Chapter 1

Introduction

1.1 Fractional calculus

1.1.1 Brief history

In 1695, G. F. A. L'Hospital inquired to G. W. Leibniz about the meaning of the symbol $d^n y / dx^n$ when $n = 1/2$. In a letter dated September 30, 1695, Leibniz replied " Thus it follows that $d^{1/2}x$ will be equal to $x\sqrt{dx : x}$. This is an apparent paradox from which, one day, useful consequences will be drawn". This discussion leads to a new branch of mathematics which deals with derivatives and integrals of arbitrary order and is known as fractional calculus (Miller and Ross (1993)). Nowadays, not only fractions but also arbitrary real and even complex numbers are considered as order of differentiation. The theory of fractional calculus gives us flexibility for the generalisation of the order of the derivative and integration from integer to any real number, and even complex number. Nevertheless, the name "fractional calculus" is kept for the general theory. S. F. Lacroix (1819) was first to define the derivative of arbitrary order in his book wherein he obtained the following formula

$$\frac{d^{1/2}}{dx^{1/2}}(x^a) = \frac{\Gamma(a+1)}{\Gamma(a+\frac{1}{2})} x^{a-\frac{1}{2}} \quad .$$

Further Joseph B. J. Fourier, who in 1822 derived the following integral representation of $f(x)$, where

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\,dy \int_{-\infty}^{\infty} \cos z(x-y)\,dz \ .$$

Using the representation, he obtained

$$\frac{d^q f(x)}{dx^q} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\,dy \int_{-\infty}^{\infty} z^q \cos\left(z(x-y) + \frac{q\pi}{2}\right) dz \ .$$

Joseph Fourier stated that: "The number q that appears in the above will be regarded as any quantity whatsoever, positive or negative".

First application (Ross (1975)) of fractional derivative was given by Niels Henrik Abel (1823). He used fractional calculus while solving an integral equation arising in tautochrone problem. He solved the following integral equation for $q = -\frac{1}{2}$,

$$k = \int_0^x (x-t)^q \ f(t)\,dt \ , \qquad (1.1)$$

where $f(t)$ is unknown. For determining f , Abel wrote the right hand side of the Equation (1.1) as $\sqrt{\pi}\left(\frac{d^{-1/2} f(x)}{dx^{-1/2}} \right)$ and applied $\frac{d^{1/2} f(x)}{dx^{1/2}}$ on both sides of Equation (1.1) to obtain

$$\frac{d^{1/2} k}{dx^{1/2}} = \sqrt{\pi} \ f(x) \ ,$$

as the fractional operators (with suitable conditions on f) have the property that,

$$\frac{d^{1/2}}{dx^{1/2}}\left[\frac{d^{-1/2} f(x)}{dx^{-1/2}} \right] = \frac{d^0 \ f(x)}{dx^0} = f(x) \ .$$

First explicit definition of fractional derivative was given by J. Liouville (1832). He expanded the function $f(x)$ as the series as

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}$$

and assumed

$$D^q f(x) = \sum_{n=0}^{\infty} c_n a_n^q e^{a_n x}$$

for those values of order q for which the series is convergent.

G. Boole (1844) developed symbolic method for solving linear differential equations with constant coefficients using fractional calculus.

In 1847 Bernhard Riemann (Ross 1975) proposed the following definition of fractional integration

$$D^{-q} f(x) = \frac{1}{\Gamma(q)} \int_c^x (x-t)^{q-1} f(t) dt + \psi(x) ,$$

where $\psi(x)$ is Riemann's complementary function.

Oliver Heaviside (1892) introduced fractional derivative in electromagnetic theory. H. Weyl (1917) and G. H. Hardy (1917) studied some properties of fractional derivative/integrals. A. Erdelyi (1939) and T. J. Oslar (1970) defined differ-integrals of arbitrary functions. Though the mathematics of fractional calculus existed in the literature for more than 300 years, its utility has been realised rather recently. Currently many scientists are working on fractional calculus and its applications in different areas.

1.1.2 Applications

Due to its non-locality, fractional derivative operator (FDO) D^q differs from the ordinary differential operator (ODO) D in many respects. The product rule and chain rule take complicated form in case of FDO. Unlike ODO, the FDO does not have physical meaning, though some efforts have been made in this direction recently (Podlubny (2002)). Moreover there are several inequivalent definitions of FDO. Hence many mathematicians were sceptical about this field.

During the last decade fractional calculus has been applied to almost every field of Science, Engineering and Mathematics. It is now realised that the non-locality is not a drawback but it leads FDO to model many natural phenomena containing long memory. Examples of such anomalous systems are abundant in nature. Few of those are atmospheric diffusion of pollution, cellular diffusion processes, network traffic, dynamics of visco-elastic materials, etc.

All such systems have non-local dynamics involving long memory which cannot be modelled using classical calculus. In fact ODO models the ideal behaviour and FDO models the real behaviour. Thus fractional differential equations (FDE) are useful for the modelling of many anomalous phenomena in nature and in the theory of complex systems. Some applications of FDO are listed as follows.

Visco-elasticity: For an elastic spring, the stress (force) is proportional to the strain (extension). If m is elastic modulus (depend on the material of spring) then Hooke's law states

$$\sigma(t) = m\ \varepsilon(t)\ , \tag{1.2}$$

where $\sigma(t)$ and $\varepsilon(t)$ denote stress and strain respectively. This rule, however, does not carry over to liquids. For viscous liquids Newton proposed the model obeying the law

$$\sigma(t) = b\frac{d}{dt}\varepsilon(t) \; , \tag{1.3}$$

where b is the viscosity coefficient. Though the laws (1.2) and (1.3) are useful in theory and model the ideal behaviour have limitations in real practice. There were unsuccessful attempts to combine these laws. In fact, behaviour of the real materials lies somewhere between ideal solids and ideal fluids. G. W. Scott Blair (1947) proposed that for intermediate materials, stress is proportional to the intermediate derivative of strain, i.e.,

$$\sigma(t) = E\,D^{q}\varepsilon(t) \; , \qquad 0 < q < 1 \; , \tag{1.4}$$

where E and q are constants depend on material under study. The remarkable contributors in this field are A. Gemant (1950), A. N. Gerasimov (1948), R. L. Bagley and P. J. Torvik (1984) and many others. M. Caputo and F. Mainardi (1971) used Caputo fractional derivatives to give more realistic models. Detailed discussion on this topic is now available in the recent book by Mainardi (2010).

Diffusion-wave equation: The time fractional diffusion-wave equation is given by

$$D_t^q\, u(x,\, t) = k\frac{\partial^2}{\partial x^2}u(x,\, t) \; , \qquad 0 < q \le 2 \; . \tag{1.5}$$

If $0 < q \le 1$, the equation represents fractional diffusion equation and for $1 < q \le 2$ it is fractional wave equation. It is useful in modelling many anomalous phenomena such as porous, amorphous through fractals, percolation clusters, diffusion through disordered media, polymers and biological systems etc.

F. Mainardi (1996) obtained the fundamental solution for the fractional diffusion-wave equation in one space-dimension. W. Wyss (1986) used Mellin transform to solve Cauchy problem. R. W. Schneider and W. Wyss (1989) converted the diffusion-wave equation into the integro-differential equation and found the corresponding Green functions in terms of Fox functions. Y. Fujita (1990) has presented the existence and uniqueness of the solution of the space-time fractional diffusion equation.

Electrical circuits: Classical electrical circuit contains inductor, capacitor and resistor described by integer-order models. A. Le Mehaute and G. Crepy proposed a circuit containing a fractance whose properties lie between resistance and capacitance and is modelled by fractional derivative. Thus this circuit has more flexibility. It can also be used for analogue fractional differentiation and integration. T. Hartley et al. (1995) have shown that the Hartley-Chua circuit of system order less than three exhibits chaos.

Control theory: Fractional derivatives are widely used in control theory because of their realistic approach. Major contribution in fractional control theory is by A. Oustaloup (1983). He developed a CRONE (Commande Robuste d'Ordre Non Entier) controller and showed that it works better than classical PID controller.

Biology: The concept of membrane reactance was given by K. S. Cole (1933). It is used in the conductance of membranes of cells of organisms. Membrane reactance is given by

$$X(\omega) = X_0 \, \omega^{-q} \; .$$

Cole presented the experimentally obtained values of q for different cases such as guinea pig lever and muscle $(q = 0.45)$, potato $(q = 0.25)$, etc.

Bio-engineering: Bio-engineering is a branch of life science which deals with the design, manufacture and maintenance of engineering equipments used in biosynthetic processes. In medical science it plays an important role in the design of artificial limbs, artificial pacemaker and so on. Bio-engineering strives to develop new mathematical tools for describing the complexity of cells and tissues. Fractional order operations are useful in encoding the multi-scale pattern arising in the muscle fibres and nerve fibres. The resulting dynamics of such multi-scale processes are expressed through fractional order differential equations. Various such applications of fractional calculus in bio-engineering are described in the recent book by R. Magin (2006).

