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A robust numerical method for a two-parameter singularly perturbed time delay parabolic problem

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Abstract

In this article, we consider a class of singularly perturbed two-parameter parabolic partial differential equations with time delay on a rectangular domain. The solution bounds are derived by asymptotic analysis of the problem. We construct a numerical method using a hybrid monotone finite difference scheme on a rectangular mesh which is a product of uniform mesh in time and a layer-adapted Shishkin mesh in space. The error analysis is given for the proposed numerical method using truncation error and barrier function approach, and it is shown to be almost second- and first-order convergent in space and time variables, respectively, independent of both the perturbation parameters. At the end, we present some numerical results in support of the theory.

Keywords Singular perturbation · Delay differential equation · Shishkin mesh · Hybrid scheme · Uniform convergence

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1 Introduction

Singularly perturbed delay differential equations often arise in modeling of various physical, biological and chemical systems such as in population dynamics, variational problems in control theory, epidemiology, circadian rhythms, respiratory system, chemostat models, tumor growth and neural networks. The delay terms in these models enable us to include some past behavior to get more practical models for the phenomena. For example, in population ecology, time delay represents the hatching period or duration of gestation; in genetic repression modeling, time delays play an important role in processes of transcription and translation as well as spatial diffusion of reactants and in control systems, delay terms account for the time delay in actuation and in information transmission and processing. Many other examples can be found in Wu (2012).

In this paper, we consider a singularly perturbed delay initial-boundary value problem in one space dimension with two small parameters. Defining $\bar{G} = G \cup \partial G$, where $G = (0, 1) \times (0, T]$ and $\partial G = \Gamma_b \cup \Gamma_r \cup \Gamma_l$ with $\Gamma_b = [0, 1] \times [-\tau, 0]$, $\Gamma_l = \{0\} \times (0, T]$, and $\Gamma_r = \{1\} \times (0, T]$, we consider

$$\begin{cases} Lu \equiv L_{\varepsilon,\mu}u - u_t = -cu(x, t - \tau) + f(x, t) & \text{in } G, \\ u\big|_{\Gamma_b} = \varphi_b(x, t), \\ u\big|_{\Gamma_l} = \varphi_l(t), \\ u\big|_{\Gamma_r} = \varphi_r(t), \end{cases} \tag{1.1}$$

where $L_{\varepsilon,\mu}u := \varepsilon u_{xx} + \mu a u_x - bu$ with parameters ε and μ such that $0 < \varepsilon \le 1,\ 0 \le \mu \le 1$. The coefficients are such that

$$0 < \alpha \le a(x, t),\quad 0 < \beta \le b(x, t),\quad 0 < \gamma \le c(x, t),\quad (x, t) \in \bar{G}.$$

Further, sufficient regularity and compatibility conditions are assumed on the data of problem (1.1) (cf. Ladyzhenskaya et al 1968). For the sake of simplicity we take $T = k\tau$ for some definite natural number k. Such problems demand uniformly convergent numerical methods, that is, methods that converge independently of singular perturbation parameters. Our main interest is in developing such a numerical method for problem (1.1).

The nature of singularly perturbed two-parameter problems changes according to the values of perturbation parameters ε and μ; from reaction–diffusion equation for $\mu = 0$ to convection–diffusion equation for $\mu = 1$. O'Malley studied such problems in ordinary differential equations asymptotically in O'malley (1967), O'malley (1969) and O'Malley (1967) and identified that the nature of these problems is quite affected by the choice of ratio of μ^2 to ε. Later some works have been done in the direction of development of uniformly convergent numerical methods, see Gracia et al. (2006), Shishkin and Titov (1976), Stynes and Tobiska (1998), Roos and Uzelac (2003), Patidar (2008), Brdar and Zarin (2016) and O'Riordan and Pickett (2019) for singularly perturbed two-parameter problems in ordinary differential equations and O'Riordan et al. (2006), Kadalbajoo and Yadaw (2012), Munyakazi (2015), Chandru et al. (2018) and Gupta et al. (2019) for singularly perturbed two-parameter problems in partial differential equations.

Singularly perturbed delay differential equations have attracted many researchers in recent years due to their widespread applications. Some uniformly convergent numerical methods for singularly perturbed delay differential equations have been developed in Erdogan and Cen (2018), Cen (2010), Kumar and Kumar (2014), Singh et al. (2018), Ansari et al. (2007), Bashier and Patidar (2011), Kaushik et al. (2010) and Kumar and Kumar (2017). Recently, in Govindarao et al. (2019), a first-order uniformly convergent method is given for problem

(1.1) using an upwind finite difference scheme on Shishkin type meshes. High-order numerical methods are very interesting as they provide good numerical approximations with low computational cost (for example, see Kumar and Kumar (2014) for time delay singularly perturbed reaction–diffusion problems, Dehghan (2004) for advection–diffusion equations, and Dehghan (2006) for two-dimensional time-dependent Schrodinger equation). However, we do not know of any high-order numerical method for problem (1.1). Thus, the aim of the paper is threefold: first, to derive a priori bounds on the solution derivatives of problem (1.1) and further provide a decomposition of the solution into smooth and layer components; second, to construct a hybrid finite difference scheme for the solution of problem (1.1); and third, to provide a uniform convergence analysis of the proposed hybrid finite difference scheme.

The outline of the paper is as follows. In Sect. 2, we derive a priori bounds on the solution derivatives of problem (1.1) and further provide a decomposition of the solution into smooth and layer components. In Sect. 3, we describe the construction of a layer adapted Shishkin mesh and the hybrid finite difference discretization of problem (1.1). Section 4 is concerned with uniform convergence analysis of the proposed method. In Sect. 5 some numerical results are presented in support of our theory. Finally, in Sect. 6, we provide conclusion of the paper.

Notation: We shall use C as the generic positive constant throughout the paper, which is independent of perturbation parameters ε and μ, and discretization parameters M and N. The maximum norm is denoted by $||.||_D$, where D is any bounded and closed subset of $[0,1]\times[0,T]$. When the domain has no particular significance, we simply use $||.||$. Similarly, we use $||.||_{D^{N,M}}$ to denote the discrete maximum norm. We also define $\eta = \min\limits_{\overline{G}} \frac{b}{a}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

2 Solution bounds for continuous problem

We start this section with a minimum principle for the differential operator L defined by $Lu = L_{\varepsilon,\mu}u - u_t$. The proof of this minimum principle can be done in a standard way. Using the minimum principle for the operator L, we shall derive derivative bounds for the solution which are required for convergence analysis of the proposed method.

Lemma 1 (Minimum principle) *If* $\psi\big|_{\partial G} \geq 0$ *and* $\left(L_{\varepsilon,\mu} - \frac{\partial}{\partial t}\right)\psi\big|_G \leq 0$, *then* $\psi\big|_{\bar{G}} \geq 0$.

Lemma 2 *The solution u of problem* (1.1) *satisfies*

$$||u|| \leq C. \tag{2.1}$$

Proof Consider the function r defined by $r(x,t) = u(x,t) - \varphi_b(x,0)$, $0 \leq t \leq T$, and $r(x,t) = u(x,t) - \varphi_b(x,t)$, $-\tau \leq t \leq 0$, which satisfies $L_{\varepsilon,\mu}r - r_t = s$ in G, where

$$\begin{aligned} s(x,t) &= -cu(x,t-\tau) + f - \left(L_{\varepsilon,\mu} - \frac{\partial}{\partial t}\right)\varphi_b(x,0) \\ &= f - \varepsilon(\varphi_b)_{xx}(x,0) - \mu a(\varphi_b)_x(x,0) + b\varphi_b(x,0) - cu(x,t-\tau) \end{aligned}$$

with $r\big|_{\Gamma_b} = 0$, $r\big|_{\Gamma_l} = \varphi_l(t) - \varphi_b(0,0)$, $r\big|_{\Gamma_r} = \varphi_r(t) - \varphi_b(1,0)$. Choosing C sufficiently large, set a barrier function $q(x,t) = Ct$, $t \in [0,T]$, and $q(x,t) = 0$, $t \in [-\tau, 0]$ satisfying the equation

$$\left(L_{\varepsilon,\mu} - \frac{\partial}{\partial t}\right)q = -C(1+bt)$$

with $q\big|_{\Gamma_b} = 0$ and $q\big|_{\Gamma_l} = q\big|_{\Gamma_r} = Ct$, so that we have

$$\left|\left(L_{\varepsilon,\mu} - \frac{\partial}{\partial t}\right) r\right| \geq \left(L_{\varepsilon,\mu} - \frac{\partial}{\partial t}\right) q \text{ on } G \quad \text{and} \quad |r(x,t)| \leq q(x,t) \text{ on } \partial G.$$

Now using Lemma 1, we have $|r(x,t)| \leq q(x,t),\ (x,t) \in \bar{G}$, and hence

$$|u(x,t) - \varphi_b(x,0)| \leq Ct, \quad (x,t) \in \bar{G}.$$

As $\varphi_b(x,0)$ is a smooth function and $t \in (0,T]$ on $\bar{G}$. Hence, we have the desired result. □

Lemma 3 *For $i, j \in \mathbb{N}_0$ satisfying $0 \leq i + 2j \leq 4$, bounds of the derivatives of the solution to problem (1.1) are given by*

$$\left\| \frac{\partial^{i+j} u}{\partial x^i \partial t^j} \right\| \leq \begin{cases} C \frac{1}{(\sqrt{\varepsilon})^i}, & \text{when } \alpha\mu^2 \leq \eta\varepsilon, \\ C(\frac{\mu}{\varepsilon})^i (\frac{\mu^2}{\varepsilon})^j, & \text{when } \alpha\mu^2 \geq \eta\varepsilon. \end{cases}$$

Proof To prove the result, we consider two cases: $\alpha\mu^2 \leq \eta\varepsilon$ and $\alpha\mu^2 \geq \eta\varepsilon$. First, consider the case when $\alpha\mu^2 \leq \eta\varepsilon$. Consider the stretched variable $\hat{x} = x/\sqrt{\varepsilon}$ corresponding to the variable x to transform problem (1.1) to

$$\begin{cases} \hat{u}_{\hat{x}\hat{x}} + \frac{\mu}{\sqrt{\varepsilon}} \hat{a}\hat{u}_{\hat{x}} - \hat{b}\hat{u} - \hat{u}_t = -\hat{c}\hat{u}(\hat{x}, t - \tau) + \hat{f} \text{ in } \hat{G}, \\ \hat{u}\big|_{\hat{\Gamma}_b} = \hat{\varphi}_b(\hat{x}, t), \\ \hat{u}\big|_{\hat{\Gamma}_l} = \hat{\varphi}_l(t), \\ \hat{u}\big|_{\hat{\Gamma}_r} = \hat{\varphi}_r(t), \end{cases}$$

where $\hat{G} = (0, 1/\sqrt{\varepsilon}) \times (0, T]$ and $\hat{\Gamma}$ is boundary of $\hat{G}$ corresponding to Γ. Now we use the method of steps and the result in Ladyzhenskaya et al. (1968, Theorem 10.1) to obtain, for $i, j \in \mathbb{N}_0$ satisfying $0 \leq i + 2j \leq 4$, the following bounds:

$$\left\| \frac{\partial^{i+j} \hat{u}}{\partial \hat{x}^i \partial t^j} \right\|_{N_{\lambda,\xi}} \leq C(1 + ||\hat{u}||_{\bar{\hat{G}}}),$$

where $N_{\lambda,\xi}$ is the rectangle $(\xi - \lambda, \xi + \lambda) \times (0, T]) \cap \hat{G}$ for any $\xi \in (0, 1/\sqrt{\varepsilon})$ and $\delta > 0$. Now we return back to the original variable to get the desired result. Next we consider the case $\alpha\mu^2 \geq \eta\varepsilon$. In this case, we use stretched variable in time also. We consider $\tilde{x} = \mu x/\varepsilon$ and $\tilde{t} = \mu^2 t/\varepsilon$ and obtain the transformed problem

$$\begin{cases} \tilde{u}_{\tilde{x}\tilde{x}} + \tilde{a}\tilde{u}_{\tilde{x}} - \frac{\varepsilon}{\mu^2} \tilde{b}\tilde{u} - \tilde{u}_{\tilde{t}} = -\tilde{c}\tilde{u}(\tilde{x}, \tilde{t} - \tilde{\tau}) + \tilde{f} \text{ in } \tilde{G}, \\ \tilde{u}\big|_{\tilde{\Gamma}_b} = \tilde{\varphi}_b(\tilde{x}, \tilde{t}), \\ \tilde{u}\big|_{\tilde{\Gamma}_l} = \tilde{\varphi}_l(\tilde{t}), \\ \tilde{u}\big|_{\tilde{\Gamma}_r} = \tilde{\varphi}_r(\tilde{t}), \end{cases}$$

where $\tilde{G} = (0, \mu/\varepsilon) \times (0, T\mu^2/\varepsilon]$ and $\tilde{\Gamma}$ is the boundary of $\tilde{G}$ corresponding to Γ. Repeating the previous argument we get the desired result. □

