# Chapter 6

# Exact Solution of a Two Phase Stefan Problem Including Moving Phase Change Material

## 6.1 Introduction

Phase-change problems or Stefan problems encounter in many aspects of natural and industrial phenomena. These problems are particular cases of the moving boundary problems, where prior information about the location of the moving boundary is not known but one has to calculate it as it is a part of solution. Due to many practical applications in the field of science, engineering and industries, Stefan problems have been attracting more interest of many researchers for over a century. The occurrence of Stefan problems can be seen in many specific fields of engineering and physical science (Tarzia, 1990; Gotz and Zaltzmans, 1995; Soni, 1999; Swenson, 2000; Yi, 2002; Mitchell and Vynnycky, 2014; Li and Sun, 2015; Fan et al., 2015; Mitchell and Vynnycky, 2016; Bollati et al., 2018; Li et al., 2018) such as melting or solidification process, crystal growth process, thermal energy storage process, metal casting, shoreline problem, and in many more areas. Stefan problems are inherently non-linear because of the presence of unknown moving boundary in the problem.

In general, classical Stefan problems (Crank, 1987; Gupta, 2017) involve constant thermal coefficients which do not always occur with the many materials. By taking this fact into consideration, Stefan problems have been modified in different ways by many researchers (Rogers, 1988; Oliver and Sunderland, 1987; Ramos et al., 1994; Broadbridge and Pincombe, 1996; Briozzo et al., 2007; Singh et al., 2007; Briozzo and Natale, 2015) to include new physical behaviour in the mathematical models of the problem. Besides these works, Ceretani et al. (2018); Kumar et al. (2018a, 2018b) are also presented few recent work associated to Stefan problems with variable thermal coefficients. In the field of phase change problem, the establishment of the exact solution to a complicated Stefan problem is a remarkable area of research due to presence of moving boundary and its non-linearity nature. From the literature, some exact solutions to the Stefan problems are presented by Solomon et al. (1982), Zubair and Chaudhry (1994), Voller et al. (2004), Trueba and Voller (2010), Zhou et al. (2018), Bollati and Tarzia (2018), Ceretani et al. (2018), Khalida et al. (2019).

In many physical processes, the phase change material is allowed to move when phase change occurs (Fila and Souplet, 2001), and this physical situation is not included in most of the available literatures. Recently, Turkyilmazoglu (2018) established analytical solutions for some mathematical models of melting and solidification processes including moving phase change material (PCM) and constant thermal coefficients. Singh et al. (2018a, 2018b) presented a mathematical model of one phase Stefan problem that simultaneously includes variable thermal coefficients and moving phase change material. In this chapter, a two phase Stefan problem with variable thermal coefficients and moving phase change material is presented that includes Dirichlets as well as convective types of boundary conditions. The thermal coefficients are assumed as the linear functions of temperature. Here, we establish analytical solution to the problem with the aid of similarity variables.

## 6.2 Mathematical Model of the Problem

Let us consider that the initial temperature of the phase change material in a semi-infinite domain $0 \le x < \infty$ is $T_s$ which is less than its melting temperature $T_m$. Here, we assume that the thermal conductivity and specific heat linearly depend on temperature. At time $t$ = 0, it is also assumed that the surface $x$ = 0 is subjected to either a temperature $T_w > T_m$ or a convective boundary condition. As time proceeds, the melting process starts and a moving interface between liquid and solid is formed which propagates in the positive $x$ direction. The mathematical model of this two-phase Stefan problem is given below:

$$\rho c_1(T_1)\left(T_{1t} + uT_{1x}\right) = \left(k_1(T_1)T_{1x}\right)_x, \qquad 0 < x < s(t),\ t > 0, \tag{6.1}$$

$$\rho c_2(T_2)\left(T_{2t} + uT_{2x}\right) = \left(k_2(T_2)T_{2x}\right)_x, \qquad s(t) < x,\ t > 0, \tag{6.2}$$