1.1.3 Fractional derivative

1.1.3.1 Grunwald-Letnikov fractional derivative

Successive differentiations of function $f(t)$ are given by

$$f^{(1)}(t) = \lim_{h\to 0}\frac{f(t)-f(t-h)}{h},$$

$$f^{(2)}(t) = \lim_{h\to 0}\frac{f^{(1)}(t)-f^{(1)}(t-h)}{h} = \lim_{h\to 0}\frac{f(t)-2f(t-h)+f(t-2h)}{h^2},$$

In general,

$$f^{(n)}(t) = D^n f(t) = \lim_{h\to 0}\frac{1}{h^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(t-kh), \tag{1.6}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is a binomial coefficient. For a non-integer $q > 0$, we can write $\binom{q}{k} = \frac{\Gamma(q+1)}{k!\Gamma(q-k+1)}$.

The Grunwald-Letnikov definition is the generalisation of the Definition (1.6) to a non-integer $q > 0$.

$$ {}^{GL}D_a^q f(t) = \lim_{h\to 0}\frac{1}{h^q}\sum_{k=0}^{\left[\frac{t-a}{h}\right]}(-1)^k \frac{\Gamma(q+1)}{k!\Gamma(q-k+1)}\; f(t-kh)\;. \tag{1.7} $$

Fractional integral of order $q > 0$ is defined by

$$ {}^{GL}D_a^{-q} f(t) = \lim_{h\to 0} h^q \sum_{k=0}^{\left[\frac{t-a}{h}\right]} \frac{\Gamma(q+k)}{k!\Gamma(q)}\; f(t-kh)\;. \tag{1.8} $$

1.1.3.2 Riemann-Liouville fractional derivative

Riemann-Liouville fractional integral operator is a direct generalization of the Cauchy's formula for an $n-$fold integral as

$$ \underbrace{\int_a^x dt \int_a^{x_1} dt \cdots \int_a^{x_{n-1}} f(t)\, dt}_{n-\text{times}} = \frac{1}{(n-1)!}\int_a^x \frac{f(t)}{(x-t)^{1-n}}\,dt \tag{1.9} $$

Definition 1.1 If $f(x) \in C[a, b]$ and $q > 0$, then

$$ I_{a^+}^q f(x) := \frac{1}{\Gamma(q)}\int_a^x \frac{f(t)}{(x-t)^{1-q}}dt\;, \quad x > a\;, $$

$$ I_{b^-}^q f(x) := \frac{1}{\Gamma(q)}\int_x^b \frac{f(t)}{(x-t)^{1-q}}dt\;, \quad x < b\;, \tag{1.10} $$

are called as the left sided and the right sided Riemann-Liouville fractional integral of order q, respectively.

Definition 1.2 $$ {}^{RL}D_a^q f(x) := \frac{1}{\Gamma(1-q)}\frac{d}{dx}\int_a^x \frac{f(t)}{(x-t)^q}dt = D\, I_a^{1-q} f(x)\;,\; 0<q<1, \tag{1.11} $$

is called the left side Riemann-Liouville fractional derivative of order q whenever the RHS exists.

The definition and properties of the Riemann-Liouville fractional derivative for arbitrary value of $q > 0$ are as follows.

Definition 1.3 Let $n-1 < q \le n$ then the left sided and right sided Riemann-Liouville fractional derivatives of order q are defined as

$$ {}^{RL}D_{a^+}^{q} f(x) := \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{q+1-n}} dt = D^n I_{a^+}^{n-q} f(x) , \qquad x > a , $$

$$ {}^{RL}D_{b^-}^{q} f(x) := \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_x^b \frac{f(t)}{(x-t)^{q+1-n}} dt = D^n I_{b^-}^{n-q} f(x) , \qquad x < b, \tag{1.12} $$

respectively, whenever the RHS exist.

In further discussion, unless mentioned otherwise, we denote ${}^{RL}D_{a^+}^{q} f(x)$ by ${}^{RL}D_{a}^{q} f(x)$ and $I_{a^+}^{q} f(x)$ by $I_{a}^{q} f(x)$, respectively. Also ${}^{RL}D^{q} f(x)$ and $I^{q} f(x)$, refer to ${}^{RL}D_{0^+}^{q} f(x)$ and $I_{0^+}^{q} f(x)$, respectively.

Properties: (i) The Riemann-Liouville fractional derivative of constant is not zero.

$$ {}^{RL}D^{q} C = \frac{C\,t^{-q}}{\Gamma(1-q)} \neq 0 . \tag{1.13} $$

(ii) Initial value problem (IVP) containing Riemann-Liouville fractional derivative requires initial conditions of the form ${}^{RL}D^{q-j} f(0)$ i.e.,

$$ I^{q}\left({}^{RL}D^{q} f(x)\right) = f(t) - \sum_{j=1}^{n} {}^{RL}D^{q-j} f(0) \frac{t^{q-j}}{\Gamma(q-j+1)} , \quad n-1 \le q < n , \tag{1.14} $$

which is not useful in real phenomena. To overcome these drawbacks, M. Caputo (1966) proposed a new definition of derivatives which allows the formulation of initial conditions for fractional IVPs in a form involving only the limit values of integer order derivatives at the lower terminal.

1.1.3.3 Caputo fractional derivative

The definition and properties of the Caputo fractional derivative are given as follows.

Definition 1.4 Let $f \in C^n[a,b]$ and $n-1<q<n$ then

$$D_a^q f(x)=\frac{1}{\Gamma(n-q)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{(q-n+1)}}dt\ , \qquad a<x<b\ . \tag{1.15}$$

Properties: (i) $D_a^q C=0$, C is a constant. (1.16)

(ii) $\lim\limits_{q\to n} D_a^q f(x)=f^{(n)}(x)$. (1.17)

1.1.3.4 Relation between Riemann-Liouville and Caputo derivatives

Theorem 1.5 Let $f \in C^n[a,b]$ and $n-1<q<n$. Then R-L and Caputo fractional derivatives are connected by the relation

$$^{RL}D_a^q f(x)=D_a^q f(x)+\sum_{k=0}^{n-1}\frac{f^{(k)}(a^+)}{\Gamma(1+k-q)}(x-a)^{k-q}\ . \tag{1.18}$$

Proof: $^{RL}D^q f(x)=D^n I^{n-q} f(x)=D^n\left[I^{n-q}\left(I^n f^{(n)}(x)+\sum_{k=0}^{n-1}\frac{f^{(k)}(a^+)}{k!}(x-a)^k\right)\right]$

$$=I^{n-q} f^{(n)}(x)+D^n I^{n-q}\sum_{k=0}^{n-1}\frac{f^{(k)}(a^+)}{k!}(x-a)^k \tag{1.19}$$

$$=D_a^q f(x)+\sum_{k=0}^{n-1}\frac{f^{(k)}(a^+)}{\Gamma(1+k-q)}(x-a)^k\ . \tag{1.20}$$

From the above theorem, we get the follow result:

(i) If $q = n \in N$, then ${}^{RL}D_a^q f(x) = D_a^q f(x) = D^n f(x)$.

(ii) If $f^{(k)}(a) = 0$ for $k = 0, 1, \cdots, n-1$, then ${}^{RL}D_a^q f(x) = D_a^q f(x)$.

(iii) If $0 < q < 1$, then ${}^{RL}D_a^q f(x) = D_a^q f(x) + \dfrac{f(a)}{\Gamma(1-q)}(x-a)^{-q}$.

Theorem 1.6 Let $f \in C^n[a, b]$ and $n-1 < q < n$, then

$$I_a^q D_a^q f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a^+)}{k!}(x-a)^k, \qquad x \geq a . \tag{1.21}$$

Proof: If $I_a^q D_a^q f(x) = I_a^q I_a^{n-q} f^{(n)}(x) = I^{(n)} f^{(n)}(x) = f(x) - \sum_{k=0}^{n-1} \dfrac{f^{(k)}(a)}{k!}(x-a)^k$, $x \geq a$, then Equation (1.21) is a particular case of the more general property

$$I_a^q D_a^r f(x) = I_a^q I_a^{m-r} f^{(m)}(x) = I_a^{(q-r)}\left(I^{(n)} f^{(n)}(x)\right) \qquad q > r , \tag{1.22}$$

$$= I_a^{q-r} f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(q-r+k+1)}(x-a)^{q-r+k}, \quad x \geq a , \quad m-1 < r < m .$$

1.1.4 Leibniz rule

(Podlubny (2002)) Leibnitz rule for fractional derivative q of the functions $f(x)$ and $g(x)$ is given by

$$_aD_t^q\big(g(t)\, f(t)\big) = \sum_{k=0}^{\infty} \binom{q}{k} g^{(k)}(t)\, D^{q-k} f(t) - R_n^q(t). \tag{1.23}$$

where $R_n^q(t) = \dfrac{1}{n!\Gamma(-q)} \displaystyle\int_a^t \frac{f(\tau)}{(t-\tau)^{q+1}} d\tau \int_\tau^t g^{(n+1)}(\xi)(\tau-\xi)^n d\xi$.

1.1.5 Fractional differential equation

In this section, some results are discussed on existence and uniqueness of fractional differential equations involving Riemann-Liouville derivative ${}^{RL}D$ and Caputo derivative D.

