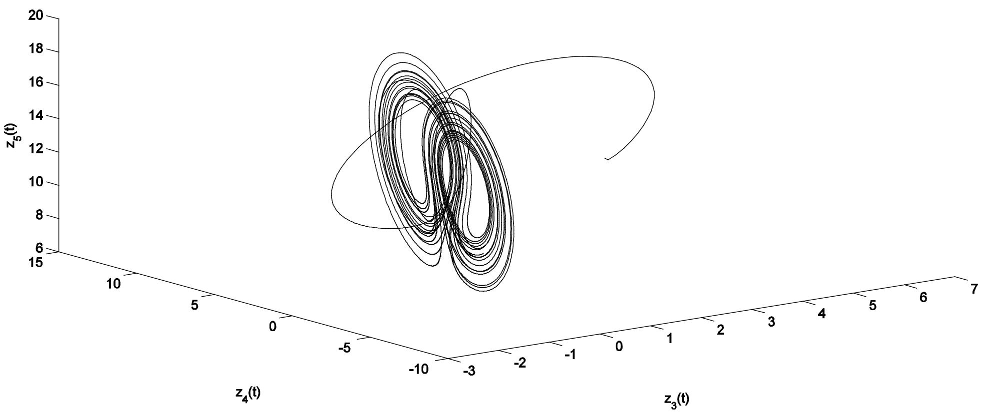


Figure 2.3 (f)

Figure 2.3: Phase portraits of the complex T system for the order of derivative $q = 0.94$ in **(a)** $z_1 - z_2 - z_3$ space; **(b)** $z_1 - z_2 - z_4$ space; **(c)** $z_1 - z_2 - z_5$ space; **(d)** $z_2 - z_3 - z_4$ space; **(e)** $z_2 - z_3 - z_5$ space; **(f)** $z_3 - z_4 - z_5$ space.

2.4 Synchronization between fractional order complex Lorenz and Lu systems

In this section, the synchronization between fractional order complex Lorenz and Lu systems is studied. Let us consider that the Lorenz system (2.10) drives the Lu system (2.13). The response system is re-written as

$$D^q y_1 = b_1(y_3 - y_1) + u_1(t) \,,$$

$$D^q y_2 = b_1(y_4 - y_2) + u_2(t) \,,$$

$$D^q y_3 = b_2 y_3 - y_1 y_5 + u_3(t) \,,$$

$$D^q y_4 = b_2 y_4 - y_2 y_5 + u_4(t) \,,$$

$$D^q y_5 = y_1 y_3 + y_2 y_4 - b_3 y_5 + u_5(t) \,. \tag{2.17}$$

In order to estimate the control functions $u_1(t)$, $u_2(t)$, $u_3(t)$, $u_4(t)$ and $u_5(t)$, the error functions between the Lorenz system (2.10) and the controlled Lu system (2.17) are defined as

$$e_1' = e_1 + i\,e_2 = y_1' - x_1', \; e_2' = e_3 + i\,e_4 = y_2' - x_2' \text{ and } e_3' = e_5 = y_3' - x_3'$$

so that

$$e_1 = y_1 - x_1 \,, \; e_2 = y_2 - x_2 \,, \; e_3 = y_3 - x_3 \,, \; e_4 = y_4 - x_4 \,, \; e_5 = y_5 - x_5 \,. \tag{2.18}$$

Subtracting system (2.10) from system (2.17) and using the notations given in Equation (2.18), we get

$$D^q e_1 = b_1(e_3 - e_1) + (b_1 - a_1)(x_3 - x_1) + u_1 \,,$$

$$D^q e_2 = b_1(e_4 - e_2) + (b_1 - a_1)(x_4 - x_2) + u_2 \,,$$

$$D^q e_3 = b_2 e_3 - a_2 x_1 + (b_2 + 1)x_3 + x_1 x_5 - y_1 y_5 + u_3 \,,$$

$$D^q e_4 = b_2 e_4 - a_2 x_2 + (b_2 + 1)x_4 + x_2 x_5 - y_2 y_5 + u_4 \,,$$

$$D^q e_5 = -b_3 e_5 + (a_3 - b_3)x_5 - x_1 x_3 - x_2 x_4 + y_1 y_3 + y_2 y_4 + u_5 \ . \tag{2.19}$$

Defining the active control functions $u_i(t)$ as

$$u_1 = V_1 - (b_1 - a_1)(x_3 - x_1) \ ,$$

$$u_2 = V_2 - (b_1 - a_1)(x_4 - x_2) \ ,$$

$$u_3 = V_3 + a_2 x_1 - (b_2 + 1)x_3 - x_1 x_5 + y_1 y_5 \ ,$$

$$u_4 = V_4 + a_2 x_2 - (b_2 + 1)x_4 - x_2 x_5 + y_2 y_5 \ ,$$

$$u_5 = V_5 - (a_3 - b_3)x_5 + x_1 x_3 + x_2 x_4 - y_1 y_3 - y_2 y_4 \ , \tag{2.20}$$

where the terms $V_i(t)$ are linear functions of the error terms $e_i(t)$. The error system (2.19) is reduced to

$$D^q e_1 = b_1(e_3 - e_1) + V_1 \ ,$$

$$D^q e_2 = b_1(e_4 - e_2) + V_2 \ ,$$

$$D^q e_3 = b_2 e_3 + V_3 \ ,$$

$$D^q e_4 = b_2 e_4 + V_4 \ ,$$

$$D^q e_5 = -b_2 e_5 + V_5 \ .$$

Let us design an appropriate feedback control which stabilizes the system so that $e_i(t)$, $i = 1,\ 2,\ 3,\ 4,\ 5$ converge to zero as time t becomes large. There are many possible choices for the control inputs $V_i(t)$. Let us choose

$$V(t) = A\ e(t), \tag{2.21}$$

where $V(t) = [V_1(t),\ V_2(t),\ V_3(t),\ V_4(t),\ V_5(t)]^T$ and $e(t) = [e_1(t),\ e_2(t),\ e_3(t),\ e_4(t),\ e_5(t)]^T$ and A is 5×5 constant matrix. In order to make the closed loop system stable, the matrix should be selected in such a way that the feedback system has eigenvalues λ_i

of A satisfy the condition $|\arg(\lambda_i)| > q\pi/2$, $i = 1, 2, 3, 4, 5$. There is no unique choice for matrix A. Let the matrix A is chosen in the form

$$A = \begin{bmatrix} -1+b_1 & 0 & -b_1 & 0 & 0 \\ 0 & -1+b_1 & 0 & -b_1 & 0 \\ 0 & 0 & -1-b_2 & 0 & 0 \\ 0 & 0 & 0 & -1-b_2 & 0 \\ 0 & 0 & 0 & 0 & -1+b_3 \end{bmatrix}.$$

In this particular choice, the closed loop system has the eigenvalues -1, -1, -1, -1, and -1. This choice will lead to the error states $e_i(t)$, $i = 1, 2, 3, 4, 5$ converge to zero as time t tends to infinity and thus the synchronization between Lorenz system and Lu system is achieved.