$$T_1(0,t) = T_w > T_m, \quad t > 0, \tag{6.3a}$$

$$k_1(T_1)T_{1x}(0,t) = -c\,t^{-\frac{1}{2}}(T_1(0,t) - T_w), \quad t > 0, \tag{6.3b}$$

$$T_1(s(t),t) = T_m,\ t > 0, \tag{6.4}$$

$$T_2(s(t),t) = T_m,\ t > 0, \tag{6.5}$$

$$T_2(x,0) = T_2(\infty,t) = T_s < T_m,\ t > 0, \tag{6.6}$$

$$k_1(T_m)T_{1x}(s(t)^-,t) - k_2(T_m)T_{2x}(s(t)^+,t) = -\rho l s'(t),\ \ t > 0, \tag{6.7}$$

$$s(0) = 0, \tag{6.8}$$

where $T_1$, $T_2$ are the temperatures in the liquid and solid regions, $x$ is the position, $t$ is the time, $T_m$ is the constant melting temperature, $T_w > T_m$ is the constant temperature at $x = 0$, $T_s < T_m$ is the constant temperature at $t = 0$, $\rho$ is the density, $s(t)^-, s(t)^+$ are the positions of interface in the liquid and solid regions, $l$ is the latent heat, $c$ is the heat flux constant.

The variable thermal conductivity $k_1(T_1),\ k_2(T_2)$ and specific heat $c_1(T_1), c_2(T_2)$ are defined as follows:

$$k_1(T_1) = k_{01}\left(1+\alpha_1 \frac{T-T_w}{\Delta T_w}\right), k_2(T_2) = k_{02}\left(1+\alpha_2 \frac{T-T_s}{\Delta T_s}\right) \tag{6.9}$$

and

$$c_1(T_1) = c_{01}\left(1+\alpha_1 \frac{T-T_w}{\Delta T_w}\right), c_2(T_2) = c_{02}\left(1+\alpha_2 \frac{T-T_s}{\Delta T_s}\right), \tag{6.10}$$

where $k_{01}, k_{02}, c_{01}, c_{02}$ are positive constants, $\Delta T_w (= T_m - T_w)$, $\Delta T_s (= T_m - T_s)$ are the reference temperatures and $\alpha_i > 0;\ i = 1, 2$.

## 6.3 Solution of the Problem

First of all, we consider the following transformation:

$$\theta_1(\eta_1) = \frac{T_1 - T_w}{\Delta T_w}, \quad \theta_2(\eta_2) = \frac{T_2 - T_s}{\Delta T_s}, \tag{6.11}$$

$$\left.\begin{array}{l} \eta_1 = \dfrac{x}{2\sqrt{\nu_1 t}},\ \eta_2 = \dfrac{x}{2\sqrt{\nu_2 t}},\ Ste_1 = -\dfrac{c_{01}\Delta T_w}{l},\ Ste_2 = \dfrac{c_{02}\Delta T_s}{l}, \\ \nu = \sqrt{\dfrac{\nu_1}{\nu_2}},\quad \nu_1 = \dfrac{k_{01}}{\rho c_{01}},\quad \nu_2 = \dfrac{k_{02}}{\rho c_{02}},\quad u = Pe\sqrt{\dfrac{\nu_1}{t}} \end{array}\right\} \tag{6.12}$$

and $$s(t) = 2\lambda\sqrt{v_1 t}\ , \tag{6.13}$$

where $\lambda$ is a constant parameter, $Pe$ is the Peclet number and $Ste_1, Ste_2$ are Stefan numbers.