Theorem 1.7 (Daftardar-Gejji and Babakhani (2004)) The unique solution of the initial value problem (IVP) is

$$ {}^{RL}D^q[\bar{x}(t)-\bar{x}(0)] = A\bar{x}(t)\ ,\quad \bar{x}(0)=\bar{x}_0\ ,\quad 0<q<1\ ,\quad t\in[0,\ \chi]\ ,\quad \chi>0\ , \qquad (1.24)$$

where A is an $n\times n$ matrix, is $E_q(t^qA)\bar{x}_0$.

Theorem 1.8 (Daftardar-Gejji and Babakhani (2004)) Let $f_i : W \to R$ be continuous, $i=1,\ 2,\ \cdots,\ n$, where

$$W=[0,\ \chi^*]\times\Pi_{j=1}^{n}[x_j(0)-l_j\ ,\ x_j(0)+l_j]\ ,\qquad \chi^*>0\ ,\quad l_j>0\ ,\quad \forall\ j\ ,$$

and $\bar{f}=(f_1\ ,\ f_2\ ,\cdots,\ f_n)$. Then the non-autonomous IVP

$$ {}^{RL}D^q[\bar{x}(t)-\bar{x}(0)] = \bar{f}(t,\ \bar{x})\ ,\qquad \bar{x}(0)=\bar{x}_0\ ,\qquad 0<q<1\ , \qquad (1.25)$$

has a solution $\bar{x}(t):[0,\ \chi]\to R^n$, where

$$\chi=\min\left\{\chi^*,\ \left(\frac{l\,\Gamma(q+1)}{\|\bar{f}\|_\infty}\right)^{1/q}\right\},\qquad l=\min\{l_1\ ,\ l_2\ ,\cdots,\ l_n\}.$$

Theorem 1.9 (Daftardar-Gejji and Jafari (2007)) Let $f=(f_1\ ,\ f_2\ ,\cdots,\ f_n):W\to R^n$ be C^1(continuously differentiable), where

$$W=[0,\ \chi^*]\times\Pi_{j=1}^{n}[x_j(0)-l_j\ ,x_j(0)+l_j]\ ,\qquad \chi^*>0\ ,\quad l_j>0\ ,\quad \forall\ j\ .$$

Then the system of non-autonomous equations

$$D^q x_i(t) = f_i(t\ ,\ x_1\ ,\ x_2\ ,\ \cdots,x_n)\ ,\quad x_i^{(k)}(0) = C_k^i\ ,\quad 1 \le i \le n\ ,\quad 1 \le k \le m_i\ , \tag{1.26}$$

where $m_i < q_i < m_i + 1$ has a unique solution $\bar{x}(t) : [0\ ,\ \chi] \to R^n$, where

$$\chi = \min\left\{ \chi^*\ ,\left(\frac{l\,\Gamma(q_i+1)}{[1+q]\|f\|} \right)^{1/q_i}\ ,\left(\frac{lk!}{[1+q]\left|C_k^i\right|} \right)^{1/k} \right\},$$

$$i = 1\ ,\ 2\ ,\ \cdots\ ,\ n\ ,\quad k = 1\ ,\ 2\ ,\ \cdots\ ,\ m_i\ ,$$

$$l = \min\{l_1\ ,\ l_2\ ,\ \cdots\ ,\ l_n\}\ ,\quad q = \max\{q_1\ ,\ q_2\ ,\ \cdots\ ,\ q_n\}\ ,$$

and $[1+q]$ denotes integral part of $1+q$.

Lemma 1.10 (Norelys et al. (2014)) Let $f(t) \in R$ be a continuous and derivable function. Then for any time instant $t \ge t_0$,

$$\frac{1}{2} D^q\ f(t) \le f(t)\ D^q f(t),\quad \forall\ \ q \in (0,\ 1)\ . \tag{1.27}$$

1.2 Dynamical system

A dynamical system is a system which changes over time according to a set of fixed rules that determine how one state of the system moves to another state.

A dynamical system has two parts:

(i) a state vector which describes exactly the state of some real or hypothetical system,

(ii) a function which tells us that for a given current state, what the state of the system will be in the next instant of time.

Mathematically, a dynamical system is described by an initial value problem. In this way, dynamical system can be considered to be a model describing the temporal evolution of a system. Modelling is a powerful analytical tool for understanding and predicting behaviour of physical and artificial systems that changes over time and thus has a history of success.

The definition of dynamical system is as follows:

Definition 1.11 A continuously differentiable function $f : R \times R^n \to R^n$ is called dynamical system if it satisfies the following properties:

(i) $f(0, x) = x$, for all $x \in R^n$

(ii) $f(t, f(s, x)) = f(t+s, x)$, for all $x \in R^n$ and $t, s \in R$,

where R^n is the state space, a member $x \in R^n$ is a state of the system, and $f(t, x)$ is the state, to which the system arrives after time t starting from the state x.

1.2.1 Classification of dynamical systems

There are two main types of dynamical systems: differential equations and iterated maps (also known as difference equations). Differential equations describe the evolution of system in continuous time, whereas iterated maps arise in problem where time is discrete.

1.3 Chaos

Chaos is written the Greek word 'Χάος', significances a state without order or predictability. According to ancient Greek mythology, chaos is the "primeval emptiness preceding the genesis of the universe, turbulent and disordered, mixing all the elements". The motions of physical systems are modelled using differential equations. When solutions of these equations are bounded, they either settle down to a fixed state or oscillate in a periodic (or quasi-periodic) state. There are some systems whose solutions do not fall in any of these categories. These solutions exhibit aperiodic (or irregular) motion for all time and never settle. Moreover, these solutions are highly sensitive to initial conditions, i.e., nearby starting trajectories separates exponentially. Thus it is difficult to predict the behaviour of the solution for a long time. Such systems are called chaotic dynamical systems.

James Clerk Maxwell was probably the first who observed the chaos. In 1860, he studied the motion of two colliding gas particles in a box which was unpredictable for long duration. Henry Poincare was the first person to glimpse the possibility of chaos in 1890. The major breakthrough in chaos theory and nonlinear dynamics was after the discovery of high speed computers in 1950’s. In late 1950’s, the meteorologist Edward Lorenz acquired the LGP-30 computer having internal memory 16 KB. Using computer, in 1963, he discovered the chaotic motion on a strange attractor. The model studied by Lorenz arising in weather prediction was consisting of autonomous system of three ordinary differential equations containing nonlinear terms. The solutions were aperiodic and sensitive to initial conditions. A simple electronic circuit resulting chaotic attractor was given by T. Matsumoto et al. (1985). R.M. May (1976) studied one dimensional maps (difference equations) modelling population dynamics. He observed that a very simple model can generate extremely complicated dynamics. This was the pioneering work in the study of chaos in maps. Thus, the chaos can occur in (i) one dimensional maps, (ii) nonlinear, autonomous system of differential equations (DE) of order three and higher and (iii) nonlinear, non-autonomous system of DEs of order two.

1.3.1 Definition of chaos

Despite the fact that there is no unified, universally accepted, rigorous definition of chaos in the literature survey, however a commonly used definition confines the following nature of chaos, which everyone will agree with the following quotation as mentioned by Steven H. Strogatz in in his monograph (1994).

"Chaos is aperiodic long-term behaviour, in a deterministic system that exhibits sensitive dependence on initial conditions".

The three properties of chaos mentioned in the definition might be explained as follows:

(i) "Aperiodic long-term behaviour" means that there are trajectories which do not settle down to fixed points, periodic orbits, or quasi-periodic orbits as time becomes large.

(ii) "Deterministic" means that the system has no random or noisy inputs or parameters. This irregular behaviour arises from the system's nonlinearity, rather than from noisy driving forces.

(iii) "Sensitive dependence on initial conditions" means that nearby trajectories separate exponentially fast, i.e., the system has a positive Lyapunov exponent.

1.3.2 Attractor and strange attractor

An attractor is a set to which all neighbouring trajectories converge. Stable fixed points and stable limit cycles are examples. More precisely, we define an attractor to be a closed set A with the following properties (Strogatz (1994))

(i) A is an invariant set: any trajectory $x(t)$ that starts in A stays in A for all time.

(ii) A attracts an open set of initial conditions: there is an open set U containing A such that if $x(0) \in U$, then the distance from $x(t)$ to A tends to zero as $t \to \infty$. This means that A attracts all trajectories that start sufficiently close to it. The largest such U is called the basin of attraction of A.

(iii) A is minimal: there is no proper subset of A that satisfies conditions (i) and (ii).

Finally, a strange attractor is an attractor that exhibits sensitive dependence on initial conditions and is called strange because it is often fractal sets. Nowadays this geometric property is regarded as less important than the dynamical property of sensitive dependence on initial conditions.

1.3.3 Chaos in differential equations

Here some examples of chaotic differential dynamical systems are discussed.

Example 1.12 The Lorenz system (Lorenz (1963)) represents convective motion of fluid which is cooled from above and warmed from below. It is represented by following set of equations:

$$\dot{x} = \sigma(y - x) ,$$

$$\dot{y} = rx - y - xz ,$$

$$\dot{z} = xy - \mu z , \qquad (1.28)$$

where σ is the Prandtl number, r is the Rayleigh number over the critical Rayleigh number and μ gives the size of the region approximated by the system.