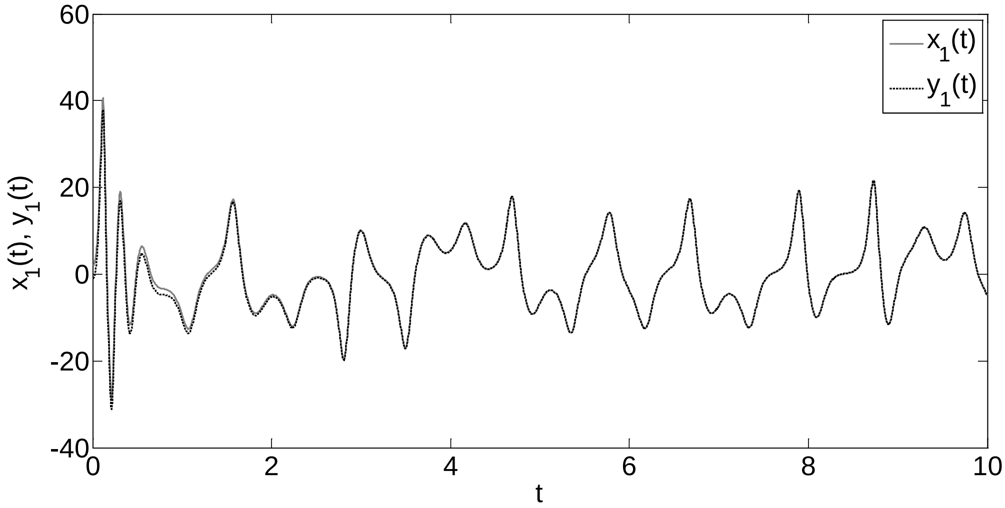


Figure 2.4 (a)

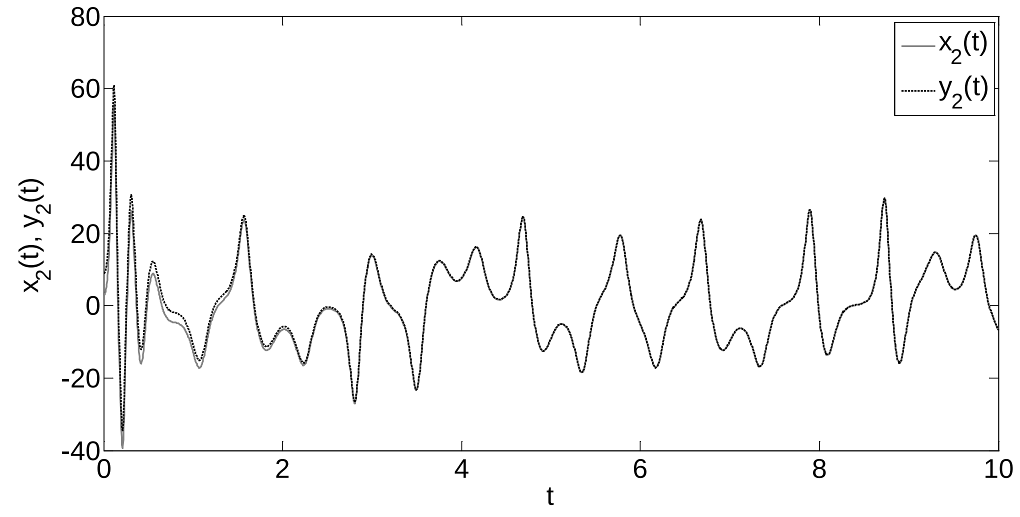


Figure 2.4 (b)

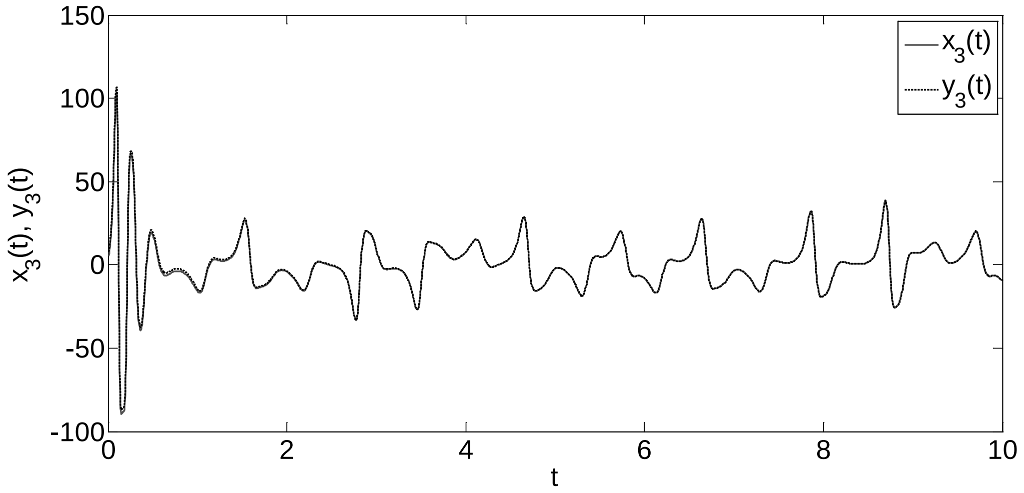


Figure 2.4 (c)

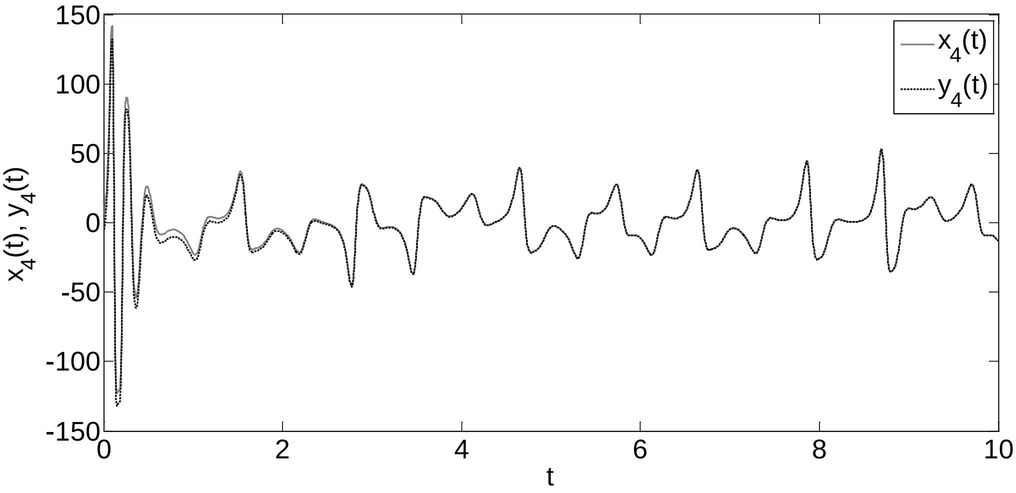


Figure 2.4 (d)