For error analysis, we also need decomposition of u as

$$u = v + w_L + w_R, \tag{2.2}$$

where v is the smooth component, and w_L and w_R are the left and right singular components, respectively. We shall derive the bounds for all these components separately.

Theorem 2.1 *For $i, j \in \mathbb{N}_0$ satisfying $0 \le i + 2j \le 4$, derivative bounds for v are given by*

$$\left\| \frac{\partial^{i+j} v}{\partial x^i \partial t^j} \right\| \le \begin{cases} C, & \textit{when } \alpha\mu^2 \le \eta\varepsilon, \\ C(1 + (\frac{\varepsilon}{\mu})^{3-i} (\frac{\mu^2}{\varepsilon})^j), & \textit{when } \alpha\mu^2 \ge \eta\varepsilon. \end{cases}$$

Proof To prove the result, we consider two cases: $\alpha\mu^2 \le \eta\varepsilon$ and $\alpha\mu^2 \ge \eta\varepsilon$. First, when $\alpha\mu^2 \le \eta\varepsilon$, we use the domain extension approach. We smoothly extend the solution of problem (1.1) to a sufficiently large neighborhood of the domain beyond Γ_l and Γ_r, denoted by $\bar{G}^\star$. On $\bar{G}$ the data of the extended problem are same as for problem (1.1). The smooth component v is the restriction (on $\bar{G}$) of the extended problem solution. Thus, using the argument in Hemker et al. (2001), for $i, j \in \mathbb{N}_0$ satisfying $0 \le i + 2j \le 4$, we obtain

$$\left\| \frac{\partial^{i+j} v}{\partial x^i \partial t^j} \right\| \le C.$$

For the case $\alpha\mu^2 \ge \eta\varepsilon$ we use asymptotic expansion approach. We express the smooth component as

$$v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3, \tag{2.3}$$

where

$$\begin{cases} \mu a \frac{\partial v_0}{\partial x} - b v_0 - \frac{\partial v_0}{\partial t} = f - c v_0(x, t - \tau), \\ v_0\big|_{\Gamma_b} = \varphi_b, \\ v_0\big|_{\Gamma_r} \text{ to be chosen,} \end{cases} \tag{2.4}$$

$$\begin{cases} \mu a \frac{\partial v_1}{\partial x} - b v_1 - \frac{\partial v_1}{\partial t} = -\frac{\partial^2 v_0}{\partial x^2} - c v_1(x, t - \tau), \\ v_1\big|_{\Gamma_b} = 0, \\ v_1\big|_{\Gamma_r} \text{ to be chosen,} \end{cases} \tag{2.5}$$

$$\begin{cases} \mu a \frac{\partial v_2}{\partial x} - b v_2 - \frac{\partial v_2}{\partial t} = -\frac{\partial^2 v_1}{\partial x^2} - c v_2(x, t - \tau), \\ v_2\big|_{\Gamma_b} = 0, \\ v_2\big|_{\Gamma_r} = 0, \end{cases} \tag{2.6}$$

$$\begin{cases} (L_{\varepsilon,\mu} - \frac{\partial}{\partial t}) v_3 = -\frac{\partial^2 v_2}{\partial x^2} - c v_3(x, t - \tau), \\ v_3\big|_{\partial G} = 0. \end{cases} \tag{2.7}$$

The following result can be established through contradictory argument:

$$\text{If } \mu a \frac{\partial y}{\partial x} - b y - \frac{\partial y}{\partial t}\bigg|_D \le 0 \text{ and } y\bigg|_{\Gamma_b \cup \Gamma_r} \ge 0 \text{ then } y\bigg|_{\bar{D}} \ge 0, \tag{2.8}$$

where $D = [0, 1) \times [0, T]$.

Further, v_0 is expressed as

$$v_0 = \sigma_0 + \mu\sigma_1 + \mu^2\sigma_2 + \mu^3\sigma_3, \tag{2.9}$$

where

$$\begin{cases} -b\sigma_0 - \frac{\partial \sigma_0}{\partial t} = f(x, t) - c\sigma_0(x, t - \tau), \\ \sigma_0\big|_{\Gamma_b} = \varphi_b, \end{cases} \tag{2.10}$$

$$\begin{cases} -b\sigma_1 - \frac{\partial \sigma_1}{\partial t} = -a\frac{\partial \sigma_0}{\partial x} - c\sigma_1(x, t-\tau), \\ \sigma_1\big|_{\Gamma_b} = 0, \end{cases} \tag{2.11}$$

$$\begin{cases} -b\sigma_2 - \frac{\partial \sigma_2}{\partial t} = -a\frac{\partial \sigma_1}{\partial x} - c\sigma_2(x, t-\tau), \\ \sigma_2\big|_{\Gamma_b} = 0, \end{cases} \tag{2.12}$$

$$\begin{cases} \mu a\frac{\partial \sigma_3}{\partial x} - b\sigma_3 - \frac{\partial \sigma_3}{\partial t} = -a\frac{\partial \sigma_2}{\partial x} - c\sigma_3(x, t-\tau), \\ \sigma_3\big|_{\Gamma_b} = 0, \\ \sigma_3\big|_{\Gamma_r} = 0. \end{cases} \tag{2.13}$$

It can be easily observed that the derivatives of $\sigma_0,\ \sigma_1$ and σ_2 involved in the expression of v_0 are bounded independently of μ. Thus, for $i, j \in \mathbb{N}_0$ satisfying $0 \le i + j \le 3$, and $k = 0, 1, 2,$ we have

$$\left\| \frac{\partial^{i+j}\sigma_k}{\partial x^i \partial t^j} \right\| \le C. \tag{2.14}$$

We have $v_0\big|_{\Gamma_r} = \sigma_0 + \mu\sigma_1 + \mu^2\sigma_2$, since $\sigma_3\big|_{\Gamma_r\cup\Gamma_b} = 0$. Now using (2.8) and method of steps, it can be easily deduced that for $i, j \in \mathbb{N}_0$ satisfying $0 \le i + j \le 3$,

$$\left\| \frac{\partial^{i+j}\sigma_3}{\partial x^i \partial t^j} \right\| \le \frac{C}{\mu^i}. \tag{2.15}$$

Thus, combining all the bounds on $\sigma_0, \sigma_1, \sigma_2$ and σ_3 for $i, j \in \mathbb{N}_0$ satisfying $0 \le i + j \le 3$, we get $\left\| \frac{\partial^{i+j}v_0}{\partial x^i \partial t^j} \right\| \le C$. Furthermore, differentiating Eq. (2.4) and using method of steps, for $i, j \in \mathbb{N}_0$ satisfying $0 \le i + j \le 7$, we get

$$\left\| \frac{\partial^{i+j}v_0}{\partial x^i \partial t^j} \right\| \le C\left(1 + \frac{1}{\mu^{i-3}}\right). \tag{2.16}$$

Now we express v_1 as

$$v_1 = \nu_0 + \mu\nu_1 + \mu^2\nu_2, \tag{2.17}$$

where

$$\begin{cases} -b\nu_0 - \frac{\partial \nu_0}{\partial t} = -\frac{\partial^2 \nu_0}{\partial x^2} - c\nu_0(x, t-\tau), \\ \nu_0\big|_{\Gamma_b} = 0, \end{cases} \tag{2.18}$$

$$\begin{cases} -b\nu_1 - \frac{\partial \nu_1}{\partial t} = -a\frac{\partial \nu_0}{\partial x} - c\nu_1(x, t-\tau), \\ \nu_1\big|_{\Gamma_b} = 0, \end{cases} \tag{2.19}$$

$$\begin{cases} \mu a\frac{\partial \nu_2}{\partial x} - b\nu_2 - \frac{\partial \nu_2}{\partial t} = -a\frac{\partial \nu_1}{\partial x} - c\nu_2(x, t-\tau), \\ \nu_1\big|_{\Gamma_b} = 0, \\ \nu_r\big|_{\Gamma_r} = 0. \end{cases} \tag{2.20}$$

We have $v_1\big|_{\Gamma_r} = \nu_0 + \mu\nu_1$. The problems for ν_0 and ν_1 are independent of small parameters. Thus, for $i, j \in \mathbb{N}_0$ satisfying $0 \le i + j \le 2$, we get

$$\left\| \frac{\partial^{i+j}\nu_0}{\partial x^i \partial t^j} \right\| \le C\left(1 + \frac{1}{\mu^{i-1}}\right) \text{ and } \left\| \frac{\partial^{i+j}\nu_1}{\partial x^i \partial t^j} \right\| \le \frac{C}{\mu^i}. \tag{2.21}$$

The bounds for ν_2 are obtained using (2.8) and method of steps. For $i, j \in \mathbb{N}_0$ satisfying $0 \le i + j \le 2$, we get

$$\left\| \frac{\partial^{i+j} \nu_2}{\partial x^i \partial t^j} \right\| \le \frac{C}{\mu^{i+1}}. \tag{2.22}$$

Thus, $\left\| \frac{\partial^{i+j} v_1}{\partial x^i \partial t^j} \right\| \le C\left(1 + \frac{1}{\mu^{i-1}}\right)$ for $i, j \in \mathbb{N}_0$ satisfying $0 \le i + j \le 2$. Further, higher derivatives are obtained by differentiating Eq. (2.5) and method of steps. For $i, j \in \mathbb{N}_0$ satisfying $0 \le i + j \le 5$, we get

$$\left\| \frac{\partial^{i+j} v_1}{\partial x^i \partial t^j} \right\| \le C\left(1 + \frac{1}{\mu^{i-1}}\right). \tag{2.23}$$

Now to obtain bounds on v_2 we use Eq. (2.6), result (2.8), the bounds on the derivatives of v_1, and previous arguments. For $i, j \in \mathbb{N}_0$ satisfying $0 \le i + j \le 4$, we get

$$\left\| \frac{\partial^{i+j} v_2}{\partial x^i \partial t^j} \right\| \le C\left(1 + \frac{1}{\mu^{i+1}}\right). \tag{2.24}$$