With the aid of Eqs. (6.11)-(6.13), the Eq. (6.1) converts to the following ODE:

$$(1+\alpha_1\,\theta_1(\eta_1))\big(\theta_1''(\eta_1) - 2Pe\theta_1'(\eta_1)\big) + \big(2\eta_1 + \alpha_1\,\theta_1'(\eta_1) + 2\alpha_1\,\eta_1\,\theta_1(\eta_1)\big)\theta_1'(\eta_1) = 0. \tag{6.14}$$

Eq. (6.2) becomes

$$(1+\alpha_2\,\theta_2(\eta_2))\big(\theta_2''(\eta_2) - 2Pev\theta_2'(\eta_2)\big) + \big(2\eta_2 + \alpha_2\,\theta_2'(\eta_2) + 2\alpha_2\,\eta_2\,\theta_2(\eta_2)\big)\theta_2'(\eta_2) = 0 \tag{6.15}$$

and the boundary conditions (6.3a), (6.3b), (6.4), (6.5) and (6.6) turn into, respectively

$$\theta_1(0) = 0, \tag{6.16a}$$

$$\theta_1'(0) + \alpha_1\theta_1(0)\,\theta_1'(0) - \gamma\,\theta_1(0) = 0\,, \tag{6.16b}$$

$$\theta_1(\lambda) = 1, \tag{6.17}$$

$$\theta_2(\lambda v) = 1, \tag{6.18}$$

$$\theta_2(\infty) = 0. \tag{6.19}$$

Also, the Eqs. (6.11)-(6.13) and the Stefan condition (6.7) give rise to

$$Ste_1(1+\alpha_1)\frac{d\theta_1}{d\eta_1} + \frac{Ste_2}{v}(1+\alpha_2)\frac{d\theta_2}{d\eta_2} = 2\lambda\ , \tag{6.20}$$

where $\gamma = 2Bi > 0,\ Bi = \dfrac{c_{01}\sqrt{v_1}}{k_{01}}$ (generalized Biot number).

Now, we present the solutions of the problem in the following two cases:

**6.3.1 Case1:** Solution of the problem with Dirichlet boundary condition (6.3a)

The solution of Eq. (6.14), with the conditions (6.16a) and (6.17) is given by

$$\theta_1(\eta_1) = \frac{1}{\alpha_1}\left(\left(1+\frac{\alpha_1(2+\alpha_1)(erf\,(Pe)-erf\,(Pe-\eta_1))}{(erf\,(Pe)-erf\,(Pe-\lambda))}\right)^{\frac{1}{2}}-1\right), \tag{6.21}$$

where $erf\,(.)$ is well known error function.

The solution of Eq. (6.15) with the conditions (6.18) and (6.19) is calculated as

$$\theta_2(\eta_2) = \frac{1}{\alpha_2}\left(\left(1+\frac{\alpha_2(2+\alpha_2)(1+erf\,(Pe\,v-\eta_2))}{(1+erf\,(Pe-\lambda)v)}\right)^{\frac{1}{2}}-1\right). \tag{6.22}$$

In this case, the location of the moving interface is determined by $s(t) = 2\lambda\sqrt{v_1 t}$, where $\lambda$ can be calculated by the following transcendental equation:

$$f_1(\lambda) = -\frac{(2+\alpha_1)Ste_1}{e^{(Pe-\lambda)^2}(erf\,(Pe)-erf\,(Pe-\lambda))}+\frac{(2+\alpha_2)Ste_2}{e^{(Pe-\lambda)^2 v^2}v(1+erf\,(Pe-\lambda)v)}+2\sqrt{\pi}\lambda = 0\,. \tag{6.23}$$

From Eq. (6.23), it is obvious that $f_1(\lambda)$ is defined and continuous on $(0,\infty)$ and $\lim_{\lambda\to 0^+} f_1(\lambda) < 0$, and $\lim_{\lambda\to\infty} f_1(\lambda) = \infty$, for all positive parameters $(\alpha_1, \alpha_2, Pe, Ste_1, Ste_2$ and $v)$ which show that there always exists at least one solution of Eq. (6.23).