Lorenz observed that the system behaves chaotically whenever $\sigma = 10$, $\mu = 8/3$ and $r > 24.74$. Figures 1.1 (a) show time-series $x(t)$, $y(t)$ and $z(t)$ and Figures 1.1 (b)-(e) show different phase portraits for the case $\sigma = 10$, $\mu = 8/3$ and $r = 28$.

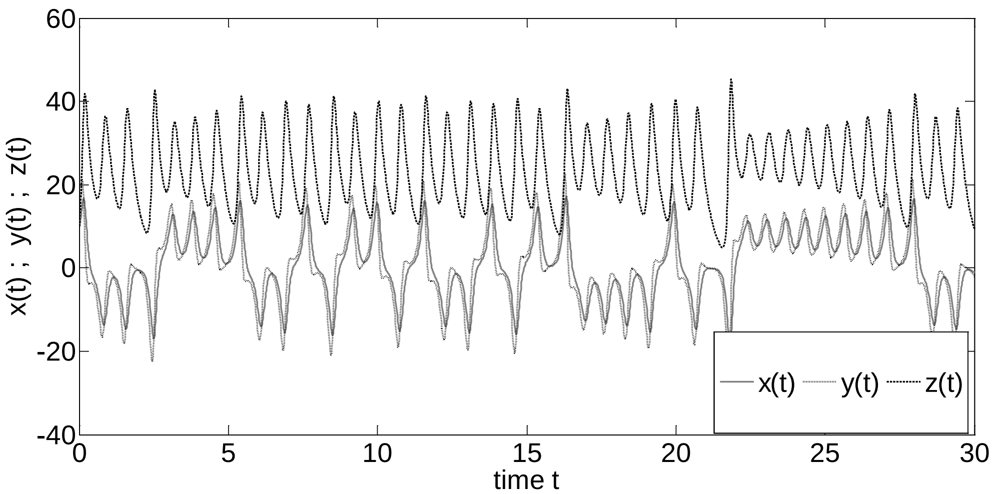

Figure 1.1 (a): Time series-Lorenz system

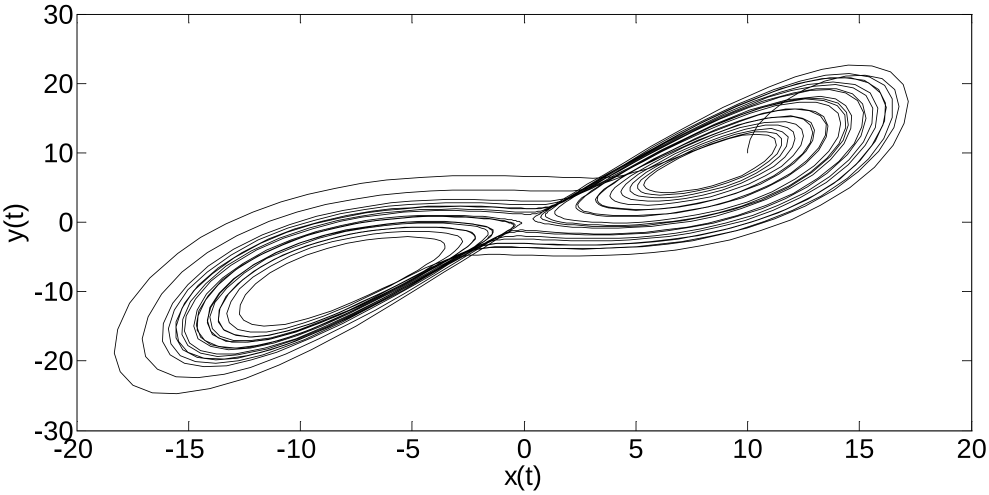

Figure 1.1 (b): Phase Portrait of Lorenz system in $x-y$ space.

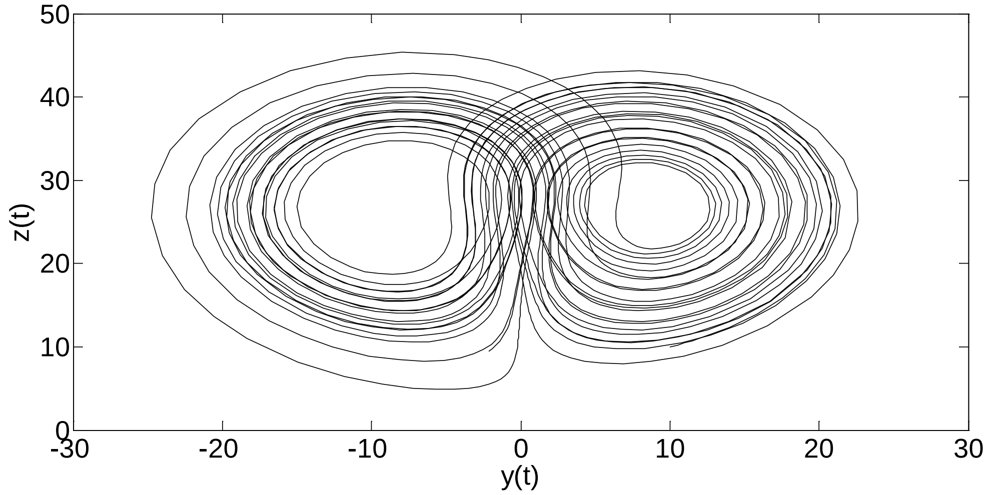

Figure 1.1 (c): Phase Portrait of Lorenz system in $y-z$ space.

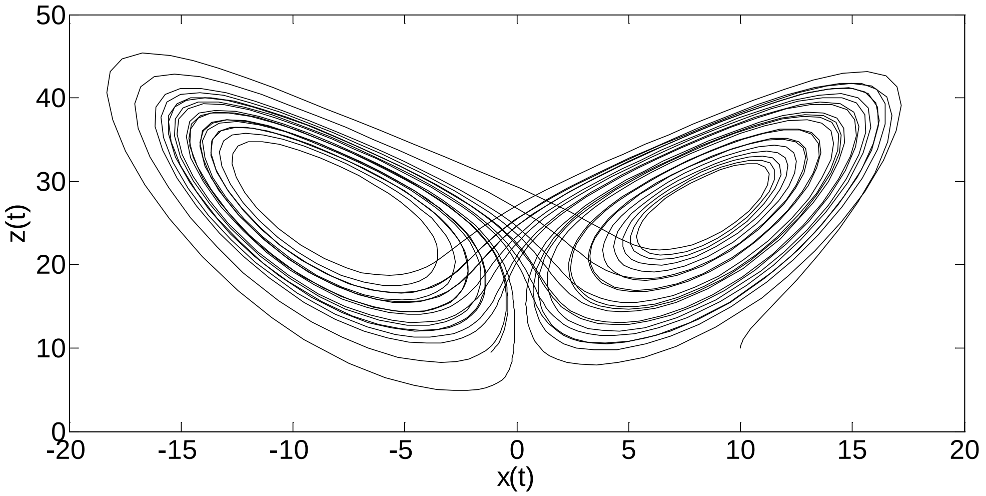

Figure 1.1 (d): Phase Portrait of Lorenz system in $x - z$ space.

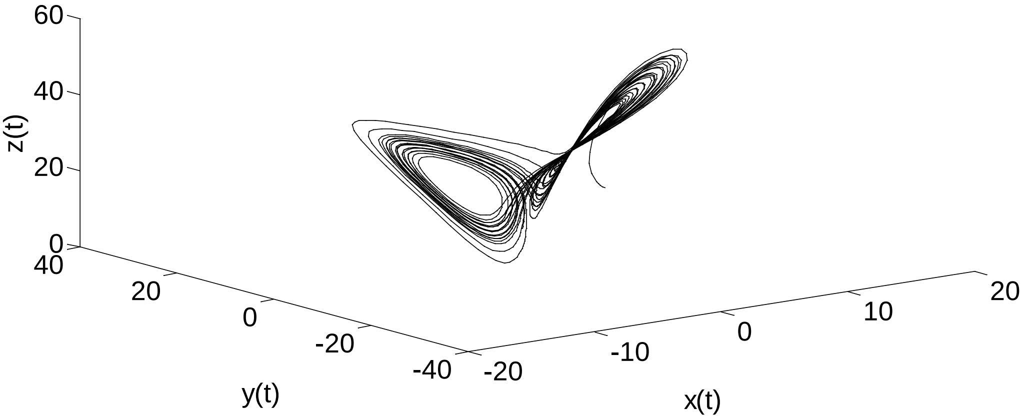

Figure 1.1 (e): Phase Portrait of Lorenz system in $x - y - z$ space.

Figure 1.1 Lorenz system

1.3.4 Chaos in fractional order systems

Chaotic fractional order dynamical systems are obtained by replacing the derivative in the system by fractional derivative. Fractional order in this case works as a chaos controller, i.e., the chaotic system can be made regular by appropriate choice of fractional derivative. It is observed that there is a critical value of the fractional order below which the system is regular and for the higher values chaotic. There is one more reason to study fractional chaotic systems that these are very useful in secure communications. Secure codes can be made using fractional chaotic systems which are difficult to break. Fractional order derivative acts as additional parameter which works as a key. An illustrative example is presented below.