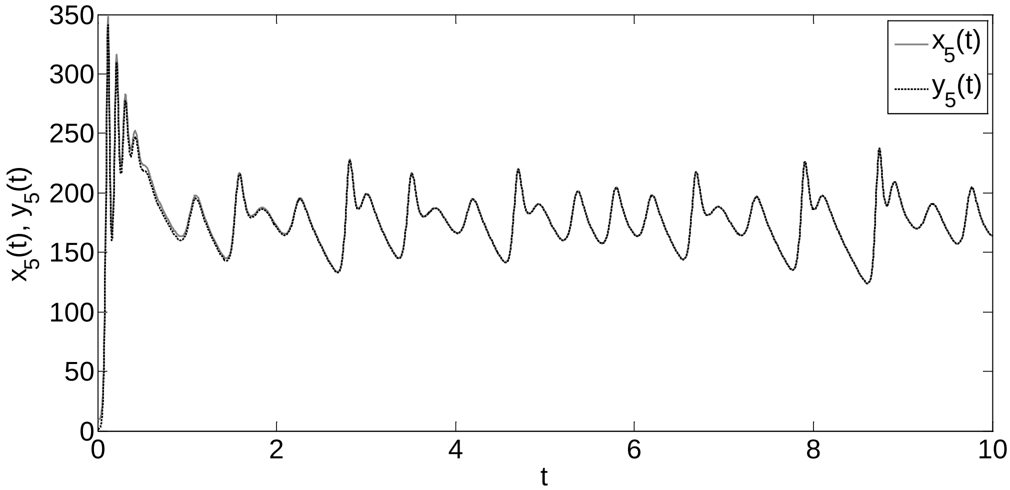


Figure 2.4 (e)

Figure 2.4: Plots of state trajectories of system (2.10) and system (2.17) for standard order $q = 1$ between **(a)** $x_1(t)$ and $y_1(t)$; **(b)** $x_2(t)$ and $y_2(t)$; **(c)** $x_3(t)$ and $y_3(t)$; **(d)** $x_4(t)$ and $y_4(t)$; **(e)** $x_5(t)$ and $y_5(t)$.

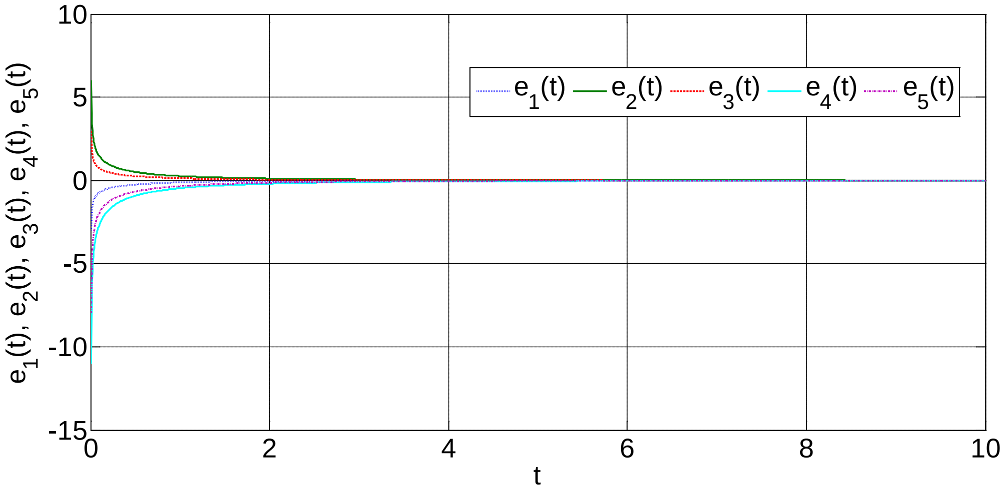


Figure 2.5 (a)

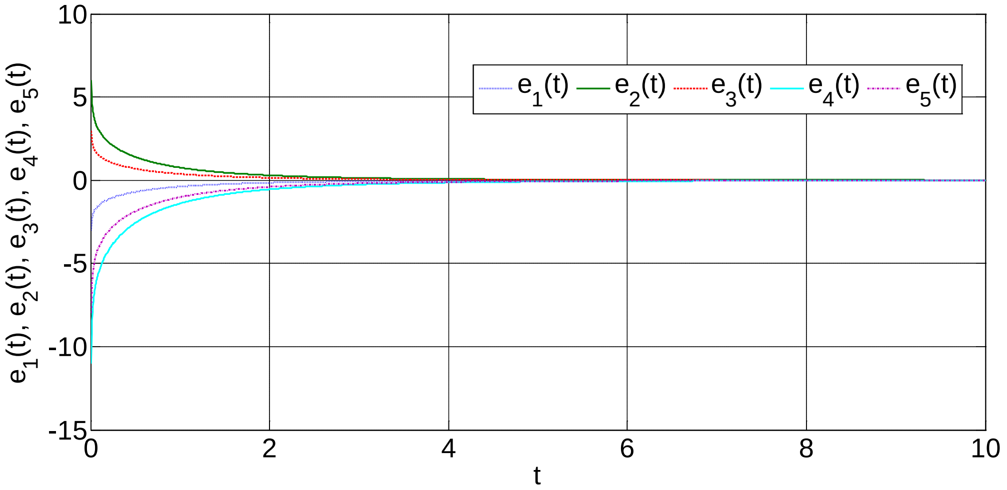


Figure 2.5 (b)

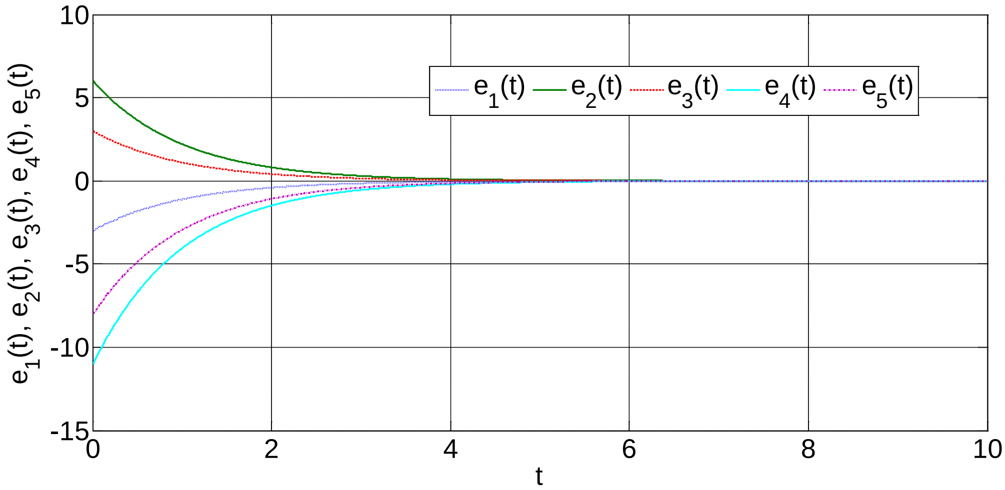


Figure 2.5 (c)

Figure 2.5: Plots of error functions between system (2.10) and system (2.17) for the order of the derivatives **(a)** $q = 0.70$; **(b)** $q = 0.85$; **(c)** $q = 1$.