Clearly, (2.7) is similar to problem (1.1). Therefore, using arguments in Lemma 3 and bounds on v_2, we get

$$\left\| \frac{\partial^{i+j} v_3}{\partial x^i \partial t^j} \right\| \le C\left(\frac{\mu}{\varepsilon}\right)^i \left(\frac{\mu^2}{\varepsilon}\right)^j \frac{1}{\mu^3}. \tag{2.25}$$

Substituting all these estimates from (2.16), (2.23), (2.24) and (2.25) into Eq. (2.3) and using $\alpha\mu^2 \ge \eta\varepsilon$, we get the required bounds for v. □

When $\alpha\mu^2 \ge \eta\varepsilon$, note that

$$||v_{tt}|| \le C(1 + \varepsilon^3\mu^{-3}\mu^4\varepsilon^{-2}) \le C. \tag{2.26}$$

Next we obtain bounds on w_L and w_R that satisfy

$$\begin{cases} (L_{\varepsilon,\mu} - \frac{\partial}{\partial t})w_L = -cw_L(x, t-\tau) \;\text{ in } G, \\ w_L\big|_{\Gamma_b \cup \Gamma_r} = 0, \\ w_L\big|_{\Gamma_l} = u - v - w_R, \end{cases} \tag{2.27}$$

$$\begin{cases} (L_{\varepsilon,\mu} - \frac{\partial}{\partial t})w_R = -cw_R(x, t-\tau) \;\text{ in } G, \\ w_R\big|_{\Gamma_b} = 0, \\ w_R\big|_{\Gamma_r} = u - v. \end{cases} \tag{2.28}$$

When $\alpha\mu^2 \le \eta\varepsilon,\ w_R\big|_{\Gamma_l} = 0$, otherwise for $\alpha\mu^2 \ge \eta\varepsilon,\ w_R\big|_{\Gamma_l}$ is defined in (2.29)–(2.33).

For the case $\alpha\mu^2 \le \eta\varepsilon$, w_L and w_R satisfy the bounds in Lemma 3. If $\alpha\mu^2 \ge \eta\varepsilon$, we consider the decomposition

$$w_R = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \varepsilon^3\omega_3, \tag{2.29}$$

where $v = v_0 + \varepsilon v_1,\ (x, t) \in \Gamma_r$ is given in (2.3) and

$$\begin{cases} \mu a \frac{\partial \omega_0}{\partial x} - b\omega_0 - \frac{\partial \omega_0}{\partial t} = -c\omega_0(x, t-\tau), \\ \omega_0\big|_{\Gamma_b} = 0, \\ \omega_0\big|_{\Gamma_r} = u - v, \end{cases} \tag{2.30}$$

$$\begin{cases} \mu a \frac{\partial \omega_1}{\partial x} - b\omega_1 - \frac{\partial \omega_1}{\partial t} = -\frac{\partial^2 \omega_0}{\partial x^2} - c\omega_1(x, t-\tau), \\ \omega_1\big|_{\Gamma_b} = 0, \\ \omega_1\big|_{\Gamma_r} = 0, \end{cases} \tag{2.31}$$

$$\begin{cases} \mu \frac{\partial \omega_2}{\partial x} - b\omega_2 - \frac{\partial \omega_2}{\partial t} = -\frac{\partial^2 \omega_1}{\partial x^2} - c\omega_2(x, t-\tau), \\ \omega_2\big|_{\Gamma_b} = 0, \\ \omega_2\big|_{\Gamma_r} = 0, \end{cases} \tag{2.32}$$

$$\begin{cases} (L_{\varepsilon,\mu} - \frac{\partial}{\partial t})\omega_3 = -\frac{\partial^2 \omega_2}{\partial x^2} - c\omega_3(x, t-\tau), \\ \omega_3\big|_{\Gamma_b} = 0, \\ \omega_3\big|_{\Gamma_r} = 0. \end{cases} \tag{2.33}$$

Lemma 4 *The singular components w_L and w_R satisfy*

$$|w_L(x,t)| \le Ce^{-\theta_L x}, \quad |w_R(x,t)| \le Ce^{-\theta_R(1-x)},$$

where

$$\theta_L = \begin{cases} \frac{\sqrt{\eta\alpha}}{\sqrt{\varepsilon}}, & \alpha\mu^2 \le \eta\varepsilon, \\ \frac{\alpha\mu}{\varepsilon}, & \alpha\mu^2 \ge \eta\varepsilon, \end{cases} \qquad \theta_R = \begin{cases} \frac{\sqrt{\eta\alpha}}{2\sqrt{\varepsilon}}, & \alpha\mu^2 \le \eta\varepsilon, \\ \frac{\eta}{2\mu}, & \alpha\mu^2 \ge \eta\varepsilon. \end{cases}$$

Proof To prove the required bound on w_L we consider the barrier function $\pi^{\pm} = Ce^{-\theta_L x} \pm w_L(x,t)$, and use the minimum principle and the method of steps. The proof is similar for for w_R, when $\alpha\mu^2 \le \eta\varepsilon$. When $\alpha\mu^2 \ge \eta\varepsilon$, using the arguments similar to O'Riordan et al. (2006) and decomposition (2.29) in the previous lemma, we obtain $|w_R(0,t)| \le e^{-2Bt}e^{-\frac{\eta}{\mu}}$, where $B < A = \min\{0, a(\frac{1}{a})_t\}$. Now consider the barrier function $\pi^{\pm} = Ce^{-2At}e^{-\frac{\eta}{2\mu}(1-x)} \pm w_R(x,t)$, use the minimum principle and the method of steps to obtain the required bound on w_R. □

Lemma 5 *For the case $\alpha\mu^2 \ge \eta\varepsilon$, the solution of* (2.27) *satisfies*

$$\left\| \frac{\partial^i w_L}{\partial x^i} \right\| \le C\left(\frac{\mu}{\varepsilon}\right)^i, \; 0 \le i \le 4, \quad \left\| \frac{\partial^2 w_L}{\partial t^2} \right\| \le C\left(1 + \frac{\mu^2}{\varepsilon}\right).$$

Proof The proof follows using the method of steps and the arguments in O'Riordan et al. (2006, Lemma 3.9). □

Lemma 6 *For the case $\alpha\mu^2 \ge \eta\varepsilon$, the solution of* (2.28) *satisfies*

$$\left\| \frac{\partial^i w_R}{\partial x^i} \right\| \le \frac{C}{\mu^i}, \; 0 \le i \le 3, \quad \left\| \frac{\partial^4 w_R}{\partial x^4} \right\| \le C(\mu^{-4} + \mu^{-2}\varepsilon^{-1}), \quad \left\| \frac{\partial^2 w_R}{\partial t^2} \right\| \le C.$$

Proof We consider the decomposition (2.29) and obtain bounds separately on ω_0, ω_1, and ω_2 using (2.8) and method of steps. We obtain

$$\left\| \frac{\partial^{i+j}\omega_0}{\partial x^i \partial t^j} \right\| \le \frac{C}{\mu^i}, \; 0 \le i+j \le 6, \; \left\| \frac{\partial^{i+j}\omega_1}{\partial x^i \partial t^j} \right\| \le \frac{C}{\mu^{i+2}}, \; 0 \le i+j \le 5, \; \left\| \frac{\partial^{i+j}\omega_2}{\partial x^i \partial t^j} \right\| \le \frac{C}{\mu^{i+4}}, \; 0 \le i+j \le 4.$$

Using Lemma 1 and the bounds on ω_2, we get

$$\|\omega_3\| \le \frac{C}{\mu^6}.$$

Now we use Lemma 3 to obtain

$$\left\| \frac{\partial^{i+j}\omega_3}{\partial x^i \partial t^j} \right\| \leq C\mu^{-6}\left(\frac{\mu}{\varepsilon}\right)^i \left(\frac{\mu^2}{\varepsilon}\right)^j.$$

The required bounds on w_R can now be achieved by substituting these bounds in (2.29) and noting that $\alpha\mu^2 \geq \eta\varepsilon$. □

3 Numerical discretization

3.1 Mesh generation

We shall discretize problem (1.1) on a tensor product mesh $\bar{G}^{N,M} = \bar{G}^N \times \bar{G}^M$, where $\bar{G}^N$ is a piecewise uniform Shishkin mesh in space and $\bar{G}^M$ is a uniform mesh in time. The uniform mesh in time is formed by taking m sub-intervals of equal length Δt in $[0, T]$ such that $t_j = j\Delta t,\ j = 0, 1, \ldots, m$. Further, $\tau = M\Delta t$ for some positive integer M such that $m = kM$. To define the mesh in space we consider two parameters ρ_1 and ρ_2 as

$$\rho_1 = \begin{cases} \min\{\frac{1}{4}, \frac{4\sqrt{\varepsilon}}{\sqrt{\eta\alpha}} \ln N\}, & \alpha\mu^2 \leq \eta\varepsilon, \\ \min\{\frac{1}{4}, \frac{4\varepsilon}{\mu\alpha} \ln N\}, & \alpha\mu^2 \geq \eta\varepsilon, \end{cases} \qquad \rho_2 = \begin{cases} \min\{\frac{1}{4}, \frac{4\sqrt{\varepsilon}}{\sqrt{\eta\alpha}} \ln N\}, & \alpha\mu^2 \leq \eta\varepsilon, \\ \min\{\frac{1}{4}, \frac{4\mu}{\eta} \ln N\}, & \alpha\mu^2 \geq \eta\varepsilon, \end{cases}$$

and partition the interval $[0, 1]$ into three subintervals $[0, \rho_1]$, $[\rho_1, 1-\rho_2]$ and $[1-\rho_2, 1]$. Now we construct the spatial mesh $\bar{G}^N$ by placing $N/4+1$, $N/2+1$ and $N/4+1$ mesh points in these three subintervals, respectively. Moreover, defining $\hat{h}_1 = \frac{\rho_1}{(N/4)}$, $\hat{h}_2 = \frac{1-\rho_1-\rho_2}{(N/2)}$, and $\hat{h}_3 = \frac{\rho_2}{(N/4)}$, the mesh points in space are given by $\bar{G}^N = \{x_i, i = 0, 1, \ldots, N\}$, where

$$x_i = \begin{cases} i\hat{h}_1, & 0 \leq i \leq N/4, \\ \rho_1 + (i - N/4)\hat{h}_2, & N/4 < i \leq 3N/4, \\ (1-\rho_2) + (i - 3N/4)\hat{h}_3, & 3N/4 < i \leq N. \end{cases}$$

We also define $h_i = x_i - x_{i-1},\ i = 1, \ldots, N$ and $\hbar_i = \frac{h_i + h_{i+1}}{2},\ i = 1, \ldots, N-1$. Now the discretized boundary is given as $\partial G^{N,M} = \Gamma_b^{\text{dis}} \cup \Gamma_l^{\text{dis}} \cup \Gamma_r^{\text{dis}}$, where $\Gamma_b^{\text{dis}} = \bar{G}^N \times \bar{G}^{-M}$, $\Gamma_l^{dis} = \bar{G}^N \cap \Gamma_l$, $\Gamma_r^{\text{dis}} = \bar{G}^N \cap \Gamma_r$ and $\bar{G}^{-M}$ denotes the uniform time mesh in $[-\tau, 0]$ with $M+1$ mesh points. Further, we divide the discretized domain into k sub-domains as $\bar{G}^{M,N} = \cup_{r=1}^{k} \bar{G}_r^{M,N}$, where $\bar{G}_r^{M,N} = \bar{G}^N \times \bar{G}_r^M$ and $\bar{G}_r^M$ denotes $M+1$ uniform mesh points in $[(r-1)\tau, r\tau]$.