**6.3.2 Case 2:** Solution of the problem with convective boundary condition (6.3b)

With the help of Eqs. (6.15b) and (6.16), the solution of Eq. (6.14) is given by

$$\theta_1(\eta_1)=\frac{1}{\alpha_1}\left(\left\{(1+\alpha_1)^2+\frac{1}{2}e^{Pe^2}\sqrt{\pi}\left(erf\,(Pe-\eta_1)-erf\,(Pe-\lambda)\right)\right.\right.$$
$$\left.\left.(2\gamma+e^{Pe^2}\sqrt{\pi}\,\gamma^2\left(erf\,(Pe)-erf\,(Pe-\lambda)\right)-A(\lambda))\right\}^{1/2}-1\right), \tag{6.24}$$

where $A(\lambda)=\left(4\alpha_1\gamma^2(2+\alpha_1)+\left(2\gamma+e^{Pe^2}\sqrt{\pi}\,\gamma^2\left(erf\,(Pe)-erf\,(Pe-\lambda)\right)\right)^2\right)^{\frac{1}{2}}.$

The solution of Eq. (6.15), i.e. $\theta_2(\eta_2)$ with the conditions (6.18) and (6.19) will remain same as given in Eq. (6.22).

In this case, moving interface constant $\lambda$ can be calculated from the following transcendental equation:

$$f_2(\lambda)=\frac{Ste_1}{2\alpha_1\,e^{(\lambda-2Pe)\lambda}}\left(2\gamma+e^{Pe^2}\sqrt{\pi}\,\gamma^2\left(erf\,(Pe)-erf\,(Pe-\lambda)\right)-A(\lambda)\right)$$
$$+\frac{(2+\alpha_2)Ste_2}{e^{(Pe-\lambda)^2v^2}\sqrt{\pi}\,v(1+erf\,(Pe-\lambda)v)}+2\lambda=0. \tag{6.25}$$

From Eq. (6.25), it is obvious that $f_2(\lambda)$ is defined and continuous on $(0,\infty)$ and $\lim_{\lambda\to 0^+} f_2(\lambda)<0,$ and $\lim_{\lambda\to\infty} f_2(\lambda)=\infty,$ for all positive parameters which indicate that there exists at least one solution of Eq. (6.25).

## 6.4 Discussion

In this study, Wolfram Research (8.0.1) software is used for all the computational work and the numerical results are presented through the different figures.

Figs. 6.1 and 6.2 are plotted for $s(t)$ against $t$ for different values of $\alpha_1\,(\alpha_1=0.2,\ 0.5,\ 1.0)$ and $\alpha_2\,(\alpha_2=0.2,\ 1.0,\ 2.0)$, respectively for Dirichlet boundary condition at $Pe=2,\ v=v_1=0.5$ and $Ste_1=Ste_2=0.5$. Fig. 6.1 portrays that the velocity of moving interface enhances when we increase $\alpha_1$ while Fig. 6.2 shows the reverse behaviour, i.e. velocity of the interface decreases with the increment in $\alpha_2$. Hence, the increment in $\alpha_1$ improves the melting process and the growth in $\alpha_2$ leads the melting process slow.

Figs. 6.3 and 6.4 are graphed for the tracking of $s(t)$ versus $t$ for various values of $\alpha_1\,(\alpha_1 = 1.0,\ 0.5,\ 0.2)$ and $\alpha_2\,(\alpha_2 = 0.2,\ 0.5,\ 1.0)$, respectively in the case of convective boundary condition at $Pe = 2,\ v = v_1 = 0.5,\ \gamma = 1$ and $Ste_1 = Ste_2 = 0.5$. These figures demonstrate the similar behaviour of the melting process with respect to the velocity of the moving interface as we observed in case of the Dirichlet boundary condition at the fixed face $x = 0$. Fig. 6.5 depicts the impact of Biot number on the trajectory of moving interface for the different Biot number at $\alpha_1 = \alpha_2 = 0.5,\ Pe = 2, v = v_1 = 0.5$ and $Ste_1 = Ste_2 = 1.5$. From this figure, it is clear that the melting becomes fast when we increase the Biot number in the case of convective boundary condition.