2.4.1 Numerical simulation and results

In this section, the numerical simulation for synchronization of fractional order complex Lorenz and Lu systems is done. The initial conditions are taken as before and thus the initial error is $e'(0) = [-3+6i\ , 3-11i\ , -8]^T$. Time step size is taken as 0.005. Figure 2.4 depicts the variations of state trajectories during synchronization at $q = 1$. From Figure 2.5, it is seen that considered complex chaotic systems are synchronized after a small time duration at fractional orders $q = 0.70$, $q = 0.85$ and also at the integer order $q = 1$. It is observed from the figures that the time taken for synchronization of systems decreases as order q approaches from integer order to fractional order system.

2.5 Synchronization between fractional order complex Lu and T systems

To study the synchronization of fractional order complex Lu system and complex T system, it is assumed that the Lu system (2.13) drives the T system (2.16). Let us define the response system as

$$D^q z_1 = c_1(z_3 - z_1) + u_1 ,$$

$$D^q z_2 = c_1(z_4 - z_2) + u_2 ,$$

$$D^q z_3 = (c_2 - c_1)z_1 - c_1 z_1 z_5 + u_3 ,$$

$$D^q z_4 = (c_2 - c_1)z_2 - c_1 z_2 z_5 + u_4 ,$$

$$D^q z_5 = z_1 z_3 + z_2 z_4 - c_3 z_5 + u_5 , \tag{2.22}$$

where $u_i(t)$ are control functions.

Defining error functions as

$$e'_1 = e_1 + i\,e_2 = z'_1 - y'_1,\ e'_2 = e_3 + i\,e_4 = z'_2 - y'_2 \text{ and } e'_3 = e_5 = z'_3 - y'_3$$

and proceeding as the previous section, we get

$$D^q e_1 = c_1(e_3 - e_1) + (c_1 - b_1)(y_3 - y_1) + u_1 ,$$

$$D^q e_2 = c_1(e_4 - e_2) + (c_1 - b_1)(y_4 - y_2) + u_2 ,$$

$$D^q e_3 = (c_2 - c_1)e_1 - c_1 z_1 z_5 + (c_2 - c_1)y_1 - b_2 y_3 + y_1 y_5 + u_3 ,$$

$$D^q e_4 = (c_2 - c_1)e_2 - c_1 z_2 z_5 + (c_2 - c_1)y_2 - b_2 y_4 + y_2 y_5 + u_4 ,$$

$$D^q e_5 = -c_3 e_5 + z_1 z_3 + z_2 z_4 - (c_3 - b_3)y_5 - y_1 y_3 - y_2 y_4 + u_5 . \tag{2.23}$$

Now defining the active control functions $u_i(t)$ as

$$u_1 = V_1 - (c_1 - b_1)(y_3 - y_1) ,$$

$$u_2 = V_2 - (c_1 - b_1)(y_4 - y_2) ,$$

$$u_3 = V_3 + c_1 z_1 z_5 - (c_2 - c_1) y_1 + b_2 y_3 - y_1 y_5 ,$$

$$u_4 = V_4 + c_1 z_2 z_5 - (c_2 - c_1) y_2 + b_2 y_4 - y_2 y_5 ,$$

$$u_5 = V_5 - z_1 z_3 - z_2 z_4 + (c_3 - b_3) y_5 + y_1 y_3 + y_2 y_4 , \qquad (2.24)$$

where terms $V_i(t)$ are linear functions of the error terms $e_i(t)$, the error system finally reduces to

$$D^q e_1 = c_1(e_3 - e_1) + V_1 ,$$

$$D^q e_2 = c_1(e_4 - e_2) + V_2 ,$$

$$D^q e_3 = (c_2 - c_1) e_1 + V_3 ,$$

$$D^q e_4 = (c_2 - c_1) e_2 + V_4 ,$$

$$D^q e_5 = -c_3 e_5 + V_5 . \qquad (2.25)$$

Considering $V(t) = Ae(t)$, where $V(t)$ and $e(t)$ are given in system (2.21), and choosing the matrix A as

$$A = \begin{bmatrix} -1 + c_1 & 0 & -c_1 & 0 & 0 \\ 0 & -1 + c_1 & 0 & -c_1 & 0 \\ -(c_2 - c_1) & 0 & -1 & 0 & 0 \\ 0 & -(c_2 - c_1) & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

so that the feedback system has eigenvalues λ_i of A satisfy the condition $|\arg(\lambda_i)| > q\pi/2$, $i = 1, 2, 3, 4, 5$. Thus the synchronization of the systems is achieved.

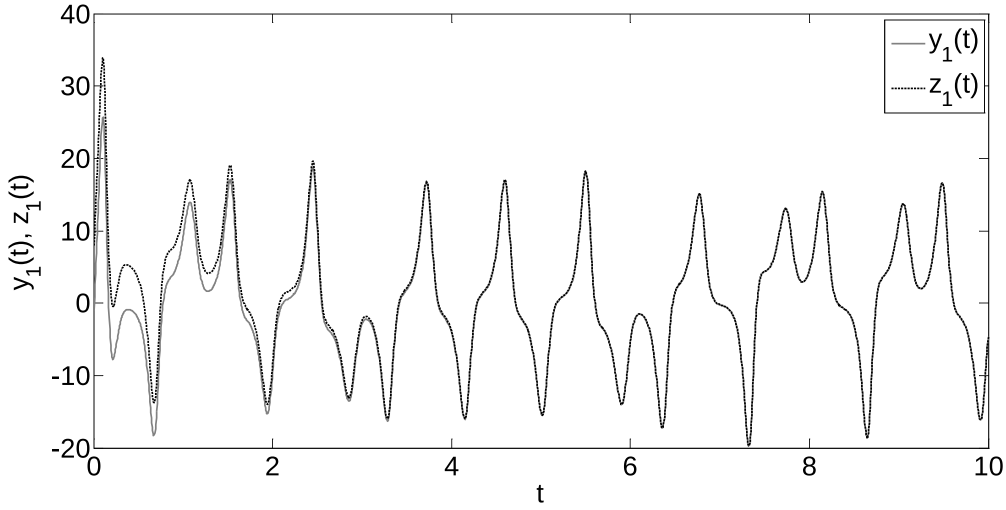


Figure 2.6 (a)

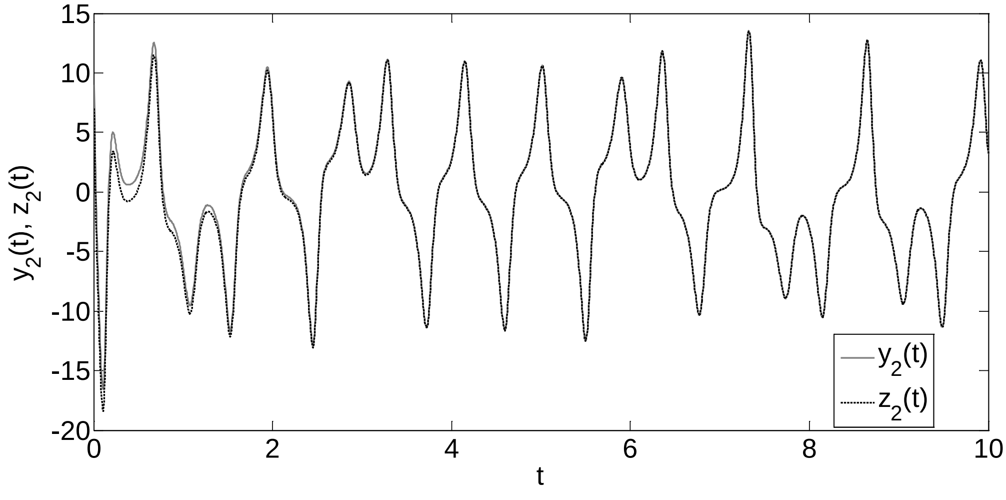


Figure 2.6 (b)