Example 1.13 The fractional order Lorenz system (Luo and Wang (2013b)) is given by

$$D^{q_1} x = 10(y - x) ,$$

$$D^{q_2} y = 28x - y - xz ,$$

$$D^{q_3} z = xy - \frac{8}{3} z , \tag{1.29}$$

where $0 < q_i \le 1 \ (i = 1 , 2 , 3)$. If $q_1 = q_2 = q_3$ then we call the system (1.29) as commensurate system otherwise incommensurate. Defining $\sum = q_1 + q_2 + q_3$ as a system order, it can be shown using numerical experiments that the chaos exists for the system order $\sum < 3$. System (1.29) is chaotic for the commensurate order $q_1 = q_2 = q_3 = 0.99$, which is shown in Figure 1.2 (a) and stable for $q_1 = q_2 = q_3 = 0.98$ shown through Figure 1.2 (b).

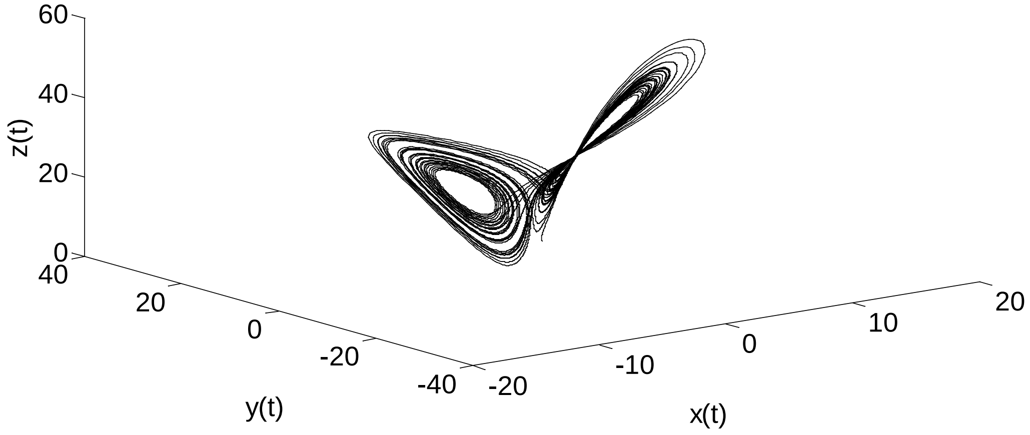

Figure 1.2 (a)

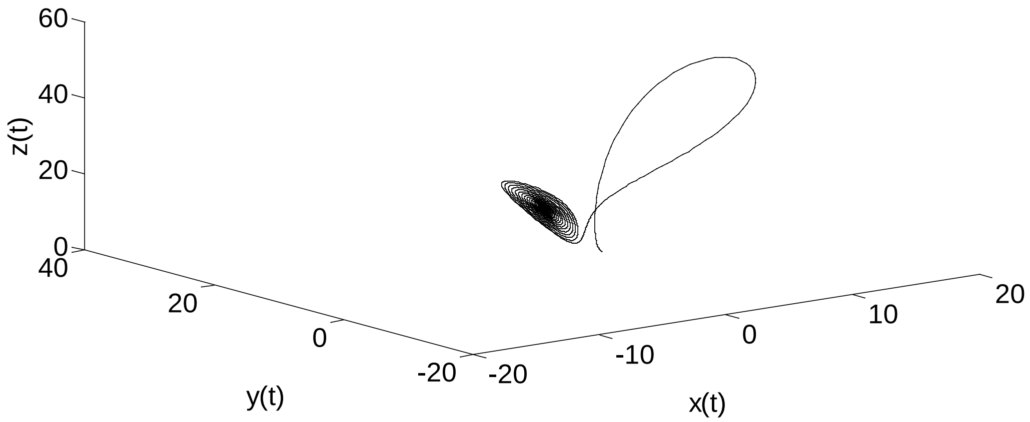

Figure 1.2 (b)

Figure 1.2: Phase portrait of the fractional order Lorenz system at the derivative order **(a)** $q_1 = q_2 = q_3 = 0.99$; **(b)** $q_1 = q_2 = q_3 = 0.98$

1.3.5 Asymptotic stability of commensurate fractional order system

Consider the following commensurate fractional ordered system

$$D^q x_i = f_i(x_1, x_2, x_3), \qquad 1 \le i \le 3. \tag{1.30}$$

Let $p \equiv (x_1^*, x_2^*, x_3^*)$ be an equilibrium point of the system (1.30), i.e. $f_i(p) = 0$, $1 \le i \le 3$ and $\xi_i = x_i - x_i^*$ a small disturbance from a fixed point. Then

$$\begin{aligned} D^q \xi_i &= D^q x_i, \\ &= f_i(x_1, x_2, x_3) = f_i(\xi_1 + x_1^*, \xi_2 + x_2^*, \xi_3 + x_3^*) \\ &= f_i(x_1^*, x_2^*, x_3^*) + \xi_1 \frac{\partial f_i(p)}{\partial x_1} + \xi_2 \frac{\partial f_i(p)}{\partial x_2} + \xi_3 \frac{\partial f_i(p)}{\partial x_3} + \text{higher ordered terms} \\ &\approx \xi_1 \frac{\partial f_i(p)}{\partial x_1} + \xi_2 \frac{\partial f_i(p)}{\partial x_2} + \xi_3 \frac{\partial f_i(p)}{\partial x_3}. \end{aligned} \tag{1.31}$$

System (1.31) can be written as

$$D^q \xi = J\xi, \tag{1.32}$$

where $\xi = [\xi_1, \xi_2, \xi_3]^T$ and

$$J = \begin{bmatrix} \frac{\partial f_1(p)}{\partial x_1} & \frac{\partial f_1(p)}{\partial x_2} & \frac{\partial f_1(p)}{\partial x_3} \\ \frac{\partial f_2(p)}{\partial x_1} & \frac{\partial f_2(p)}{\partial x_2} & \frac{\partial f_2(p)}{\partial x_3} \\ \frac{\partial f_3(p)}{\partial x_1} & \frac{\partial f_3(p)}{\partial x_2} & \frac{\partial f_3(p)}{\partial x_3} \end{bmatrix}. \tag{1.33}$$

The linear autonomous system

$$D^q \xi = J\xi, \qquad \xi(0) = \xi_0, \tag{1.34}$$

where J is $n \times n$ matrix and $0 < q < 1$ is asymptotically stable if and only if $\left|\arg(\lambda)\right| > q\pi/2$ for all eigenvalues λ of J. In this case, each component of solution $\xi(t)$ decays towards zero like t^{-q}.

This shows that if $\left|\arg(\lambda)\right| > q\pi/2$ for all eigenvalues λ of J, then the solution $\xi_i(t)$ of the system (1.32) tends to zero as t becomes large. Thus the equilibrium point p of the system is asymptotically stable if $\left|\arg(\lambda)\right| > q\pi/2$, for all eigenvalues λ of J i.e., if $\min_i \left|\arg(\lambda_i)\right| > q\pi/2$.

1.4 Synchronization

Fujisaka and Yamada (1983) paved the way with their pioneering studies on chaos synchronization, but it was not until 1990 when Pecora and Carroll (1990) introduced their method of chaotic synchronization and suggested application to secure communications that the subject received considerable attention within the scientific community. L. M. Pecora and T. L. Carroll wrote that: "Chaotic systems would seem to be dynamical systems that defy synchronization. Two identical autonomous chaotic systems started at nearly the same initial points-in phase space have trajectories which quickly become uncorrelated, even though each maps out the same attractor in phase space. It is thus practically impossible to construct, identical, chaotic, synchronized system in laboratory".

In chaos synchronization, two or more chaotic systems are coupled, or one chaotic system drives another system. The research article of Pecora and Carrol (1990) was the first to introduce a method to synchronize drive and response systems of two identical or non-identical systems with different initial conditions.

It might seem that the synchronization of chaotic system is difficult to achieve due to their extremely sensitive dependence on initial conditions. The synchronization scenario has been of long standing interest and studied extensively. Synchronization of chaotic systems is defined as a process wherein two or many chaotic systems, either identical or non-identical, adjust a given property of their motion to a common behaviour due to coupling or forcing.

Synchronization of the complex chaotic systems has more applications in secure communication due to special features of chaotic systems, such as extremely sensitive to initial conditions and system parameters, double the number of variables and complex dynamical behaviours with strong unpredictability.

1.4.1 Types of synchronization

Inspired by the seminal works of Fujisaka and Yamada (1983) and of Pecora and Carroll (1990) on synchronization of chaotic systems, various types of synchronization scenario have been investigated, viz., complete synchronization, anti-synchronization, phase synchronization, hybrid synchronization, lag synchronization, generalised synchronization, projective synchronization, function projective synchronization, dual synchronization, combination synchronization, dual combination synchronization, compound synchronization, etc. These different types of synchronization are described in details in following sub-sections.