Figs. 6.6 and 6.7 represent the effect of Peclet number on the location of moving interface s($t$) for case 1 and case 2, respectively at $\alpha_1 = \alpha_2 = 0.5,$ $v = 0.5,\ v_1 = 0.5,\ Ste_1 = 0.2$ and $Ste_2 = 0.2$. From these figures, it is found that the movement of s($t$) or melting process improves if we increase the Peclet number in the both cases. It is also observed that the effect of *Pe* on melting process for the convective boundary condition is more pronounced than the case 1.

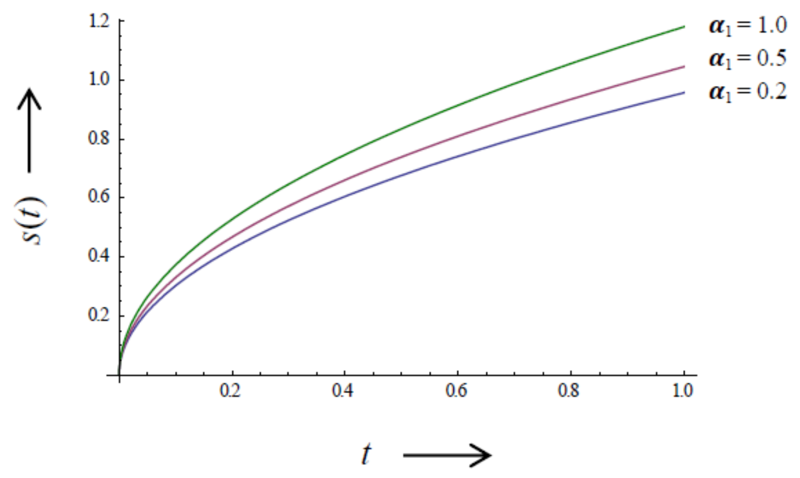


Fig.6.1. Plot of tracking of phase front $s(t)$ vs. $t$ for different $\alpha_1$ at $\alpha_2 = 0.5, Pe = 2, v = v_1 = 0.5$ and $Ste_1 = Ste_2 = 0.5$.

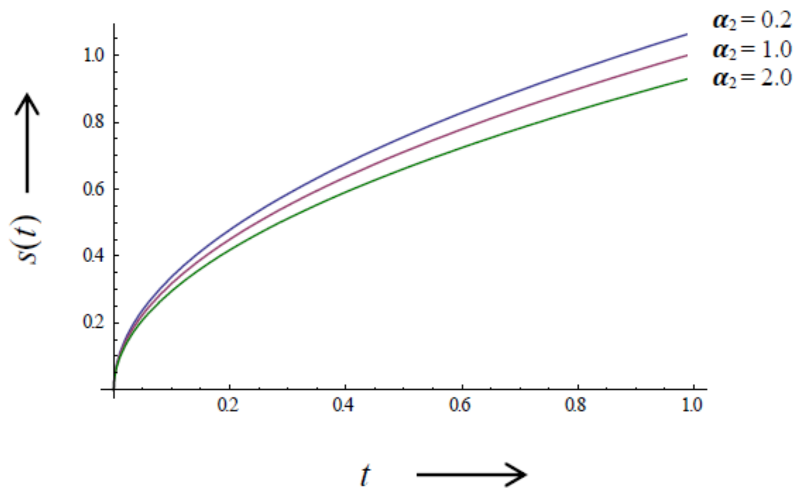


Fig.6.2. Plot of tracking of phase front $s(t)$ vs. $t$ for different $\alpha_2$ at $\alpha_1 = 0.5, Pe = 2, v = v_1 = 0.5$ and $Ste_1 = Ste_2 = 0.5$.