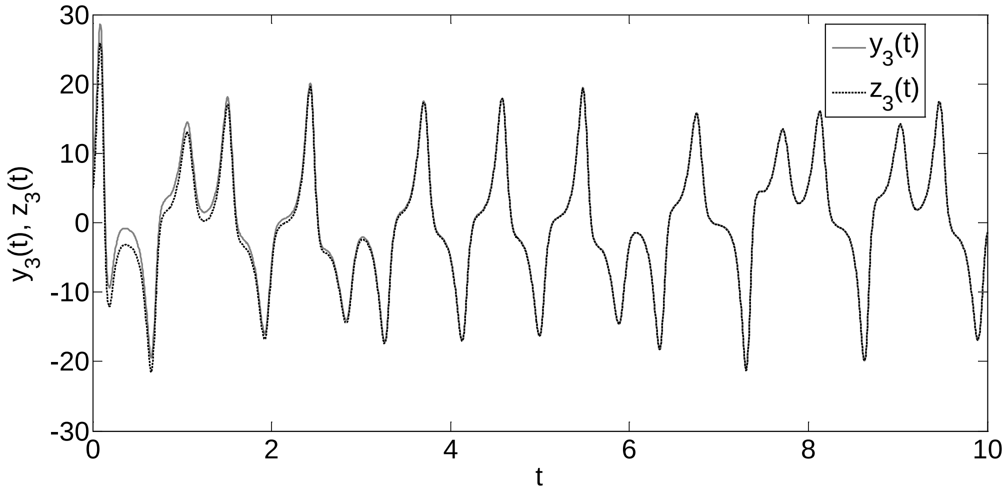


Figure 2.6 (c)

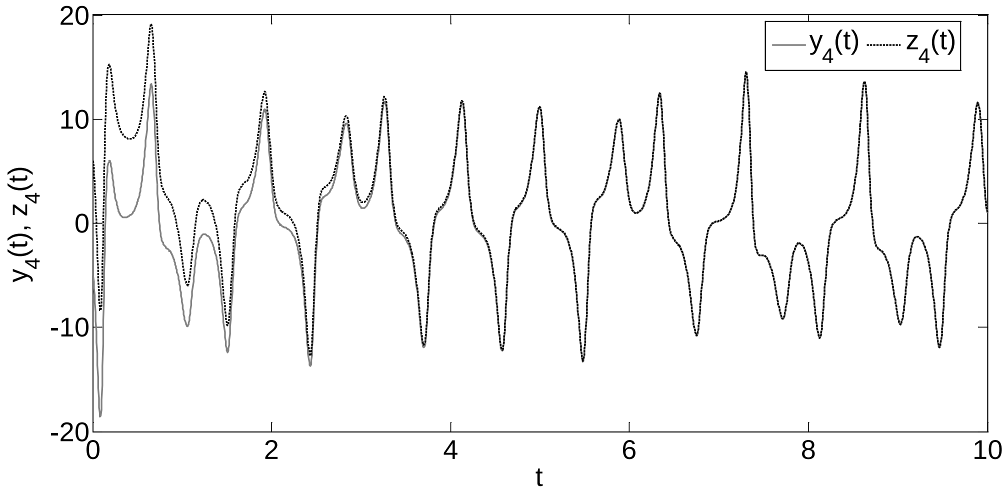


Figure 2.6 (d)

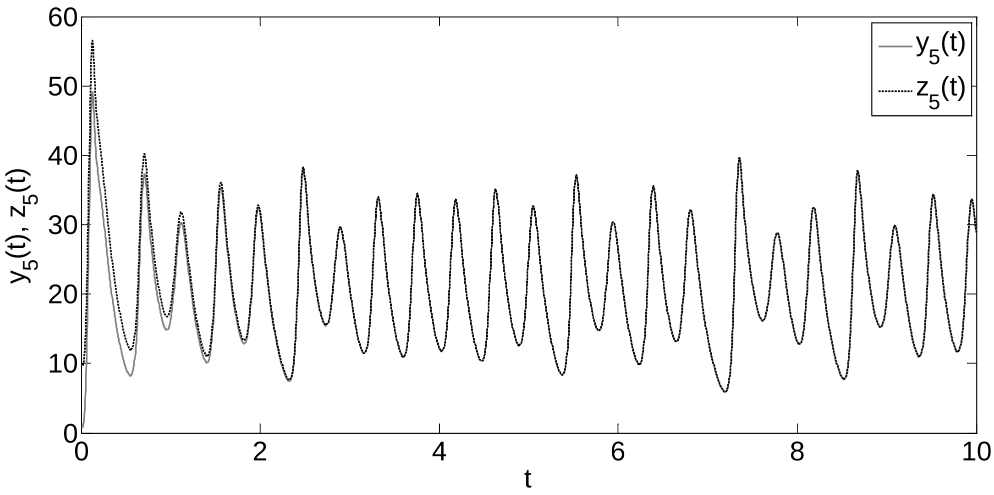


Figure 2.6 (e)

Figure 2.6: Plots of state trajectories of system (2.13) and system (2.22) for standard order $q = 1$ between: **(a)** $y_1(t)$ and $z_1(t)$; **(b)** $y_2(t)$ and $z_2(t)$; **(c)** $y_3(t)$ and $z_3(t)$; **(d)** $y_4(t)$ and $z_4(t)$; **(e)** $y_5(t)$ and $z_5(t)$.

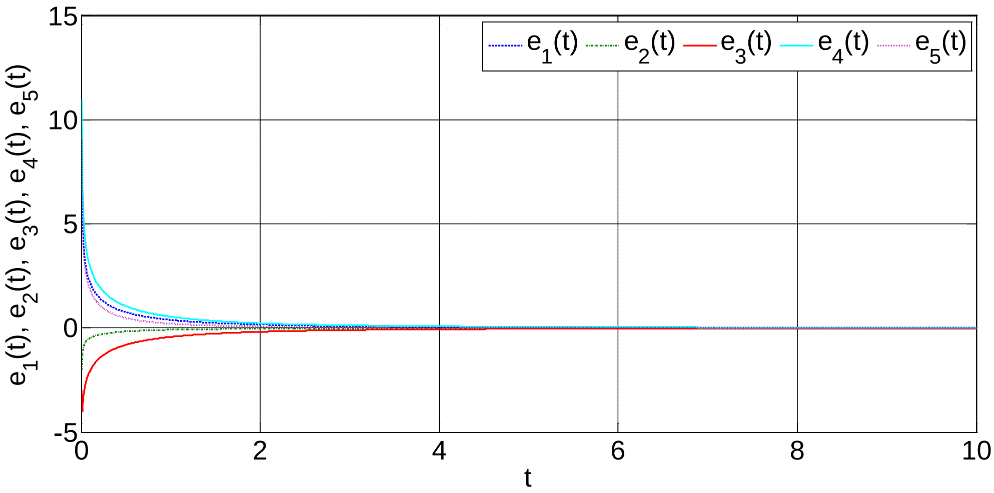


Figure 2.7 (a)