3.2 Problem discretization

On the above generated rectangular mesh, we discretize problem (1.1) by a hybrid scheme comprising an implicit Euler scheme in time direction and a combination of central difference, upwind and mid-point operators in spatial direction. We define

$$\begin{cases} [L_{\text{cen}}^{N,M} U]_i^j = \varepsilon\delta_x^2 U_i^j + \mu a_i^j D_x^0 U_i^j - b_i^j U_i^j - D_t^- U_i^j, \\ [L_{\text{up}}^{N,M} U]_i^j = \varepsilon\delta_x^2 U_i^j + \mu a_i^j D_x^+ U_i^j - b_i^j U_i^j - D_t^- U_i^j, \\ [L_{\text{mp}}^{N,M} U]_i^j = \varepsilon\delta_x^2 U_i^j + \mu \overline{a_i^j} D_x^+ U_i^j - \overline{b_i^j U_i^j} - \overline{D_t^- U_i^j}, \end{cases}$$

where

$$D_x^+ U_i^j = \frac{U_{i+1}^j - U_i^j}{h_{i+1}}, \quad D_x^0 U_i^j = \frac{U_{i+1}^j - U_{i-1}^j}{2\hbar_i}, \quad D_t^- U_i^j = \frac{U_i^j - U_i^{j-1}}{\Delta t},$$
$$\delta_x^2 U_i^j = \frac{1}{\hbar_i}\left(\frac{U_{i+1}^j - U_i^j}{h_{i+1}} - \frac{U_i^j - U_{i-1}^j}{h_i}\right),$$

and for any mesh function Z, $\overline{Z_i^j} = \frac{Z_i^j + Z_{i+1}^j}{2}$. Further, define $\hat{b} = b + 1/\Delta t$. Then the discretized problem is of the form

$$\begin{cases} [L^{N,M}U]_i^j = F_i^j, & (x_i, t_j) \in G^{N,M}, \\ U_i^j = u(x_i, t_j), & (x_i, t_j) \in \partial G^{N,M}, \end{cases} \tag{3.1}$$

where

$$[L^{N,M}U]_i^j = \begin{cases} [L_{\mathrm{cen}}^{N,M}U]_i^j, & \text{if } 1 \le i < N/4, \\ [L_{\mathrm{cen}}^{N,M}U]_i^j, & \text{if } N/4 < i < 3N/4 \text{ and } \mu\hat{h}_2\|a\| < 2\varepsilon, \\ [L_{\mathrm{mp}}^{N,M}U]_i^j, & \text{if } N/4 < i < 3N/4,\ \mu\hat{h}_2\|a\| \ge 2\varepsilon \text{ and } \hat{h}_2\|\hat{b}\| < 2\mu\alpha, \\ [L_{\mathrm{up}}^{N,M}U]_i^j, & \text{if } N/4 < i < 3N/4,\ \mu\hat{h}_2\|a\| \ge 2\varepsilon \text{ and } \hat{h}_2\|\hat{b}\| \ge 2\mu\alpha, \\ [L_{\mathrm{cen}}^{N,M}U]_i^j, & \text{if } 3N/4 < i < N \text{ and } \mu\hat{h}_3\|a\| < 2\varepsilon, \\ [L_{\mathrm{mp}}^{N,M}U]_i^j, & \text{if } 3N/4 < i < N \text{ and } \mu\hat{h}_3\|a\| \ge 2\varepsilon, \end{cases} \tag{3.2}$$

at the left transition point ρ_1, that is, if $x_i = \rho_1$,

$$[L^{N,M}U]_i^j = \begin{cases} [L_{\mathrm{cen}}^{N,M}U]_i^j, & \text{if } \rho_1 = 0.25, \\ [L_{\mathrm{mp}}^{N,M}U]_i^j, & \text{if } \rho_1 < 0.25 \text{ and } \hat{h}_2\|\hat{b}\| < 2\mu\alpha, \\ [L_{\mathrm{up}}^{N,M}U]_i^j, & \text{otherwise}, \end{cases} \tag{3.3}$$

at the right transition point ρ_2, that is, if $x_i = \rho_2$,

$$[L^{N,M}U]_i^j = \begin{cases} [L_{\mathrm{cen}}^{N,M}U]_i^j, & \text{if } 1-\rho_2 = 0.75, \text{ and } \mu\hat{h}_3\|a\| < 2\varepsilon, \\ [L_{\mathrm{mp}}^{N,M}U]_i^j, & \text{if } 1-\rho_2 = 0.75, \text{ and } \mu\hat{h}_3\|a\| \ge 2\varepsilon, \\ [L_{\mathrm{mp}}^{N,M}U]_i^j, & \text{if } 1-\rho_2 > 0.75, \text{ and } \hat{h}_3\|\hat{b}\| < 2\mu\alpha, \\ [L_{\mathrm{up}}^{N,M}U]_i^j, & \text{otherwise}, \end{cases} \tag{3.4}$$

and

$$F_i^j = \begin{cases} -c_i^j U_i^{j-m} + f_i^j, & \text{if } L^{N,M} = L_{\mathrm{cen}}^{N,M} \text{ or } L_{up}^{N,M}, \\ -\overline{c_i^j U_i^{j-m}} + \overline{f_i^j}, & \text{if } L^{N,M} = L_{\mathrm{mp}}^{N,M}. \end{cases} \tag{3.5}$$

We next establish discrete equivalent of the minimum principle. Let N_0 be the smallest positive integer such that

$$N_0/\ln N_0 > 8\max\{\|a\|/\alpha, (\|b\| + M)/(\eta\alpha)\}. \tag{3.6}$$

Lemma 7 *The discretization* (3.1) *satisfies discrete minimum principle, that is, if for any mesh function* z *with* $z_i^j|_{\partial G^{N,M}} \geq 0$ *and* $[L^{N,M} z]_i^j|_{G^{N,M}} \leq 0$*, then* $z_i^j|_{\bar{G}^{N,M}} \geq 0$.

Proof Considering (3.6), it can be easily verified that the entries of the tridiagonal system-matrix formed by the proposed difference scheme (3.1) formulate a negative M-matrix. Hence, the difference operator $L^{N,M}$ satisfies the discrete minimum principle. □

4 Error analysis

We start with the decomposition of U as

$$U = V + W_L + W_R, \tag{4.1}$$

where

$$\begin{cases} [L^{N,M} V]_i^j = -c_i^j V_i^{j-m} + f_i^j, & (x_i, t_j) \in G^{N,M}, \\ V_i^j|_{\partial G^{N,M}} = v(x_i, t_j), \end{cases} \tag{4.2}$$

$$\begin{cases} [L^{N,M} W_L]_i^j = -c_i^j (W_L)_i^{j-m}, & (x_i, t_j) \in G^{N,M}, \\ (W_L)_i^j|_{\partial G^{N,M}} = w_L(x_i, t_j), \end{cases} \tag{4.3}$$

$$\begin{cases} [L^{N,M} W_R]_i^j = -c_i^j (W_L)_i^{j-m}, & (x_i, t_j) \in G^{N,M}, \\ (W_R)_i^j|_{\partial G^{N,M}} = w_R(x_i, t_j). \end{cases} \tag{4.4}$$

Hence, we can decompose the error as

$$[U - u]_i^j = [V - v]_i^j + [W_L - w_L]_i^j + [W_R - w_R]_i^j \text{ for all } (x_i, t_j) \in \bar{G}^{N,M}. \tag{4.5}$$

Consequently, we can find error bound for each component separately. We shall use the following lemma frequently.

Lemma 8 *For* $z = v, w_L, w_R$ *defined on* $\bar{G}$ *and* $Z = V, W_L, W_R$ *defined on* $\bar{G}^{N,M}$*, the local truncation error defined by*

$$[L^{N,M}(Z - z)]_i^j = -c_i^j [Z - z]_i^{j-m} + \left[\left(L_{\varepsilon,\mu} - \frac{\partial}{\partial t}\right) z - L^{N,M} z\right]_i^j,$$

on arbitrary mesh with step sizes h_i *is given by*

$$|[L_{cen}^{N,M}(Z - z)]_i^j| \leq |[Z - z]_i^{j-m}| + C\left[\varepsilon \hbar_i \|z_{xxx}\| + \mu \hbar_i \|z_{xx}\| + M^{-1} \|z_{tt}\|\right],$$
$$|[L_{up}^{N,M}(Z - z)]_i^j| \leq |[Z - z]_i^{j-m}| + C\left[\varepsilon \hbar_i \|z_{xxx}\| + \mu h_{i+1} \|z_{xx}\| + M^{-1} \|z_{tt}\|\right],$$
$$|L_{mp}^{N,M}(Z - z)]_i^j| \leq |[Z - z]_i^{j-m}| + C\left[\varepsilon \hbar_i \|z_{xxx}\| + \mu h_{i+1}^2 (\|z_{xxx}\| + \|z_{xx}\|) + M^{-1} \|z_{tt}\|\right],$$

and on uniform mesh with step size h *is given by*

$$|L_{cen}^{N,M}(Z - z)]_i^j| \leq |[Z - z]_i^{j-m}| + C\left[\varepsilon h^2 \|z_{xxxx}\| + \mu h^2 \|z_{xxx}\| + M^{-1} \|z_{tt}\|\right],$$
$$|[L_{up}^{N,M}(Z - z)]_i^j| \leq |[Z - z]_i^{j-m}| + C\left[\varepsilon h^2 \|z_{xxxx}\| + \mu h \|z_{xx}\| + M^{-1} \|z_{tt}\|\right],$$
$$|[L_{mp}^{N,M}(Z - z)]_i^j| \leq |[Z - z]_i^{j-m}| + C\left[\varepsilon h \|z_{xxx}\| + \mu h^2 (\|z_{xxx}\| + \|z_{xx}\|) + M^{-1} \|z_{tt}\|\right].$$

Lemma 9 *For every* $(x_i, t_j) \in \overline{G}_1^{N,M}$, *the following error bound holds for the smooth component of the error*

$$|[V - v]_i^j| \le C(M^{-1} + N^{-2}).$$

Proof When mesh is uniform, we use Lemma 8, Theorem 2.1, and the discrete minimum principle Lemma 7 to get the following bound:

$$|[(V - v)]_i^j| \le C[M^{-1} + N^{-2}].$$

When mesh is not uniform, either the mid-point scheme is used at the transition points or if $h_{i+1}\|\hat{b}\| \ge 2\mu\alpha$, the upwind scheme is used. Using Lemma 8 and Theorem 2.1, we get

$$|[L^{M,N}(V - v)]_i^j| \le \begin{cases} C(M^{-1} + N^{-2}), & \text{when } i \ne N/4,\ 3N/4, \\ C(M^{-1} + N^{-1}(\varepsilon + N^{-1})), & \text{otherwise.} \end{cases}$$

Define the barrier function

$$\Theta_i^j = C(N^{-2}(\phi(x_i) + 1) + M^{-1}),$$

where

$$\phi(x_i) = \begin{cases} 1, & \text{if } 0 \le x_i \le \rho_1, \\ 1 - \frac{x_i - \rho_1}{2(1-\rho_1-\rho_2)}, & \text{if } \rho_1 \le x_i \le 1 - \rho_2, \\ \frac{1-x_i}{2\rho_2}, & \text{if } 1 - \rho_2 \le x_i \le 1. \end{cases}$$

Noting that $1/\rho_2 \ge 4$, we have

$$\varepsilon\delta_x^2\Theta_i^j = \begin{cases} 0, & \text{when } i \ne N/4,\ 3N/4, \\ O(-\varepsilon N^{-1}), & \text{otherwise,} \end{cases}$$

and $D_x^0\Theta_i^j \le 0$, $D_x^+\Theta_i^j \le 0$. Now we use Lemma 7 to get the desired result. □

We introduce two barrier functions that are essential to establish the error bounds of singular components

$$\Phi_i^j = \begin{cases} \prod_{r=1}^{i}(1 + \theta_L h_r)^{-1}, & \text{if } 1 \le i \le N, \\ 1, & i = 0, \end{cases} \qquad \Psi_i^j = \begin{cases} \prod_{r=i+1}^{N}(1 + \theta_R h_r)^{-1}, & \text{if } 0 \le i < N, \\ 1, & i = N, \end{cases} \tag{4.6}$$

where θ_L and θ_R are as defined in Lemma 4.