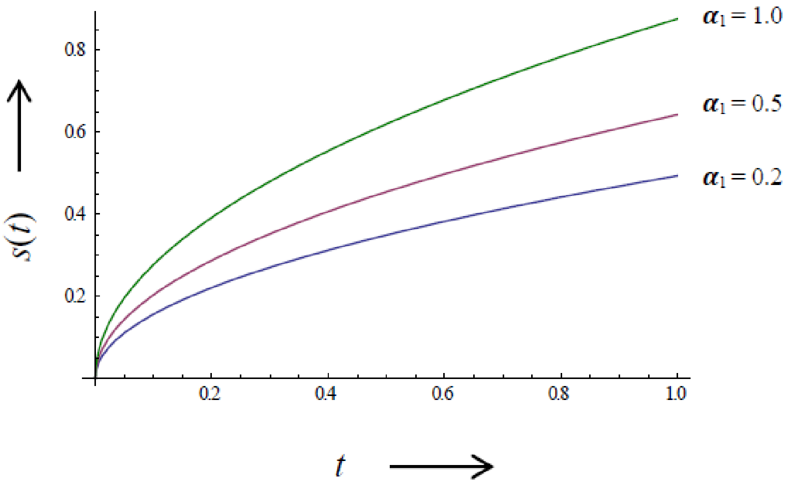


Fig.6.3. Plot of tracking of phase front $s(t)$ vs. $t$ for different $\alpha_1$ at $\alpha_2 = 0.5, Pe = 2, v = v_1 = 0.5, \gamma = 1$ and $Ste_1 = Ste_2 = 0.5$.

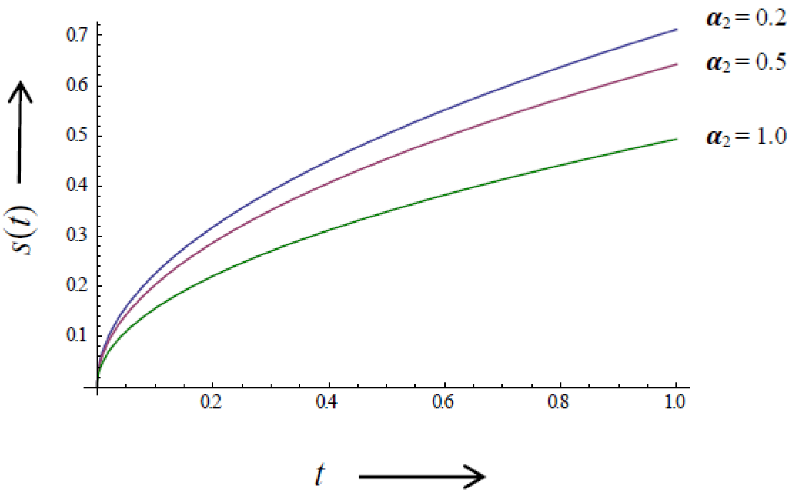


Fig.6.4. Plot of tracking of phase front $s(t)$ vs. $t$ for different $\alpha_2$ at $\alpha_1 = 0.5, Pe = 2, v = v_1 = 0.5, \gamma = 1$ and $Ste_1 = Ste_2 = 0.5$.

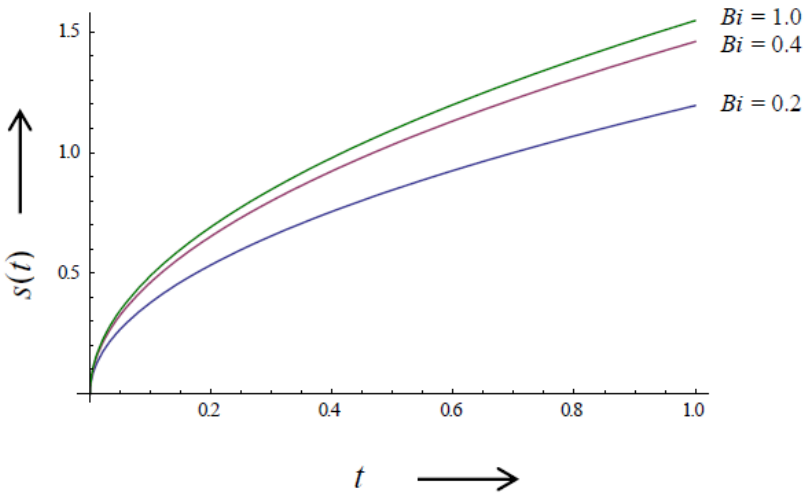


Fig.6.5. Plot of tracking of phase front $s(t)$ vs. $t$ for different $Bi$ at $\alpha_1 = \alpha_2 = 0.5, Pe = 2, v = v_1 = 0.5$ and $Ste_1 = Ste_2 = 1.5$.