1.4.1.1 Complete synchronization

Complete synchronization appears as the equality of the state variables while evolving in time. In complete synchronization the chaotic trajectories of the coupled systems remain in step with each other in course of time. This is observed in coupled chaotic systems with identical elements, i.e., each component having the same dynamics and parameter set. This kind of synchronization was first described by Pacora and Carroll (1990).

Consider two continuous time chaotic systems as

$$\dot{x}(t) = f\left(x(t)\right), \tag{1.35}$$

$$\dot{y}(t) = g\left(y(t)\right) + u\left(x(t)\ ,\ y(t)\right), \tag{1.36}$$

where $x(t)$, $y(t) \in R^n$ are the state vectors of drive system (1.35) and response system (1.36) respectively, f , g : $R^n \to R^n$ is continuous nonlinear functions, and $u\left(x(t)\ ,\ y(t)\right)$ is the control function.

The systems (1.35) and (1.36) are said to be synchronized when $\lim_{t \to \infty} \left\| y(t) - x(t) \right\| = 0$ for initial conditions $x(0)$ and $y(0)$.

Complete synchronization is also referred as convention synchronization, or identical synchronization, or simply synchronization.

1.4.1.2 Anti-synchronization

Two chaotic systems (1.35) and (1.36) are said to be anti-synchronized, when the respective states of chaotic systems $x(t)$ and $y(t)$ have the same magnitude but opposite in sign. Mathematically, anti-synchronization is achieved when $\lim_{t \to \infty} \left\| y(t) + x(t) \right\| = 0$.

1.4.1.3 Hybrid synchronization

Hybrid synchronization is an attractive case where three states are defined in such a way that first and third states of the two systems are complete synchronized, and the second state of the systems are anti-synchronized. So in hybrid synchronization, synchronization and anti- synchronization co-exist in the systems.

1.4.1.4 Generalized synchronization

Coupled chaotic systems are said to exhibit generalized synchronization if there exits some function relation between systems, i.e., $y(t) = \varphi(x(t))$, which means that the states of the two interacting systems are functionally synchronized. This type of synchronization occurs mainly when the coupled chaotic systems are different.

1.4.1.5 Anticipating and lag synchronization

In these cases, the synchronized state is characterised by a time interval τ such that the dynamical variables of the chaotic systems are related by $y(t) = x(t+\tau)$. This means that the dynamics of one of the systems follows and anticipates the dynamics of the other. These types of synchronization may occur in time-delayed chaotic systems, coupled in a drive-response configuration. In case of anticipating synchronization, the response anticipates the dynamics of the drive. In lag synchronization, $\tau < 0$ appears as the asymptotic boundedness of the difference between the output of one system at time t and the output of the other shifted in time of a lag time. In particular, if the time delay may becomes zero i.e., $\tau = 0$, the anticipating synchronization and lag synchronization are further simplified to complete synchronization.

1.4.1.6 Phase synchronization

This scenario of the synchronization occurs when the coupled chaotic systems keep their phase difference bounded by a constant while their amplitudes remain uncorrelated. This phenomenon is mostly achieved in coupled non-identical systems. In case of phase synchronization, if $\varphi_1(t)$ and $\varphi_2(t)$ denote the phases of the two coupled chaotic systems, synchronization of the phase is described by the relation $n\varphi_1(t) = m\varphi_2(t)$, with m and n whole numbers.

1.4.1.7 Projective synchronization

Mainieri and Rehacek (1999) introduced the projective synchronization in partially linear systems, where the responses of two identical systems synchronize up to a constant scaling factor.

Consider the drive system as (1.35) and response system as (1.36). Defining the error state as $e(t) = y(t) - \lambda x(t)$, where λ is the real constant, the systems (1.35) and (1.36) are said to be projective synchronized, if $\lim_{t\to\infty}\|e(t)\| = 0$.

In particular, if $\lambda = 1$ and $\lambda = -1$, the projective synchronization is further simplified to complete synchronization and anti-phase synchronization respectively.

1.4.1.8 Function projective synchronization

In function projective synchronization, drive system is synchronized with response system up to a desired scaling function. It is introduced by Chen and Li (2007). Defining the error state as $e(t) = y(t) - \lambda(t)x(t)$, where $\lambda(t)$ is the continuously differentiable function with $\lambda(t) \neq 0$, $\forall t$, the systems (1.35) and (1.36) are said to be function projective synchronized if there exists a scaling function $\lambda(t)$ such that $\lim_{t\to\infty}\|e(t)\| = 0$.

1.4.1.9 Modified projective synchronization

Modified projective was proposed in Li (2007). Defining the error state as $e(t) = y(t) - A\,x(t)$, where $A = diag[a_1 \ , a_2 \ , \cdots , a_n]$ is the scaling constant matrix such that a_i's are constatnt scaling factors $\forall\, i \in N$, the systems (1.35) and (1.36) are said to be modified projective synchronized, if there exists a constant matrix A such that $\lim_{t\to\infty}\|e(t)\| = 0$.

1.4.1.10 Modified function projective synchronization

Modified function projective synchronization is more general than function projective synchronization and modified projective synchronization. If we define the error state as $e(t) = y(t) - A(t)\,x(t)$ between the drive system (1.35) and response system (1.36), where $A(t) = diag[a_1(t) \ , a_2(t) \ , \cdots , a_n(t)]$ is the function scaling matrix such that $a_i(t) \neq 0$, $\forall\, i \in N$ are continuously differentiable function. Systems (1.35) and (1.36) are said to be modified function projective synchronized if there exists a function scaling matrix $A(t)$ such that $\lim_{t\to\infty}\|e(t)\| = 0$.

1.4.1.11 Dual synchronization

In the case of dual combination synchronization the first two drive systems are taken as

Drive systems-I:

$$\dot{X} = F(X) \,, \tag{1.37}$$

where X is state vector variable.

Drive system-II:

$$\dot{Y} = G(Y) \,, \tag{1.38}$$

where Y is state vector variable.

The linear combination of the drive systems I & II gives rise to

$$\begin{aligned} V_m &= \sum_{i=1}^{n} a_i X_i + \sum_{i=1}^{n} b_i Y_i \\ &= [a_1 ,\ a_2 , \cdots,\ a_n]X + [b_1 , b_2 , \cdots, b_n]Y \\ &= A^T X + B^T Y \\ &= [A^T\ B^T]\begin{bmatrix} X \\ Y \end{bmatrix} = C^T \xi, \end{aligned}$$

where $A = [a_1 ,\ a_2 , \cdots,\ a_n]^T$ and $B = [b_1 , b_2 , \cdots, b_n]^T$ are known and $C = [A^T\ \ B^T]^T$.

Next two response systems are considered as

Response system-I:

$$\dot{x} = f(x) + u^{(1)} \,, \tag{1.39}$$

where x is state vector variable.

Response system-II:

$$\dot{y} = g(y) + u^{(2)} \,, \tag{1.40}$$

where y is state vector variable and $u^{(1)}(t)$, $u^{(2)}(t)$ are control functions, such that $u^{(i)}(t) = [u_1^{(i)} ,\ u_2^{(i)} , \cdots ,u_n^{(i)}]^T , \quad i = 1,\ 2$.

The linear combination of the response system I & II gives rise to

$$V_s = \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i y_i$$

$$= [a_1, a_2, \cdots, a_n]x + [b_1, b_2, \cdots, b_n]y$$

$$= A^T x + B^T y$$

$$= [A^T \; B^T] \begin{bmatrix} X_m \\ Y_m \end{bmatrix} = C^T \eta.$$

The goal is to obtain the dual synchronization among drive and response systems. Let us define the error function among the drive systems (1.37), (1.38) and response systems (1.39), (1.40) as

$$e = V_s - V_m \, ,$$

The drive systems (1.37), (1.38) and response systems (1.39), (1.40) are said to be dual synchronized if $\lim_{t\to\infty} \|e\| = 0$, where $\|\cdot\|$ denotes matrix norm.

1.4.1.12 Combination synchronization

The drive systems are considered as

$$\dot{x}_1 = f_1(x_1) \tag{1.41}$$

$$\dot{x}_2 = f_2(x_2) \tag{1.42}$$

and the response system is taken as

$$\dot{y} = f(y) + U(x_1, x_2, y), \tag{1.43}$$

where $x_1 = [x_1^1, x_2^1, \cdots, x_n^1]^T$, $x_2 = [x_1^2, x_2^2, \cdots, x_n^2]^T$ and $y = [y_1, y_2, \cdots, y_n]^T$ are the state vectors of the chaotic systems. f_1, f_2, $f : R^n \to R^n$ are continuous vector functions and $U(x_1, x_2, y)$ is a control function.

Two drive systems (1.41), (1.42) and one response system (1.43) are said to be combination synchronized, if there exists three constants matrices called scaling matrices, A_1, A_2, A_3 and $A_3 \neq 0$, such that $\lim_{t\to\infty}\|A_1x_1 + A_2x_2 - A_3y\| = 0$, where $\| \,.\, \|$ represents the matrix norm.

It is noted that if $A_1 \neq 0$, $A_2 = 0$, $A_n = I$ then this problem is reduced to the projective synchronization, where I is an $n \times n$ identity matrix. If the scaling matrix A_1 is considered as a function, then synchronization problem is reduced into function projective synchronization problem.