Lemma 10 *The barrier functions* Φ *and* Ψ *defined above satisfy*

$$[L^{N,M}\Phi]_i^j \le 0, \quad [L^{N,M}\Psi]_i^j \le 0.$$

Proof Here we provide the proof for the mid-point operator. Similar arguments can be used for the central difference and upwind operators. On applying the mid-point operator to the barrier function Φ_i^j, we get

$$\begin{aligned} [L_{\text{mp}}^{N,M}\Phi]_i^j &= \varepsilon\delta_x^2\Phi_i^j + \mu\overline{a_i^j}D_x^+\Phi_i^j - \overline{b_j^i\Phi_i^j} - \overline{D_t^-\Phi_i^j} \\ &= \varepsilon\frac{\theta_L^2}{\hbar_i}h_{i+1}\Phi_{i+1}^j + \mu\overline{a_i^j}(-\theta_L\Phi_{i+1}^j) - \frac{b_i^j\Phi_i^j}{2} - \frac{b_{i+1}^j\Phi_{i+1}^j}{2} \\ &= \left[2\varepsilon\theta_L^2\left(\frac{h_{i+1}}{2\hbar_i} - 1\right) + \left(2\varepsilon\theta_L^2 - \mu\overline{a_i^j}\theta_L - \frac{b_{i+1}^j}{2}\right) - \frac{b_i^j}{2}(1 + \theta_L h_{i+1})\right]\Phi_{i+1}^j \end{aligned}$$

$$\leq \left[2\varepsilon\theta_L^2 - \mu\overline{a_i^j}\theta_L - \frac{b_{i+1}^j}{2}\right]\Phi_{i+1}^j \leq 0$$

since for $\theta_L = \frac{\sqrt{\eta\alpha}}{2\sqrt{\varepsilon}}$, we have $\left(2\varepsilon\theta_L^2 - \frac{b_{i+1}}{2}\right) = \left(\frac{\eta\alpha}{2} - \frac{b_{i+1}}{2}\right) \leq 0$ and for $\theta_L = \frac{\mu\alpha}{2\varepsilon}$, we have $\left(2\varepsilon\theta_L^2 - \mu\overline{a_i^j}\theta_L\right) = \left(\frac{\mu^2\alpha}{2\varepsilon}(\alpha - \overline{a_i^j})\right) \leq 0$. Now we apply the mid-point operator to the barrier function Ψ_i^j to get

$$\begin{aligned}
[L_{\mathrm{mp}}^{N,M}\Psi]_i^j &= \varepsilon\delta_x^2\Psi_i^j + \mu\overline{a_i^j}D_x^+\Psi_i^j - \overline{b_j^i\Psi_i^j} - \overline{D_t^-\Psi_i^j}\\
&= \varepsilon\frac{\theta_R^2}{\hbar_i}h_i\Psi_{i-1}^j + \mu\overline{a_i^j}\theta_R\Psi_i^j - \frac{b_i^j\Psi_i^j}{2} - \frac{b_{i+1}^j\Psi_{i+1}^j}{2}\\
&\leq \left[2\varepsilon\theta_R^2\left(\frac{h_i}{2\hbar_i} - 1\right) + \left(2\varepsilon\theta_R^2 + \mu\overline{a_i^j}\theta_R - \overline{b_i^j}\right)\right]\Psi_i^j\\
&\leq \left[2\varepsilon\theta_R^2 + \mu\overline{a_i^j}\theta_R - \overline{b_i^j}\right]\Psi_i^j.
\end{aligned}$$

Now for both values of θ_R, we obtain $[L_{mp}^{N,M}\Psi]_i^j \leq \left(\eta\overline{a_i^j} - \overline{b_i^j}\right)\Psi_i^j = \left[\frac{a_i^j}{2}\left(\eta - \frac{b_i^j}{a_i^j}\right) + \frac{a_{i+1}^j}{2}\left(\eta - \frac{b_{i+1}^j}{a_{i+1}^j}\right)\right]\Psi_i^j \leq 0$. □

Lemma 11 *The layer components satisfy*

$$\begin{aligned}
|(W_L)_i^j| &\leq CN^{-2},\ i = N/4, \ldots, N,\ j\Delta t \leq \tau,\\
|(W_R)_i^j| &\leq CN^{-2},\ i = 0, \ldots, 3N/4,\ j\Delta t \leq \tau.
\end{aligned}$$

Proof Defining $\Theta^{\pm}(x_i, t_j) = C\Phi_i^j \pm (W_L)_i^j$, with C chosen sufficiently large, using Lemmas 10 and 7, we get $|(W_L)_i^j| \leq C\Phi_i^j$. Further, for $i = N/4, \ldots, N$, using the fact that $\ln(1+t) > t(1 - t/2)$, we have

$$\Phi_i^j \leq \Phi_{N/4}^j = \left[\left(1 + 8N^{-1}\ln N\right)^{-N/8}\right]^2 \leq CN^{-2}.$$

The same argument can be used to bound $(W_R)_i^j$, for $i = 0, \ldots, 3N/4$, $j\Delta t \leq \tau$. □

Lemma 12 *Let W_L be the solution of* (4.3) *and w_L be the solution of* (2.27). *Then for every $(x_i, t_j) \in \overline{G}_1^{N,M}$, the error bounds are*

$$|[W_L - w_L]_i^j| \leq \begin{cases} C(M^{-1} + N^{-2}(\ln N)^2), & \text{when } \alpha\mu^2 \leq \eta\varepsilon,\\ C(M^{-1}\ln N + N^{-2}(\ln N)^3), & \text{when } \alpha\mu^2 \geq \eta\varepsilon.\end{cases}$$

Proof We first consider the uniform mesh case, that is, when $\rho_1 = \frac{1}{4}$. If $\alpha\mu^2 \leq \eta\varepsilon$,

$$|[L^{N,M}(W_L - w_L)]_i^j| \leq C\left[N^{-2}\left(\varepsilon\|(w_L)_{xxxx}\| + \mu\|(w_L)_{xxx}\|\right) + M^{-1}\|(w_L)_{tt}\|\right].$$

Using $\frac{1}{\sqrt{\varepsilon}} \leq C\ln N$ and derivative bounds of w_L, we have

$$|[L^{N,M}(W_L - w_L)]_i^j| \leq C[M^{-1} + N^{-2}(\ln N)^2].$$

If $\alpha\mu^2 \geq \eta\varepsilon$, again using $\frac{\mu}{\varepsilon} \leq C \ln N$ and derivative bounds of w_L, we have

$$|[L^{N,M}(W_L - w_L)]_i^j| \leq C[M^{-1}\ln N + N^{-2}(\ln N)^3].$$

The desired result follows using Lemma 7.

For the case when $\rho_1 < \frac{1}{4}$, the error is analysed first outside the layer region and then inside layer region. In the outside region, that is, when $(x_i, t_j) \in [\rho_1, 1) \times (0, \tau]$, w_L and W_L both are small. So, using Lemmas 4 and 11, and the fact $\rho_1\theta_L = 4\ln N$,

$$|[W_L - w_L]_i^j| \leq |(W_L)_i^j| + |(w_L)_i^j| \leq C[e^{-\theta_L x_i} + N^{-2}] \leq C[e^{-\theta_L \rho_1} + N^{-2}] \leq CN^{-2}.$$

For inside the layer region, that is, when $(x_i, t_j) \in (0, \rho_1) \times (0, \tau]$; if $\alpha\mu^2 \leq \eta\varepsilon$,

$$|[L^{N,M}(W_L - w_L)]_i^j| \leq C\left[N^{-2}\left(\varepsilon\rho_1^2\|(w_L)_{xxxx}\| + \mu\rho_1^2\|(w_L)_{xxx}\|\right) + M^{-1}\|(w_L)_{tt}\|\right].$$

Using derivative bounds of w_L and the fact that $\rho_1 \leq C\sqrt{\varepsilon}\ln N$, we get

$$\begin{aligned}|[L^{N,M}(W_L - w_L)]_i^j| &\leq C\left[(N^{-1}\ln N)^2\left(1 + \frac{\mu}{\sqrt{\varepsilon}}\right) + M^{-1}\right]\\ &\leq C[M^{-1} + N^{-2}(\ln N)^2].\end{aligned}$$

On applying Lemma 7 we get the desired result. If $\alpha\mu^2 \geq \eta\varepsilon$, using derivative bounds of w_L and the fact that $\rho_1 \leq C(\frac{\varepsilon}{\mu}\ln N)$, we get

$$|[L^{N,M}(W_L - w_L)]_i^j| \leq C\frac{\mu^2}{\varepsilon}\left[(N^{-1}\ln N)^2 + M^{-1}\right].$$

Now we consider

$$\Theta^{\pm}(x_i, t_j) = C\left[(\rho_1 - x_i)\frac{\mu}{\varepsilon}((N^{-1}\ln N)^2 + M^{-1}) + N^{-2}\right] \pm [W_L - w_L]_i^j$$

Clearly, for $(0, t_j)$, (ρ_1, t_j), $0 < t_j \leq \tau$ and for $(x_i, 0)$, $0 \leq x_i \leq \rho_1$, we have $\Theta^{\pm} \geq 0$. Also, choosing C large enough such that $[L^{N,M}\Theta^{\pm}]_i^j \leq 0$. Therefore, using Lemma 7, we get

$$\begin{aligned}|[W_L - w_L]_i^j| &\leq C\left[(\rho_1 - x_i)\frac{\mu}{\varepsilon}((N^{-1}\ln N)^2 + M^{-1}) + N^{-2}\right]\\ &\leq C[M^{-1}\ln N + N^{-2}(\ln N)^3].\end{aligned}$$

□

Lemma 13 *Suppose W_R is the solution of (4.4) and w_R is the solution of (2.28). Then for every $(x_i, t_j) \in \overline{G}_1^{N,M}$, the error bound is*

$$|[W_R - w_R]_i^j| \leq C[M^{-1} + N^{-2}(\ln N)^2].$$

Proof We shall use similar ideas as in the case of left singular component. When $(x_i, t_j) \in (0, 1 - \rho_2] \times (0, \tau]$, we have

$$[W_R - w_R]_i^j| \leq |(W_R)_i^j| + |(w_R)_i^j| \leq C[e^{-\theta_R \rho_2} + N^{-2}] \leq CN^{-2}.$$