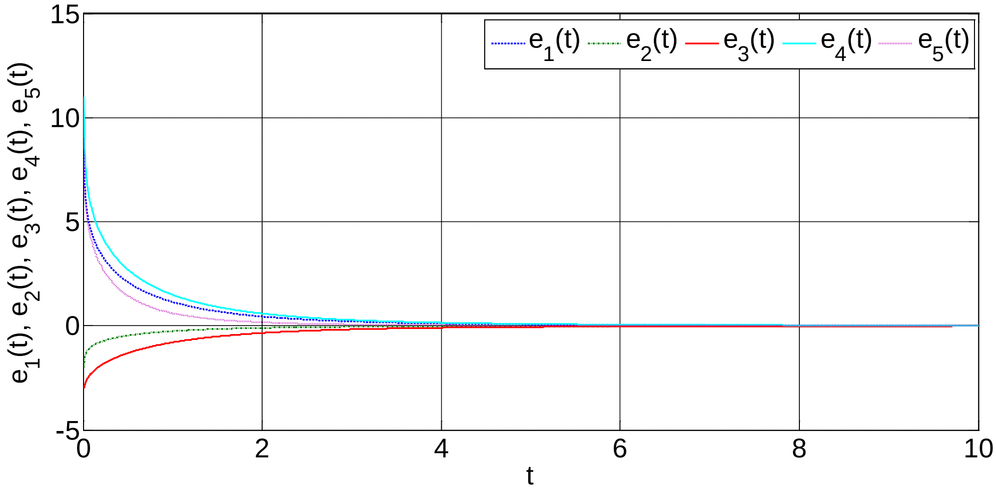


Figure 2.7 (b)

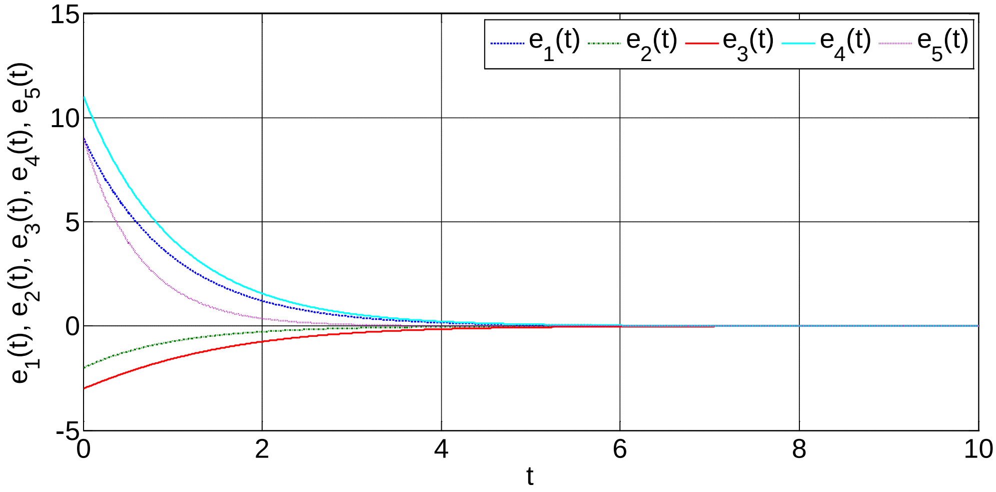


Figure 2.7 (c)

Figure 2.7: Plots of error functions between system (2.13) and system (2.22) for: **(a)** $q = 0.70$; **(b)** $q = 0.85$; **(c)** $q = 1$.

2.5.1 Numerical simulation and results

In this section, the numerical simulation for synchronization of fractional order complex Lu and T systems are done for earlier choices of parameters and for the time step size as 0.005. For the given initial conditions, the initial error is $e'(0) = [9-2i, -3+11i, 9]^T$. The variations of state trajectories during synchronization at $q=1$ are described through Figure 2.6. It is observed from Figure 2.7 that the complex systems are synchronized after a small time duration at $q=0.70$, $q=0.85$ and $q=1$. Figures depict that the systems take higher time for synchronization at standard order as compared to fractional order.

2.6 Synchronization between fractional order complex Lorenz and T systems

During the study of synchronization between fractional order Lorenz and T systems, describing the system (2.10) as drive system and system (2.22) as response system and proceeding as before, we get the error system as

$$D^q e_1 = c_1(e_3 - e_1) + V_1 \ ,$$

$$D^q e_2 = c_1(e_4 - e_2) + V_2 \ ,$$

$$D^q e_3 = (c_2 - c_1)e_1 + V_3 \ ,$$

$$D^q e_4 = -e_4 + V_4 \ ,$$

$$D^q e_5 = -c_3 e_5 + V_5 \ ,$$

where

$$u_1 = V_1 - (c_1 - a_1)(x_3 - x_1) \ ,$$

$$u_2 = V_2 - (c_1 - a_1)(x_4 - x_2) \ ,$$

$$u_3 = V_3 - (c_2 - c_1 - a_2)x_1 - x_3 - x_1x_5 + c_1z_1z_5 \ ,$$

$$u_4 = V_4 - (c_2 - c_1 - a_2)x_2 - x_4 - x_2x_5 + c_1z_2z_5 \ ,$$

$$u_5 = V_5 - (a_3 - c_3)x_5 + x_1x_3 + x_2x_4 - z_1z_3 - z_2z_4 \ ,$$

with

$$e_1 = z_1 - y_1 \,,\ e_2 = z_2 - y_2 \,,\ e_3 = z_3 - y_3 \,,\ e_4 = z_4 - y_4 \,,\ e_5 = z_5 - y_5 \,,$$

where

$$e_1' = e_1 + i\,e_2 = z_1' - y_1' \ ,\ \ e_2' = e_3 + i\,e_4 = z_2' - y_2' \text{ and } e_3' = e_5 = z_3' - y_3' \ .$$

Here taking $V(t) = A\ e(t)$, where

$$A = \begin{bmatrix} -1+c_1 & 0 & -c_1 & 0 & 0 \\ 0 & -1+c_1 & 0 & -c_1 & 0 \\ -(c_2-c_1) & 0 & -1 & 0 & 0 \\ 0 & -(c_2-c_1) & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

It is seen that the closed loop system has the eigenvalues $-1,\ -1,\ -1,\ -1,$ and $-1,$ and thus it is concluded that the required synchronization of the considered systems is achieved.