1.4.1.13 Dual combination synchronization

In this section the dual combination synchronization is proposed among four drive and two response systems. First two drive systems are defined by systems (1.37) and (1.38).

Next two drive systems are defined as

Drive systems-III:

$$\dot{X}' = f(X')\,, \tag{1.44}$$

where X' is state vector variable.

Drive system-IV:

$$\dot{Y}' = g(Y')\,, \tag{1.45}$$

where Y' is state vector variable.

The linear combination of the drive systems III & IV, gives rise to

$$V_m' = \sum_{i=1}^{n} a_i X_i' + \sum_{i=1}^{n} b_i Y_i'$$

$$= [a_1, a_2, ...a_n]X' + [b_1, b_2, ...b_n]Y'$$

$$= A^T X + B^T Y$$

$$= [A^T \; B^T] \begin{bmatrix} X' \\ Y' \end{bmatrix} = C^T \xi'.$$

Now, the corresponding two response systems with control functions are described by systems (1.39) and (1.40).

Now defining the error function among four drive systems (1.37), (1.38), (1.44), (1.45) and response systems (1.39), (1.40) as $e = V_s - V_m - V_m'$.

The drive systems (1.37), (1.38), (1.44) (1.45), and the response systems (1.39), (1.40) are said to be dual combination synchronized if $\lim_{t \to \infty} \|e\| = 0$, where $\|\cdot\|$ denotes matrix norm.

In the present thesis during synchronization of identical / non-identical chaotic systems, complete synchronization and dual combination synchronization are studied.

1.5 Methods for synchronization

The most effectively and widely studied approach is due to the discovery of Pecora and Carroll scheme proposed by L.M. Pecora and T. L. Carroll in the year 1990, where two identical chaotic systems with different initial conditions are synchronized. They have theoretically proven and experimentally demonstrated that it is possible to synchronize chaotic systems by introducing appropriate couplings between the systems. During last few years, Active control method, Adaptive control method, Tracking control method, Nonlinear control method, Backstepping method, Pecora and Carroll method, etc are used during synchronization of chaotic systems.

1.5.1 Active control method

The active control method was first proposed by E. W. Bai and K. E. Lonngren in 1997, and synchronizes the identical Lorenz chaotic system using active control method. After this, in 2000, Bai and Lonngren showed the sequential synchronization of two Lorenz systems using this method. In 2000, the active control method successfully applied for synchronization of two different chaotic systems viz., easy periodic system and Rossler system by Ho and Hung. In 2007, Li and Yan investigated chaos synchronization of fractional order Lorenz, Rossler and Chen systems taking one system as drive and other as response system. In 2008, Vincent presented chaos synchronization between two nonlinear systems using two different techniques viz., active control and back stepping control in terms of transient analysis. In the same year, Zhou and Cheng showed synchronization between different fractional order chaotic systems viz., Rossler & Chen systems and Chua & Chen systems. Recently, Srivastava et. al. (2014) have successfully applied the active control method for anti-synchronization between identical and non-identical fractional order chaotic systems. The active control method has received huge attention during the last few years.

1.5.2 Nonlinear control method

In 2005, J. H. Park studied the chaos synchronization of chaotic systems via nonlinear control method. Dong et al. studied synchronization of the hyperchaotic Rossler system with uncertain parameter using the same method in the year 2006. In 2009, Xin proposed the projective synchronization using this method. The method was successfully used by Li and Ge (2011) during the study of pragmatically adaptive synchronization of different orders chaotic systems with uncertain parameters and also by Singh et al. (2014) during synchronization and ant-synchronization of chaotic systems.

To process the method for synchronization, first consider the fractional order chaotic system as the drive system as

$$\dot{x}_i = P\,x_i + Q\,f(x_i)\,,\;\; 0 < q \le 1\;,\;\; i = 1,\,2,\,\cdots,\,n\,, \tag{1.46}$$

where $x_i = [x_1, x_2, \cdots, x_n]^T \in R^n$ is the state vector variable, P and Q are $n \times n$ matrices of the system parameters and $f : R^n \to R^n$ is a nonlinear function of the system.

Consider another fractional order chaotic system as a response system as

$$\dot{y}_i = P_1\,y_i + Q_1\,g(y_i) + u_i\;,\quad i = 1,\,2,\,\cdots,\;n\;, \tag{1.47}$$

where $y_i = [y_1,\, y_2,\, \cdots,\, y_n]^T \in R^n$, is the state vector of the system, P_1 and Q_1 are $n \times n$ matrices of the system parameters, $g : R^n \to R^n$ is a nonlinear function of the system and u_i are the control function of the system.

Defining the error states as $e_i = y_i - x_i$, $i = 1,\,2,\,\cdots,\,n$, the error system becomes

$$\dot{e}_i = P_1\,e_i + Q_1\,g(y_i) + (P_1 - P)\,x_i - Q\,f(x_i) + u_i\;. \tag{1.48}$$

During the synchronization the aim is to find the appropriate feedback controller u_i so that the dynamical error system (1.48) can be stabilized in order to get $\lim_{t\to\infty}\|e(t)\| = 0$ for all $e(0) \in R^n$.

Now defining the Lyapunov function as

$$V = \frac{1}{2} e_i^T e_i \ ,$$

with $e_i(t) = [e_1(t), e_2(t), \cdots, e_n(t)]^T$.

The derivative of $V(t)$ w. r. to t is

$$\frac{dV}{dt} = \frac{1}{2}\frac{d(e_i^T e_i)}{dt}$$

$$= \frac{1}{2}\frac{d}{dt}(e_1^2 + e_2^2 + \ldots\ldots + e_n^2)$$

$$= \sum_{i=1}^{n} \frac{1}{2}\frac{de_i^2}{dt}$$

$$= \sum_{i=1}^{n} e_i \frac{de_i}{dt}$$

Choosing the control functions as $u_i = -(P_1 + 1)e_i - (P_1 - P)x_i - Q_1\, g(y_i) + Q\, f(x_i)$, we get

$$\frac{dV}{dt} = -\sum_{i=1}^{n} e_i^2 \ , \tag{1.49}$$

which shows that the Lyapunov function $V(t)$ becomes negative definite, so as to get the required synchronization of the systems (1.46) and (1.47).

1.5.3 Backstepping method

The backstepping design is a recursive procedure which combines choice of Lyapunov function with the design of feedback control functions. There are many advantages of the method as it is a systematic procedure for controlling chaotic dynamic. It can be applied over circuits and systems. In 1999, Mascolo and Grassi have controlled chaotic dynamic using backstepping design during its applications in the Lorenz system and Chua's circuit. In 2001, Wang and Ge proposed the adaptive synchronization of uncertain chaotic systems via backstepping design. In the same year, Lu and Zhang controlled the Chen's chaotic attractors using backstepping design based on parameters identification. In the year 2003, Tan et al. synchronized the chaotic systems using backstepping design, while Yu and Zhang controlled the uncertain behaviour of chaotic systems using backstepping design. Recently, Park (2006) and Wu et al. (2009) have shown that the backstepping method is very simple, reliable and powerful for controlling the chaotic behaviour and synchronization of chaotic systems. The detail of backstepping method is given in chapter 4.

1.6 Delay differential equations

The initial value problem (IVP) given by

$$\dot{y}(t) = f\left(t,\, y(t)\right),$$

$$y(0) = y_0 \,, \tag{1.50}$$

is useful in modelling many physical phenomena. The term $y(t)$ is called state variable which represents some physical quantities those evolve with time t. However, in some situations the derivative term $\dot{y}(t)$ depends not only on the state $y(t)$ but also on some past values $y(t-\tau)$. The term τ is called a delay or a lag. Time-delay arises because finite time is required to sense information and then react to it. Corresponding differential equation is thus takes the form $\dot{y}(t) = f\big(t,\, y(t),\, y(t-\tau)\big)$, called a delay differential equation (DDE).

The solution of this equation is not uniquely determined by its initial state at a given moment, but instead the solution profile on an interval with length equal to the delay τ has to be given. Since an infinite dimensional set of initial conditions between $-\tau$ and 0 is required, therefore such problems are infinite dimensional in nature and difficult to solve.

From the literature survey it can be found that there are lot of applications of DDE in the following areas.

Population dynamics: The well-known logistic equation given by Pierre Verhulst

$$\dot{N}(t) = r\,N(t)\left(1 - \frac{N(t)}{K}\right) \tag{1.51}$$

describes the growth of population, where $N(t)$ is population at time t and $r > 0$ is Malthusian parameter describing growth rate, and K is carrying capacity. The model assumes that the population density negatively affects the per capita growth rate due to environmental degradation. Hutchinson (1948) introduced a delay into the logistic equation to account for hatching and maturation periods, which is given by

$$\dot{N}(t) = r\,N(t)\left(1 - \frac{N(t-\tau)}{K}\right)\ . \tag{1.52}$$

Virology: Culshaw and Ruan (2000) proposed a HIV model to include a time delay between virus-cell contact and subsequent infection of the $CD4+T-$cell. The model is

$$\dot{T}(t) = s - \mu_T T(t) + rT(t)\left(1 - \frac{T(t)+I(t)}{T_{\max}}\right) - k_1 T(t)V(t) \ ,$$

$$\dot{I}(t) = k_1' T(t-\tau)V(t-\tau) - \mu_1 I(t) \ ,$$

$$\dot{V}(t) = N\mu_b I(t) - k_1 T(t)V(t) - \mu_V V(t) \ , \qquad (1.53)$$

where T denotes healthy $T-$cells in the blood, I is the HIV infected $T-$cells and V is the HIV virus level in the blood.