Now we consider the case when $(x_i, t_j) \in (1-\rho_2, 1) \times (0, \tau]$. If $\rho_2 = \frac{1}{4}$ and $\alpha\mu^2 \leq \eta\varepsilon$, we use Lemma 8, bounds on the derivatives of w_R, and the inequality $\frac{1}{\sqrt{\varepsilon}} \leq C \ln N$ to bound the truncation for the central difference operator. We get

$$
\begin{aligned}
|[L^{N,M}(W_R - w_R)]_i^j| &\leq C\Big[\hat{h}_3^2\Big(\varepsilon\|(w_R)_{xxxx}\| + \mu\|(w_R)_{xxx}\|\Big) + M^{-1}\|(w_R)_{tt}\|\Big] \\
&\leq C\Big[\frac{N^{-2}}{\varepsilon} + M^{-1}\Big] \\
&\leq C[M^{-1} + N^{-2}(\ln N)^2].
\end{aligned}
$$

When $\mu\hat{h}_3\|a\| \geq 2\varepsilon$, mid-point is used. So, we use the same argument to get

$$
\begin{aligned}
|[L^{N,M}(W_R - w_R)]_i^j| &\leq C\Big[\varepsilon\hat{h}_3\|(w_R)_{xxx}\| + \mu\hat{h}_3^2\Big(\|(w_R)_{xxx}\| + \|(w_R)_{xx}\|\Big) \\
&\quad + M^{-1}\|(w_R)_{tt}\|\Big] \\
&\leq C\Big[\mu N^{-2}\Big(\|(w_R)_{xxx}\| + \|(w_R)_{xx}\|\Big) + M^{-1}\|(w_R)_{tt}\|\Big] \\
&\leq C\Big[N^{-2}\frac{\mu}{\varepsilon^{3/2}} + M^{-1}\Big] \\
&\leq C[M^{-1} + N^{-2}(\ln N)^2].
\end{aligned}
$$

Now, consider the case when $\rho_2 = \frac{1}{4}$ and $\alpha\mu^2 \geq \eta\varepsilon$. Using the arguments as above and the inequality $\frac{1}{\mu} \leq C \ln N$, the truncation error for the central difference operator is given as

$$
\begin{aligned}
|[L^{N,M}(W_R - w_R)]_i^j| &\leq C\Big[N^{-2}\Big(\varepsilon\|(w_R)_{xxxx}\| + \mu\|(w_R)_{xxx}\|\Big) + M^{-1}\|(w_R)_{tt}\|\Big] \\
&\leq C\Big[N^{-2}/\mu^2 + M^{-1}\Big] \\
&\leq C[M^{-1} + N^{-2}(\ln N)^2],
\end{aligned}
$$

and the truncation error for mid-point is given as

$$
\begin{aligned}
|[L^{N,M}(W_R - w_R)]_i^j| &\leq C\Big[\mu N^{-2}\Big(\|(w_R)_{xxx}\| + \|(w_R)_{xx}\|\Big) + M^{-1}\|(w_R)_{tt}\|\Big] \\
&\leq C\Big[N^{-2}/\mu^2 + M^{-1}\Big] \\
&\leq C[M^{-1} + N^{-2}(\ln N)^2].
\end{aligned}
$$

Next we consider the case when $\rho_2 < \frac{1}{4}$. If $\alpha\mu^2 \leq \eta\varepsilon$, then the mid-point operator is not used. So, we consider only the central difference operator. Using the derivative bounds of w_R and the fact that $\rho_2 \leq C\sqrt{\varepsilon}\ln N$, the truncation error is given as

$$
\begin{aligned}
|[L^{N,M}(W_R - w_R)]_i^j| &\leq C\Big[N^{-2}(\ln N)^2\Big(1 + \frac{\mu}{\sqrt{\varepsilon}}\Big) + M^{-1}\Big] \\
&\leq C[M^{-1} + N^{-2}(\ln N)^2].
\end{aligned}
$$

If $\alpha\mu^2 \geq \eta\varepsilon$, then using derivative bounds of w_R and the fact that $\rho_2 \leq C\mu \ln N$, the truncation error for central difference operator is given as

$$\begin{aligned}\left|[L^{N,M}(W_R - w_R)]_i^j\right| &\leq C\left[N^{-2}(\ln N)^2\left(1 + \frac{\varepsilon}{\mu^2}\right) + M^{-1}\right]\\ &\leq C[M^{-1} + N^{-2}(\ln N)^2],\end{aligned}$$

and the truncation error for mid-point operator is given by

$$\begin{aligned}\left|[L^{N,M}(W_R - w_R)]_i^j\right| &\leq C\left[N^{-2}\frac{\mu\rho^2}{\mu^3} + M^{-1}\right]\\ &\leq C[M^{-1} + N^{-2}(\ln N)^2].\end{aligned}$$

Combining the results, for all cases, we have

$$\left|[L^{N,M}(W_R - w_R)]_i^j\right| \leq C[M^{-1} + N^{-2}(\ln N)^2].$$

Now we use the discrete minimum principle to get the desired result. □

Next we obtain the error on $\overline{G}_2^{N,M}$. For $(x_i, t_j) \in G_2^{N,M}$, we note that

$$[L^{N,M}(V - v)]_i^j = -c_i^j[V - v]_i^{j-m} + \left[\left(L_{\varepsilon,\mu} - \frac{\partial}{\partial t}\right)v - L^{N,M}v\right]_i^j.$$

The first term on the left hand side is bounded using Lemma 9. To bound the second term we use arguments in Lemma 9 and then use the discrete minimum principle to get

$$|[V - v]_i^j| \leq C(M^{-1} + N^{-2}),\ (x_i, t_j) \in \overline{G}_2^{N,M}.$$

Similarly, we can bound $|[W_L - w_L]_i^j|$ and $|[W_R - w_R]_i^j|$ for $(x_i, t_j) \in \overline{G}_2^{N,M}$. Finally, applying an induction argument, we can obtain, for $(x_i, t_j) \in \overline{G}_p^{N,M}$, $p = 1, \ldots, k$, the following bounds:

$$\begin{aligned}&|[V - v]_i^j| \leq C(M^{-1} + N^{-2}),\\ &|[W_L - w_L]_i^j| \leq \begin{cases} C(M^{-1} + N^{-2}(\ln N)^2), & \text{when } \alpha\mu^2 \leq \eta\varepsilon \\ C(M^{-1}\ln N + N^{-2}(\ln N)^3), & \text{when } \alpha\mu^2 \geq \eta\varepsilon, \end{cases}\\ &|[W_R - w_R]_i^j| \leq C[M^{-1} + N^{-2}(\ln N)^2].\end{aligned}$$

Hence, combining the bounds for smooth and singular components, we have the following main result.

Theorem 4.1 *Suppose U is the solution of problem (3.1) and u is the exact solution of (1.1). Then*

$$\|U - u\|_{\overline{G}^{N,M}} \leq \begin{cases} C(M^{-1} + N^{-2}(\ln N)^2), & \text{when } \alpha\mu^2 \leq \eta\varepsilon \\ C(M^{-1}\ln N + N^{-2}(\ln N)^3), & \text{when } \alpha\mu^2 \geq \eta\varepsilon. \end{cases}$$

5 Numerical examples

To demonstrate the effectiveness of the proposed finite difference scheme (3.1) for problem (1.1), we consider three test examples. For these test examples, we generate the rectangular meshes and compute the discrete solutions. Then the tables for errors and rates of convergence are presented in support of the theoretical results presented in the previous section.

Table 1 Maximum errors $F_{\varepsilon,\mu}^{N,M}$ and $F^{N,M}$, and rates of convergence $\rho_{\varepsilon,\mu}^{N,M}$ and $\rho^{N,M}$ using scheme (3.1) for Example 1

$\mu = 10^{-3}$ ε	$N = 32$ $M = 8$	$N = 64$ $M = 16$	$N = 128$ $M = 32$	$N = 256$ $M = 64$	$N = 512$ $M = 128$	$N = 1024$ $M = 256$
$10^0 = 1$	7.3399e−3	5.7663e−3	3.6422e−3	2.1295e−3	1.1629e−3	6.0952e−4
	0.3481	0.6628	0.7743	0.8726	0.9320	
10^{-2}	3.5162e−2	1.3240e−2	5.9994e−3	3.1032e−3	1.5788e−3	7.9633e−4
	1.4091	1.1420	0.9511	0.9749	0.9874	
10^{-4}	4.3705e−2	1.6704e−2	7.3802e−3	3.7406e−3	1.8967e−3	9.5511e-4
	1.3876	1.1785	0.9803	0.9797	0.9898	
10^{-6}	4.3471e−2	1.6596e−2	7.3290e−3	3.7218e−3	1.8873e−3	9.4388e−4
	1.3892	1.1792	0.9776	0.9796	0.9997	
10^{-8}	4.3429e−2	1.6573e−2	7.3303e−3	3.7211e−3	1.8870e−3	9.5031e−4
	1.3898	1.1769	0.9781	0.9795	0.9896	
10^{-10}	4.4343e−2	1.6572e−2	7.3303e−3	3.7211e−3	1.8870e−3	9.5031e−4
	1.3898	1.1768	0.9781	0.9795	0.9896	
10^{-12}	4.4343e−2	1.6572e−2	7.3303e−3	3.7211e−3	1.8870e−3	9.5031e−4
	1.3898	1.1768	0.9781	0.9795	0.9896	
$F^{N,M}$	4.3705e−2	1.6704e−2	7.3802e−3	3.7406e−3	1.8967e−3	9.5511e−4
$\rho^{N,M}$	1.3876	1.1785	0.9803	0.9797	0.9898	

Example 5.1 Our first test example is the following time-delay problem:

$$\begin{cases} \varepsilon u_{xx}(x,t) + \mu(1+x)u_x(x,t) - u(x,t) \\ -u_t(x,t) = -u(x,t-\tau) + 16x^2(1-x)^2, \ (x,t) \in (0,1) \times (0,2], \\ u(x,t) = 0, \quad (x,t) \in [0,1] \times [-\tau, 0], \\ u(0,t) = 0, \ u(1,t) = 0, \quad t \in [0,2]. \end{cases}$$

The exact solution of this problem is not known, so the numerical errors and rates of convergence will be calculated with the help of double mesh principle. In this technique, the spatial mesh is bisected into $2N$ subintervals and time mesh into $2M$ subintervals. Then the maximum pointwise error and rate of convergence are calculated by

$$F_{\varepsilon,\mu}^{N,M} = \max_{i,j} |(U^{N,M})_i^j - (\tilde{U}^{2N,2M})_{2i}^{2j}| \quad \text{and} \quad \rho_{\varepsilon,\mu}^{N,M} = \log_2 \frac{F_{\varepsilon,\mu}^{N,M}}{F_{\varepsilon,\mu}^{2N,2M}},$$

respectively, where $U^{N,M}$ is the numerical solution on the mesh $\bar{G}^{N,M}$ and $\tilde{U}^{2N,2M}$ is the numerical solution on the mesh obtained after introductions of mid-points in $\bar{G}^{N,M}$. Now uniform error and uniform rate of convergence are calculated by

$$F^{N,M} = \max_{\varepsilon,\mu} F_{\varepsilon,\mu}^{N,M} \quad \text{and} \quad \rho^{N,M} = \log_2 \frac{F^{N,M}}{F^{2N,2M}}.$$

In Tables 1 and 2, the maximum pointwise errors and the corresponding rates of convergence for the numerical solution computed using scheme (3.1) are presented for $\mu = 10^{-3}$ and 10^{-9}, respectively, with different values of ε and discretization parameters N and M varying with the same ratio (N and M both multiplied by 2). Here, we see that the convergence rates of the scheme is near one, confirming the first order in time. In Tables 3 and