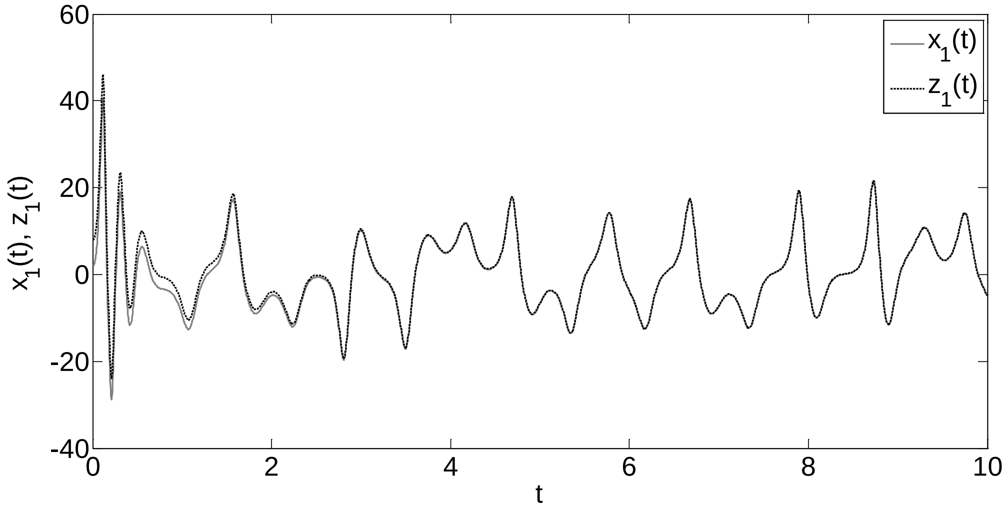


Figure 2.8 (a)

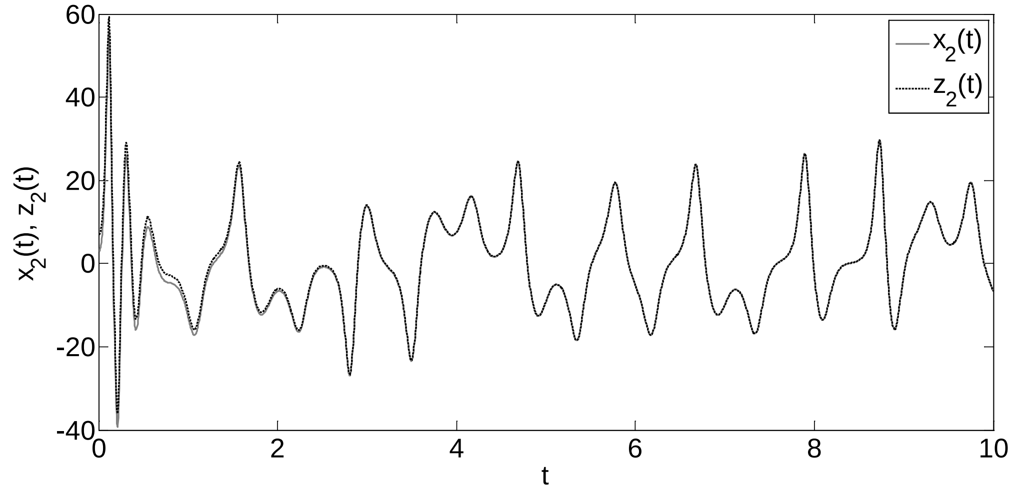
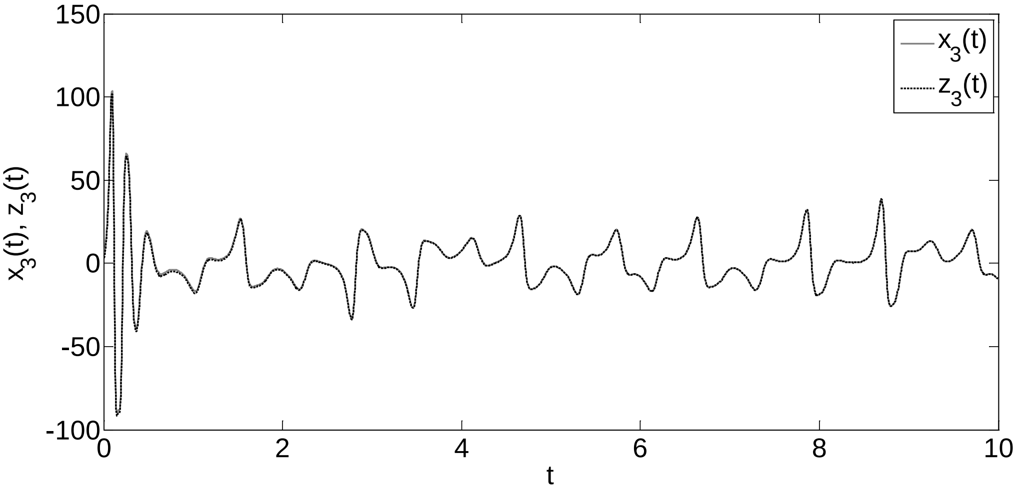


Figure 2.8 (b)



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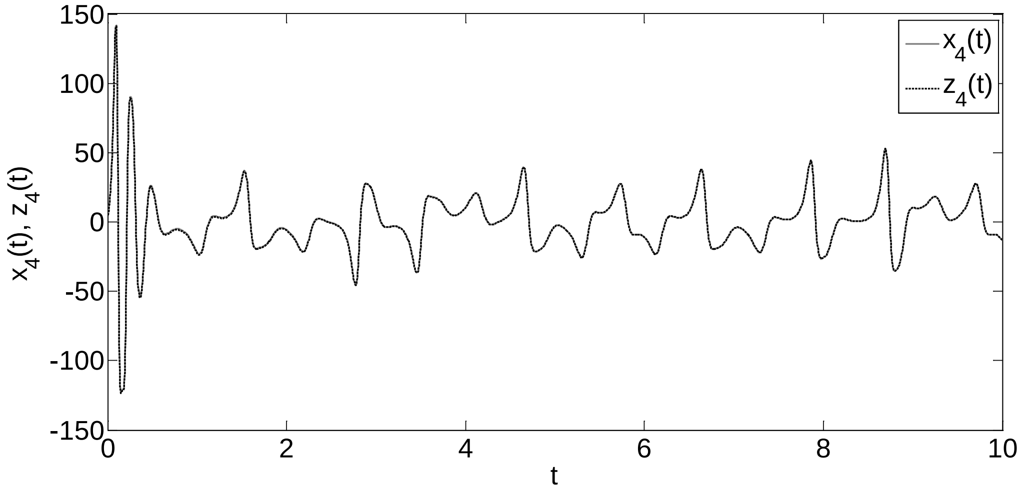

Figure 2.8 (d)

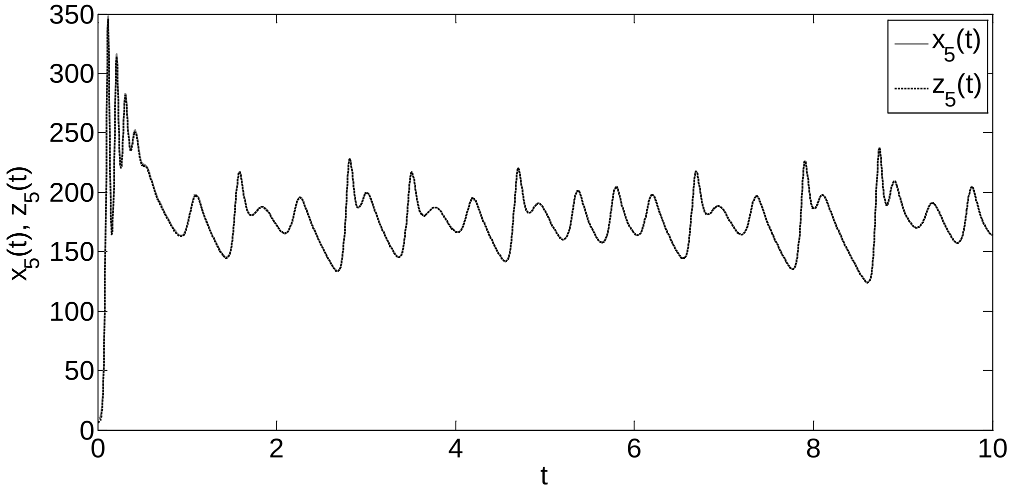

Figure 2.8 (e)