Nonlinear optics: Ikeda et al. (1980) considered a nonlinear absorbing medium containing two-level atoms placed in a ring cavity and subject to a constant input of light. The optical system undergoes a time-delayed feedback that destabilizes its steady-state output. The DDE formula is

$$\tau\dot{\phi}(t) = -\phi(t) + A^2\left[1 + 2B\cos\left(\phi(t-t_D) - \phi_0\right)\right]. \qquad (1.54)$$

1.6.1 Existence and uniqueness of a solution: Method of steps

Theorem 1.14 Let $f(t, x, y)$ and $f_x(t, x, y)$ be continuous on R^3 and let $\phi : [-\tau, 0] \to R$ be continuous. Then there exists $\sigma > 0$ and a unique solution of the IVP

$$\dot{x}(t) = f\left(t, x(t), x(t-\tau)\right), \qquad t \ge 0 \ ,$$

$$x(t) = \phi(t) \ , \qquad -\tau \le t \le 0 \qquad (1.55)$$

on $[-\tau, \sigma]$.

One need a solution $u(t)$ such that $u(t)=\phi(t)$, $-\tau \le t \le 0$ and satisfying (1.55) for $t \ge 0$. For $0 \le t \le \tau$, the function $u(t)$ must satisfy the IVP

$$\dot{x}(t) = f\big(t,\, x(t),\, \phi(t-\tau)\big) = g\big(t,\, x(t)\big),$$

$$x(0) = \phi(0)\,. \tag{1.56}$$

Since $g\big(t,\, x(t)\big) = f\big(t,\, x(t),\, \phi(t-\tau)\big)$ and $g_x\big(t,\, x(t)\big)$ are continuous, the local solution of the ODE (1.56) is guaranteed by standard results from ODE theory. If this local solution exists for the entire interval $[0,\, \tau]$ then the solution $u(t)$ is defined so far on $[-\tau,\, \tau]$ and one may repeat the above argument to extend their solution still further to the right.

1.6.2 Numerical solution of a DDE

Consider the DDE, given by (1.55) which is equivalent to

$$x(t) = x(0) + \int_0^t f\big(z,\, x(z),\, x(z-\tau)\big)dz\,. \tag{1.57}$$

Assign the step-size $h = \tau / k$, for some fixed natural number k so that $h \le \tau$. Consider the nodes $t_n = nh$, $n = 0, 1, 2, \cdots$ and denote $x_n = x(t_n)$. For $0 \le n \le k$, the term $x(t_n - \tau) =$ $x(nh - kh) =$ $x(-(k-n)h) =$ $\phi(-(k-n)h)$. For $n > k$, $x(t_n - \tau) =$ $x((n-k)h) = x_{n-k}$. Assume that we have obtained the values $x_1,\, x_2,\, \cdots,\, x_{n-1}$. For $n-$th step, can be written as

$$x_n = x_{n-1} + \int_{t_{n-1}}^{t_n} f\big(z,\, x(z),\, x(z-\tau)\big)dz\,, \tag{1.58}$$

using some suitable integration formulae.

1.6.3 Fractional delay differential equations

Many processes in nature possess memory and hereditary properties which cannot be modeled using ordinary derivative. There are two ways to include memory in the model. One is described in preceding section, called the "delay" and another is "fractional derivative (FD)". Due to its non-local nature, FD is having a long memory. If these two things unified, the resulting model may be robust. In this section, some results are studied on fractional order delay differential equations (FDDEs).

1.6.3.1 Existence and uniqueness of solution

Consider an IVP

$$D^q x(t) = A_0 x(t) - A_1 x(t-\tau) + f(t) , \qquad t \geq 0 ,$$

$$x(t) = \phi(t) , \quad t \in [-\tau , 0] , \tag{1.59}$$

where $0 < q \leq 1$, ϕ is a given continuous function on $[-\tau , 0]$, A_0 and A_1 are constant system matrices of appropriate dimensions. This system is defined on the interval $[0, T]$, $T > 0$ and $f(t)$ is a given continuous function on $[0, T]$.

Lemma 1.15 The IVP (1.59) has a continuous solution on $[-\tau , T]$ if and only if the following system

$$x(t) = A_1 \sum_{k=1}^{\infty} A_0^{k-1} I^{kq} x(t-\tau) + \sum_{k=1}^{\infty} A_0^{k-1} I^{kq} f(t) \tag{1.60}$$

has a continuous solution on $[0, T]$ which satisfies the same initial condition.

Theorem 1.16 Let ϕ be a given continuous function, fix $T>0$ and assume $f(t)$ continuous function from $[0,T]$ to R^n then there exists a unique solution $x(\phi, f)$ of IVP (1.59) defined on $[0,T]$ which coincides with ϕ on $[-\tau, 0]$ and satisfies (1.59) for $t\in[0,T]$.

Proof: We use the method of steps to prove the existence and uniqueness of solution of equivalent system (1.60). Let $x_i(t)$ be solution in the interval $[(i-1)\tau, i\tau]$ which is given by following procedure.

For $t\in[0,\tau]$, we have

$$x_1(t) = A_1\sum_{k=1}^{\infty} A_0^{k-1} I^{kq}\phi(t-\tau) + \sum_{k=1}^{\infty} A_0^{k-1} I^{kq} f(t)$$

For $t\in[\tau, 2\tau]$, we have

$$x_1(t) = A_1\sum_{k=1}^{\infty} A_0^{k-1}\frac{1}{\Gamma(kq)}\int_0^{\tau}(t-s)^{kq-1}\phi(s-\tau)\,ds$$

$$+ A_1\sum_{k=1}^{\infty} A_0^{k-1}\frac{1}{\Gamma(kq)}\int_{\tau}^{t}(t-s)^{kq-1}x_1(s-\tau)\,ds + \sum_{k=1}^{\infty} A_0^{k-1} I^{kq} f(t)$$

In general, for $t\in[(n-1)\tau, n\tau]$, we have

$$x_1(t) = A_1\sum_{k=1}^{\infty} A_0^{k-1}\frac{1}{\Gamma(kq)}\int_0^{\tau}(t-s)^{kq-1}\phi(s-\tau)\,ds$$

$$+ A_1\sum_{k=1}^{\infty} A_0^{k-1}\frac{1}{\Gamma(kq)}\int_{\tau}^{2\tau}(t-s)^{kq-1}x_1(s-\tau)\,ds$$

$$+\cdots$$

$$+ A_1\sum_{k=1}^{\infty} A_0^{k-1}\frac{1}{\Gamma(kq)}\int_{(n-1)\tau}^{t}(t-s)^{kq-1}x_{n-1}(s-\tau)\,ds + \sum_{k=1}^{\infty} A_0^{k-1} I^{kq} f(t)\,.$$

Thus the existence and uniqueness of solution of IVP (1.59) is proved.

1.6.3.2 Stability result

Deng et al. (2007) have studied the stability of $n-$dimensional linear fractional order differential equation with time delays for the following system:

$$D^{q_1}x_1(t) = a_{11}x_1(t-\tau_{11}) + a_{12}x_2(t-\tau_{12}) + \cdots + a_{1n}x_n(t-\tau_{1n}) \ ,$$

$$D^{q_2}x_2(t) = a_{21}x_1(t-\tau_{21}) + a_{22}x_2(t-\tau_{22}) + \cdots + a_{2n}x_n(t-\tau_{2n}) \ ,$$

$$\vdots$$

$$D^{q_n}x_n(t) = a_{n1}x_1(t-\tau_{n1}) + a_{n2}x_2(t-\tau_{n2}) + \cdots + a_{1n}x_n(t-\tau_{nn}) \ , \qquad (1.61)$$

where $q_i \in (0, 1)$, $x_i(t)$, $x_i(t-\tau_{ij})$ state variables and τ_{ij} are delay terms.

It is shown that if all the roots of the characteristic equation $\det(\Delta(\lambda)) = 0$, where

$$\Delta(\lambda) = \begin{bmatrix} \lambda^{q_1} - a_{11}e^{-\lambda\tau_{11}} & -a_{12}e^{-\lambda\tau_{12}} & \cdots & -a_{1n}e^{-\lambda\tau_{1n}} \\ -a_{21}e^{-\lambda\tau_{21}} & \lambda^{q_2} - a_{22}e^{-\lambda\tau_{22}} & \cdots & -a_{2n}e^{-\lambda\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1}e^{-\lambda\tau_{n1}} & -a_{n2}e^{-\lambda\tau_{n2}} & \cdots & \lambda^{q_n} - a_{nn}e^{-\lambda\tau_{nn}} \end{bmatrix},$$

have negative real parts, then the system (1.61) is asymptotically stable.

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