Table 2 Maximum errors $F_{\varepsilon,\mu}^{N,M}$ and $F^{N,M}$, and rates of convergence $\rho_{\varepsilon,\mu}^{N,M}$ and $\rho^{N,M}$ using scheme (3.1) for Example 1

$\mu = 10^{-9}$ ε	$N = 32$ $M = 8$	$N = 64$ $M = 16$	$N = 128$ $M = 32$	$N = 256$ $M = 64$	$N = 512$ $M = 128$	$N = 1024$ $M = 256$
$10^0 = 1$	7.3402e−3	5.7663e−3	3.6422e−3	2.1294e−3	1.1629e−3	6.0952e−4
	0.3481	0.6628	0.7743	0.8726	0.9320	
10^{-2}	3.5175e−2	1.3245e−2	6.0008e−3	3.1039e−3	1.5791e−3	7.9649e−4
	1.4090	1.1422	0.9510	0.9749	0.9874	
10^{-4}	4.3708e−2	1.6705e−2	7.3807e−3	3.7407e−3	1.8967e−3	9.5512e−4
	1.3875	1.1784	0.9804	0.9798	0.9897	
10^{-6}	4.3816e−2	1.6749e−2	7.4017e−3	3.7489e−3	1.9008e−3	9.5717e−4
	1.3873	1.1781	0.9813	0.9799	0.9898	
10^{-8}	4.3817e−2	1.6750e−2	7.4019e−3	3.7490e−3	1.9008e−3	9.5719e−4
	1.3873	1.1781	0.9813	0.9799	0.9898	
10^{-10}	4.3817e−2	1.6750e−2	7.4019e−3	3.7490e−3	1.9008e−3	9.5719e−4
	1.3873	1.1781	0.9813	0.9799	0.9898	
10^{-12}	4.3817e−2	1.6750e−2	7.4019e−3	3.7490e−3	1.9008e−3	9.5719e−4
	1.3873	1.1781	0.9813	0.9799	0.9898	
$F^{N,M}$	4.3817e−2	1.6750e−2	7.4019e−3	3.7490e−3	1.9008e−3	9.5719e−4
$\rho^{N,M}$	1.3873	1.1781	0.9813	0.9799	0.9898	

4, the maximum pointwise errors and the corresponding rates of convergence are presented for $\mu = 10^{-3}$ and 10^{-9}, respectively, with different values of ε taking the discretization parameters N and M varying with the ratios of 2 in earlier 4 in later. Now the results in these tables clearly show that the maximum pointwise error is uniform with second order of convergence, confirming the spatial order of convergence.

Example 5.2 Our second test example is the following time-delay problem:

$$\begin{cases} \varepsilon u_{xx}(x,t) + \mu(1 + x(1-x) + t^2)u_x(x,t) - (1+5xt)u(x,t) \\ -u_t(x,t) = -u(x,t-\tau) + x(1-x)(e^t - 1), \quad (x,t) \in (0,1) \times (0,2], \\ u(x,t) = 0, \quad (x,t) \in [0,1] \times [-\tau, 0], \\ u(0,t) = 0, \; u(1,t) = 0, \quad t \in (0,2]. \end{cases} \tag{5.1}$$

The exact solution of this problem also is not known. So, the maximum pointwise error and rate of convergence are computed as for the previous example. Then uniform error and uniform rate of convergence are calculated in a similar way. The maximum pointwise error and the rate of convergence are presented in Tables 5 and 6 for $\mu = 10^{-3}$ and 10^{-9}, respectively. In these tables, N and M are increased in the same ratio. In Tables 7 and 8, the maximum pointwise error and the rate of convergence are presented for $\mu = 10^{-3}$ and 10^{-9}, respectively, with different values of ε with the discretization parameters N and M varying with the ratios of 2 and 4 respectively. From these tables we can confirm the spatial order of convergence as proved theoretically.

In Figure 1, log–log plots for maximum pointwise error vs discretization parameter are given for Examples 5.1 and 5.2, respectively, for $N = 32, 64, 128, 256, 512$ and $M =$

Table 3 Maximum errors $F_{\varepsilon,\mu}^{N,M}$ and $F^{N,M}$, and rates of convergence $\rho_{\varepsilon,\mu}^{N,M}$ and $\rho^{N,M}$ using scheme (3.1) for Example 1

$\mu = 10^{-3}$ ε	$N = 32$ $M = 8$	$N = 64$ $M = 32$	$N = 128$ $M = 128$	$N = 256$ $M = 512$	$N = 512$ $M = 2048$	$N = 1024$ $M = 4096$
$10^0 = 1$	7.3340e−4	3.6381e−3	1.1619e−3	3.1204e−4	7.9516e−5	1.9976e−5
	1.0126	1.6467	1.8967	1.9724	1.9930	
10^{-2}	3.5163e−2	6.0222e−3	1.5861e−3	4.0183e −4	1.0079e−4	2.0239e−5
	2.5457	1.9247	1.9809	1.9952	1.9988	
10^{-4}	4.3706e−2	7.3807e−3	1.8967e−3	4.7927e −4	1.2015e−4	2.5220e−5
	2.5660	1.9602	1.9846	1.9961	1.9988	
10^{-6}	4.3471e−2	7.2591e−3	1.8455e−3	4.5342e −4	1.0720e−4	2.6321e−5
	2.5821	1.9757	2.0251	2.0806	1.9987	
10^{-8}	4.3429e−2	7.2480e−3	1.8450e−3	4.5277e −4	1.0686e−4	2.6425e−5
	2.5830	1.9739	2.0268	2.0831	1.9987	
10^{-10}	4.4342e−2	7.2479e−3	1.8450e−3	4.5277e −4	1.0685e−4	2.6425e−5
	2.5830	1.9739	2.0268	2.0831	1.9987	
10^{-12}	4.3429e−2	7.2479e−3	1.8450e−3	4.5277e −4	1.0685e−4	2.6425e−5
	2.5830	1.9739	2.0268	2.0831	1.9987	
$F^{N,M}$	4.3706e−2	7.3807e−3	1.8967e−3	4.7927e−4	1.2015e−4	2.5220e−5
$\rho^{N,M}$	2.5660	1.9602	1.9846	1.9961	1.9988	

Table 4 Maximum errors $F_{\varepsilon,\mu}^{N,M}$ and $F^{N,M}$, and rates of convergence $\rho_{\varepsilon,\mu}^{N,M}$ and $\rho^{N,M}$ using scheme (3.1) for Example 1

$\mu = 10^{-9}$ ε	$N = 32$ $M = 8$	$N = 64$ $M = 32$	$N = 128$ $M = 128$	$N = 256$ $M = 512$	$N = 512$ $M = 2048$	$N = 1024$ $M = 4096$
$10^0 = 1$	7.3402e−2	3.6381e−3	1.1619e−3	3.1203e−4	7.9515e−5	1.9975e−5
	1.0126	1.6467	1.8967	1.9724	1.9930	
10^{-2}	3.5174e−2	6.0235e−3	1.5865e−3	4.0191e −4	1.0081e−4	2.5224e−5
	2.5459	1.9247	1.9809	1.9951	1.9987	
10^{-4}	4.3708e−2	7.3811e−3	1.8968e−3	4.7929e −4	1.2014e−4	3.0057e−5
	2.5660	1.9603	1.9846	1.9961	1.9990	
10^{-6}	4.3816e−2	7.4018e−3	1.9008e−3	4.8029e −4	1.2039e−4	3.0119e−5
	2.5655	1.9612	1.9846	1.9961	1.9990	
10^{-8}	4.3817e−2	7.4019e−3	1.9009e−3	4.8030e −4	1.2039e−4	3.0120e−5
	2.5655	1.9612	1.9846	1.9961	1.9990	
10^{-10}	4.3817e−2	7.4019e−3	1.9009e−3	4.8030e −4	1.2039e−4	3.0120e−5
	2.5655	1.9612	1.9846	1.9961	1.9990	
10^{-12}	4.3817e−2	7.4019e−3	1.9009e−3	4.8030e −4	1.2039e−4	3.0120e−5
	2.5655	1.9612	1.9846	1.9961	1.9990	
$F^{N,M}$	4.3817e−2	7.4019e−3	1.9009e−3	4.8030e−4	1.2039e−4	3.0120e−5
$\rho^{N,M}$	2.5655	1.9612	1.9846	1.9961	1.9990	

Table 5 Maximum errors $F_{\varepsilon,\mu}^{N,M}$ and $F^{N,M}$, and rates of convergence $\rho_{\varepsilon,\mu}^{N,M}$ and $\rho^{N,M}$ using scheme (3.1) for Example 2

$\mu = 10^{-3}$ ε	$N = 32$ $M = 8$	$N = 64$ $M = 16$	$N = 128$ $M = 32$	$N = 256$ $M = 64$	$N = 512$ $M = 128$	$N = 1024$ $M = 256$
$10^0 = 1$	7.6576e−4	3.6191e−4	1.7499e−4	8.5923e−5	4.2625e−5	2.1245e−5
	1.0812	1.0483	1.0262	1.0113	1.0045	
10^{-2}	8.8707e−3	4.1142e−3	2.0098e−3	9.9812e −4	4.9790e−4	2.4871e−4
	1.1084	1.0335	1.0098	1.0033	1.0013	
10^{-4}	1.1161e−2	5.1087e−3	2.4749e−3	1.2214e −3	6.0706e−4	3.0264e−4
	1.1274	1.0455	1.0188	1.0086	1.0042	
10^{-6}	1.1008e−2	5.0450e−3	2.4437e−3	1.2073e −3	6.0036e−4	2.9941e−4
	1.1257	1.0457	1.0172	1.0079	1.0036	
10^{-8}	1.0941e−2	5.0426e−3	2.4442e−3	1.2071e −3	6.0016e−4	2.9929e−4
	1.1175	1.0447	1.0178	1.0080	1.0037	
10^{-10}	1.0940e−2	5.0428e−3	2.4442e−3	1.2071e −3	6.0016e−4	2.9929e−4
	1.1173	1.0448	1.0178	1.0080	1.0037	
10^{-12}	1.0940e−2	5.0428e−3	2.4442e−3	1.2071e −3	6.0016e−4	2.9929e−4
	1.1173	1.0448	1.0178	1.0080	1.0037	
$F^{N,M}$	1.1161e−2	5.1087e−3	2.4749e−3	1.2214e−3	6.0706e−4	3.0264e−4
$\rho^{N,M}$	1.1274	1.0455	1.0188	1.0086	1.0042	

Table 6 Maximum errors $F_{\varepsilon,\mu}^{N,M}$ and $F^{N,M}$, and rates of convergence $\rho_{\varepsilon,\mu}^{N,M}$ and $\rho^{N,M}$ using scheme (3.1) for Example 2