Figure 2.8: Plots of **s**tate trajectories of system (2.10) and system (2.22) for standard order $q = 1$ between: **(a)** $x_1(t)$ and $z_1(t)$; **(b)** $x_2(t)$ and $z_2(t)$; **(c)** $x_3(t)$ and $z_3(t)$; **(d)** $x_4(t)$ and $z_4(t)$; **(e)** $x_5(t)$ and $z_5(t)$.

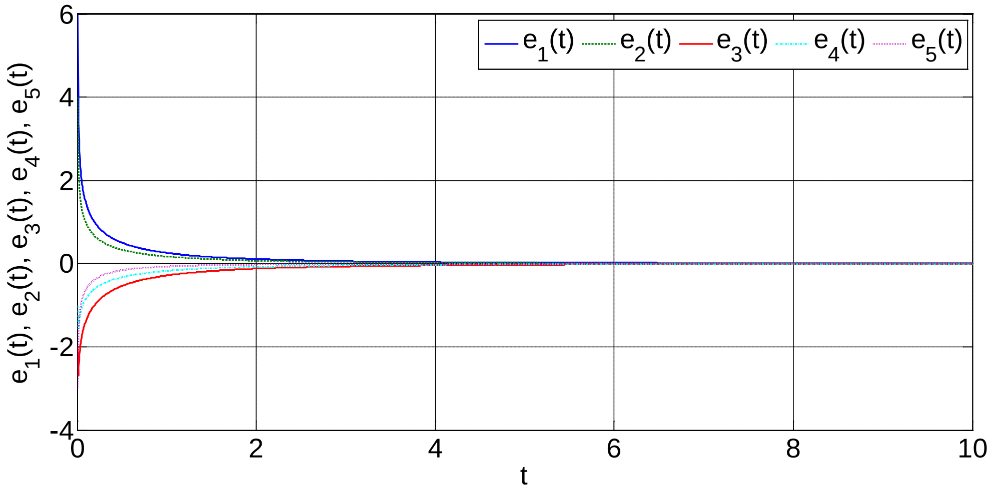

Figure 2.9 (a)

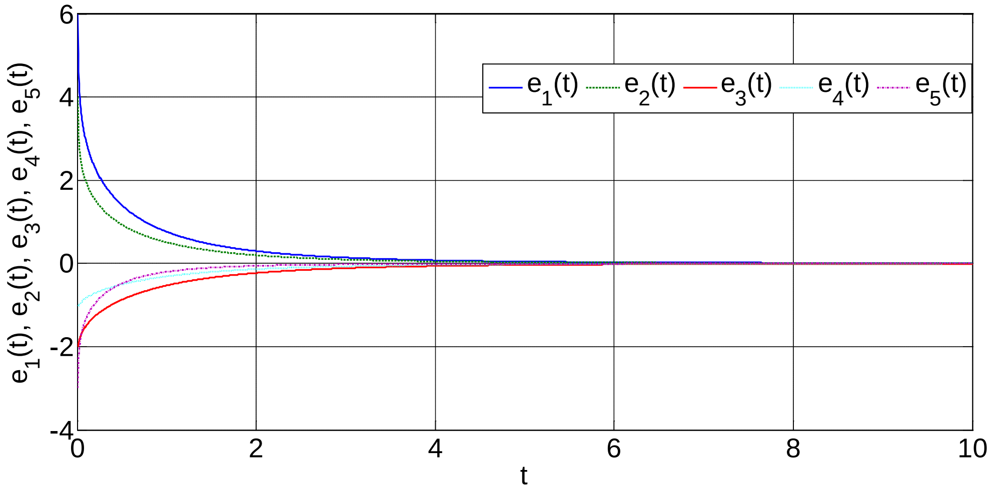

Figure 2.9 (b)

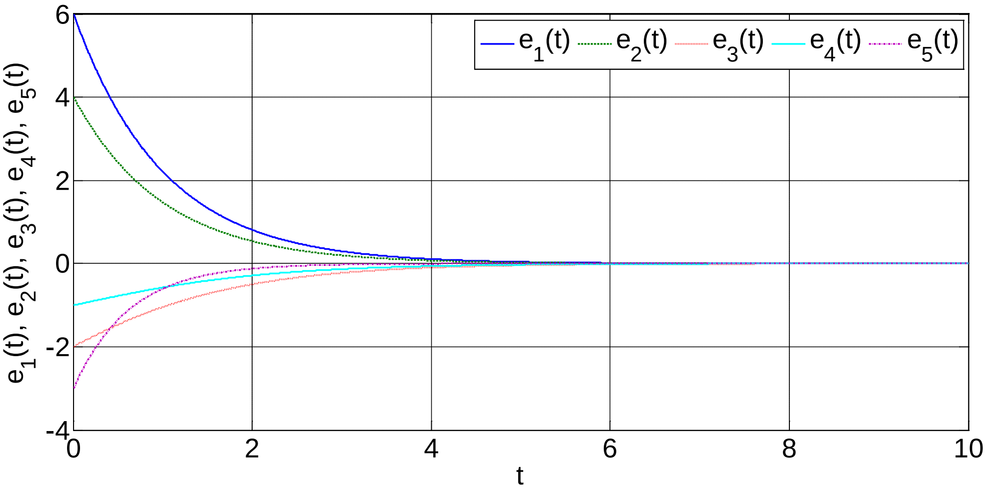

Figure 2.9 (c)

Figure 2.9: Plots of error functions between system (2.10) and system (2.22) for: **(a)** $q = 0.70$; **(b)** $q = 0.85$; **(c)** $q = 1$.

2.6.1 Numerical simulation and results

For the considered drive and response systems the initial error is $e'(0)$ = $[6+4i, -2-i, -3]^T$ and the time step size is taken as 0.005. The state trajectories at $q=1$ are shown in Figure 2.8. Again Figure 2.9 demonstrates the required time for synchronization of the considered pair of systems at $q = 0.70$, $q = 0.85$ and $q = 1$, which shows that it takes maximum time at $q = 1$ and minimum time at $q = 0.70$ for synchronization.

2.7 Conclusion

In this chapter the active control method is successfully used to achieve perfect control of a pair of fractional order complex chaotic systems along with the desired trajectory, which clearly exhibits the reliability and potential of the method even for fractional order complex systems to be synchronized. The most important part of the study is the comparison of the time of synchronization in each of three different pairs of complex systems when a pair of systems approaches from integer order to fractional order.

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