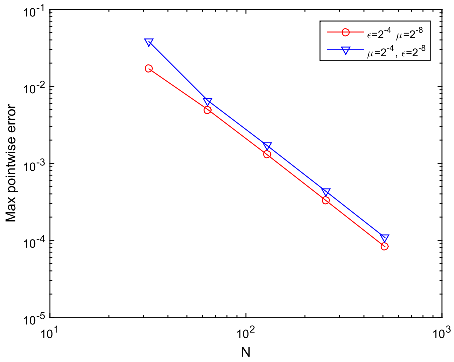
$\mu = 10^{-9}$ ε	$N = 32$ $M = 8$	$N = 64$ $M = 16$	$N = 128$ $M = 32$	$N = 256$ $M = 64$	$N = 512$ $M = 128$	$N = 1024$ $M = 256$
$10^0 = 1$	7.6573e−4	3.6189e−4	1.7498e−4	8.5918e−5	4.2622e−5	2.1244e−5
	1.0812	1.0483	1.0261	1.0113	1.0045	
10^{-2}	8.8197e−3	4.0953e−3	1.9997e−3	9.5642e−3	4.9551e−4	2.4752e−4
	1.1067	1.0341	1.0096	1.0031	1.0013	
10^{-4}	1.1053e−2	5.0755e−3	2.4578e−3	1.2132e−3	6.0309e−4	3.0069e−4
	1.1228	1.0461	1.0184	1.0084	1.0040	
10^{-6}	1.1046e−2	5.0765e−3	2.4625e−3	1.2161e−3	6.0456e−4	3.0141e−4
	1.1216	1.0437	1.0178	1.0083	1.0041	
10^{-8}	1.1100e−2	5.0838e−3	2.4627e−3	1.2161e−3	6.0457e−4	3.0141e−4
	1.1265	1.0456	1.0179	1.0082	1.0041	
10^{-10}	1.1093e−2	5.0782e−3	2.4639e−3	1.2162e−3	6.0457e−4	3.0141e−4
	1.1272	1.0433	1.0185	1.0084	1.0041	
10^{-12}	1.1092e−2	5.0775e−3	2.4640e−3	1.2162e−3	6.0457e−4	3.0142e−4
	1.1272	1.0433	1.0185	1.0084	1.0041	
$F^{N,M}$	1.1100e−2	5.0838e−3	2.4640e−3	1.2162e−3	6.0457e−4	3.0142e−4
$\rho^{N,M}$	1.1265	1.0449	1.0185	1.0084	1.0041	

Table 7 Maximum errors $F_{\varepsilon,\mu}^{N,M}$ and $F^{N,M}$, and rates of convergence $\rho_{\varepsilon,\mu}^{N,M}$ and $\rho^{N,M}$ using scheme (3.1) for Example 2

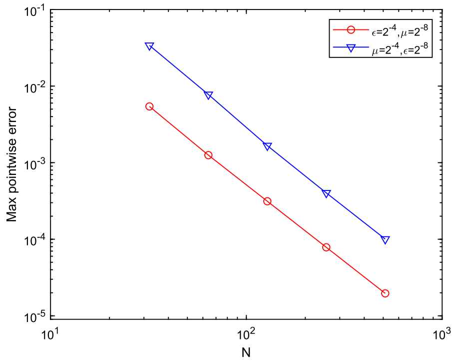
$\mu = 10^{-3}$ ε	$N = 32$ $M = 8$	$N = 64$ $M = 32$	$N = 128$ $M = 128$	$N = 256$ $M = 512$	$N = 512$ $M = 2048$	$N = 1024$ $M = 4096$
$10^0 = 1$	7.6577e−4	1.8173e−4	4.4719e−5	1.1158e −5	2.7887e−6	6.9712e−7
	2.0751	2.0228	2.0028	2.0005	2.0001	
10^{-2}	8.8707e−3	2.0119e−3	4.9889e−4	1.2455e −4	3.1129e−5	7.7817e−6
	2.1404	2.0118	2.0019	2.0004	2.0001	
10^{-4}	1.1161e−2	2.4750e−3	6.0733e−4	1.5120e −4	3.7759e−5	9.4372e−6
	2.1729	2.0269	2.0060	2.0015	2.0003	
10^{-6}	1.1008e−2	2.4058e−3	6.4802e−4	2.7303e −4	8.7917e−5	2.4212e−5
	2.1941	1.8924	1.2469	1.6348	1.8604	
10^{-8}	1.0942e−2	2.4109e−3	8.9349e−4	3.3001e −4	9.5237e−5	2.3572e−5
	2.1821	1.4320	1.4369	1.7929	2.0144	
10^{-10}	1.0941e−2	2.4109e−3	8.7733e−4	3.2286e −4	9.2375e−5	2.1949e−5
	2.1820	1.4585	1.4422	1.8054	2.0733	
10^{-12}	1.0941e−2	2.4109e−3	8.7716e−4	3.2278e −4	9.2344e−5	2.1932e−5
	2.1820	1.4586	1.4422	1.8055	2.0739	
$F^{N,M}$	1.1161e−2	2.4750e−3	8.9349e−4	3.3001e−4	9.5237e−5	2.3572e−5
$\rho^{N,M}$	2.1729	1.4699	1.4369	1.7929	2.0144	

Table 8 Maximum errors $F_{\varepsilon,\mu}^{N,M}$ and $F^{N,M}$, and rates of convergence $\rho_{\varepsilon,\mu}^{N,M}$ and $\rho^{N,M}$ using scheme (3.1) for Example 2

$\mu = 10^{-9}$ ε	$N = 32$ $M = 8$	$N = 64$ $M = 32$	$N = 128$ $M = 128$	$N = 256$ $M = 512$	$N = 512$ $M = 2048$	$N = 1024$ $M = 4096$
$10^0 = 1$	7.6573e−4	1.8172e−4	4.4716e−5	1.1157e −5	2.7884e−6	6.9707e−7
	2.0751	2.0228	2.0028	2.0005	2.0001	
10^{-2}	8.8197e−3	2.0026e−3	4.9636e−4	1.2393e −4	3.0976e−5	7.7434e−6
	2.1388	2.0124	2.0018	2.0004	2.0001	
10^{-4}	1.1053e−2	2.4577e−3	6.0306e−4	1.5012e −4	3.7490e−5	9.3702e−6
	2.1691	2.0269	2.0062	2.0015	2.0004	
10^{-6}	1.1046e−2	2.4546e−3	6.0440e−4	1.5048e −4	3.7581e−5	9.3928e−6
	2.1700	2.0219	2.0059	2.0016	2.0004	
10^{-8}	1.1100e−2	2.4588e−3	6.0443e−4	1.5046e −4	3.7582e−5	9.3932e−6
	2.1745	2.0243	2.0062	2.0013	2.0004	
10^{-10}	1.1093e−2	2.4555e−3	6.0458e−4	1.5049e−4	3.7583e−5	9.3931e−6
	2.1756	2.0220	2.0062	2.0016	2.0004	
10^{-12}	1.1092e−2	2.4551e−3	6.0458e−4	1.5049e−4	3.7583e−5	9.3931e−6
	2.1757	2.0218	2.0062	2.0016	2.0004	
$F^{N,M}$	1.1100e−2	2.4588e−3	6.0458e−4	1.5049e−4	3.7583e−5	9.3931e−6
$\rho^{N,M}$	2.1745	2.0239	2.0062	2.0016	2.0004	



(a) For Example 5.1



(b) For Example 5.2

Fig. 1 Log–log plots for maximum error vs N

$8,\ 32,\ 128,\ 512,\ 2048$ with $\varepsilon = 2^{-4},\ \mu = 2^{-8}$ and $\mu = 2^{-4},\ \varepsilon = 2^{-8}$. The slope of these plots also confirm the second-order uniform convergence of the proposed numerical method.

For both the above examples, we observe that the main contribution to the global error comes from the time discretization. So, to compare the accuracy of our proposed method with the existing method (Govindarao et al. 2019), we consider a third test example in which the main contribution to the global error comes from the spatial discretization.

Example 5.3 Our third test example is the following time-delay problem:

$$\begin{cases} \varepsilon u_{xx}(x,t) + \mu u_x(x,t) - u(x,t) - u_t(x,t) = -u(x,t-\tau) + f(x,t), & (x,t) \in (0,1) \times (0,2], \\ u(x,t) = \varphi_b(x,t), \quad (x,t) \in \ [0,1] \times [-\tau, 0], \\ u(0,t) = \varphi_l(t), \ u(1,t) = \varphi_r(t), \quad t \in (0,2], \end{cases} \tag{5.2}$$

where f, φ_b, φ_l, and φ_r are such that

$$u(x,t) = \frac{t}{\varepsilon - \mu - 1}\left[-e^{(1-x)} + \frac{e - e^{(\mu-m)/2\varepsilon}}{1 - e^{-m/2\varepsilon}} e^{-(\mu+m)x/2\varepsilon} + \frac{e^{1-(\mu+m)/2\varepsilon} - 1}{e^{-m/\varepsilon} - 1} e^{(\mu-m)/(1-x)/2\varepsilon}\right]$$

with $m = \sqrt{\mu^2 + 4\varepsilon}$.

Since the exact solution of this problem is known, the maximum pointwise error is calculated by

$$F_{\varepsilon,\mu}^{N,M} = \max_{i,j} |(U^{N,M})_i^j - u(x_i, t_j)|.$$

We then calculate uniform error and uniform rate of convergence by

$$F^{N,M} = \max_{\varepsilon,\mu} F_{\varepsilon,\mu}^{N,M} \quad \text{and} \quad \rho^{N,M} = \log_2 \frac{F^{N,M}}{F^{2N,2M}}.$$

Corresponding to $\mu = 10^{-2}$ and $\mu = 10^{-8}$ we consider $\varepsilon \in S_\varepsilon = \{10^{-4}, 10^{-5}, \ldots, 10^{-15}\}$. Table 9 shows the comparison of the proposed method and the method of Govindarao et al. (2019) in terms of uniform errors and uniform rates of convergence for Example 5.3. These results verify that the proposed method gives more accurate results with high rate of convergence than the method given in Govindarao et al. (2019).

Table 9 Comparison of the proposed method with the method in Govindarao et al. (2019) in terms of uniform errors $F^{N,M}$ and uniform rates of convergence $\rho^{N,M}$ for Example 3

$\varepsilon \in S_\varepsilon$	Method	$N = 32$ $M = 8$	$N = 64$ $M = 16$	$N = 128$ $M = 32$	$N = 256$ $M = 64$	$N = 512$ $M = 128$	$N = 1024$ $M = 256$
	Method Govindarao et al. (2019)	6.0753e−01	4.2471e−01	2.6525e−01	1.6023e−01	9.2668e−02	5.2422e−02
$\mu = 10^{-2}$		0.5165	0.6792	0.7272	0.7899	0.8219	
	Proposed method	5.6720e−01	2.0292e−01	6.1946e−02	2.0130e−02	6.3260e−03	1.9526e−03
		1.4829	1.7119	1.6217	1.6700	1.6959	
	Method Govindarao et al. (2019)	6.1057e−01	4.2782e−01	2.6757e−01	1.6177e−01	9.3592e−02	5.2952e−02
$\mu = 10^{-8}$		1.3873	1.1782	0.9814	0.9799	0.9898	
	Proposed method	5.7190e−01	2.0463e−01	6.2406e−02	2.0262e−02	6.3579e−03	1.9567e−03
		1.4828	1.7132	1.6229	1.6722	1.7002	

6 Conclusion

We proposed a finite difference numerical method for singularly perturbed delay partial differential Eq. (1.1) with two perturbation parameters ε and μ. On a rectangular mesh consisting of a layer adapted piecewise uniform Shishkin mesh in spatial direction and a uniform mesh in temporal direction, we discretized the problem with the difference operator comprising of an implicit Euler method for time and a hybrid scheme for space consisting of central difference, upwind and midpoint operators. It is shown through the truncation error and barrier function approach that the proposed numerical method is convergent with first order in time and almost second-order in space independent of both the perturbation parameters. Numerical results confirm the theoretical convergence results and demonstrate the efficiency of the proposed method

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