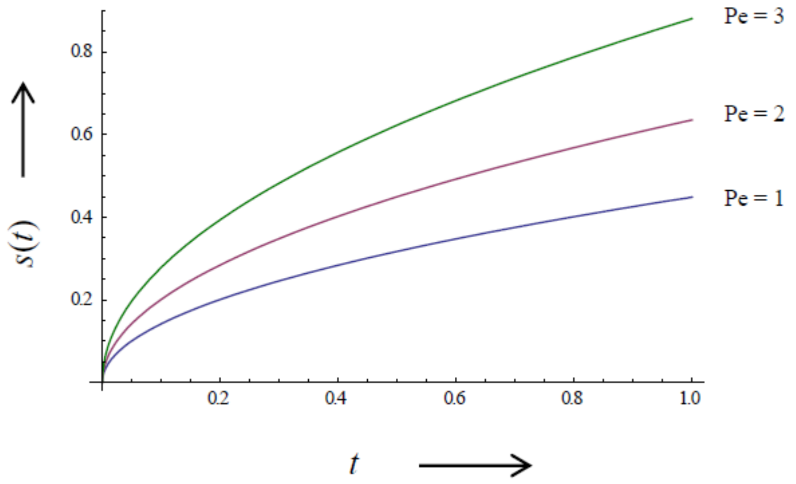


Fig.6.6. Plot of tracking of phase front $s(t)$ vs. $t$ for different $Pe$ at $\alpha_1 = \alpha_2 = v = v_1 = 0.5$ and $Ste_1 = Ste_2 = 0.2$.

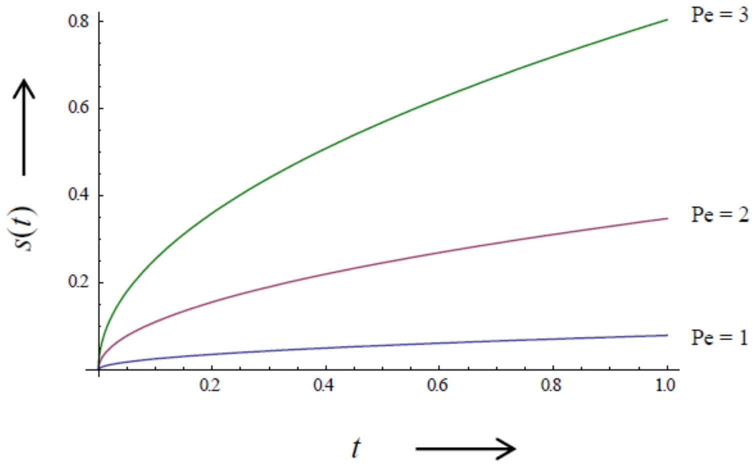


Fig.6.7. Plot of tracking of phase front $s(t)$ vs. $t$ for different $Pe$ at $\alpha_1 = \alpha_2 = v = v_1 = 0.5$, $\gamma = 2$ and $Ste_1 = Ste_2 = 0.2$.

## 6.5 Conclusion

In this chapter, we establish the exact solution to the two-phase Stefan problem with variable thermal coefficients and moving phase change material by taking Dirichlet as well as convective boundary condition separately. From this study, it is seen the location of moving interface is directly proportional to the square root of time as we have found in the classical Stefan problems (Crank, 1987). For both the cases, it is also observed that the parameters $\alpha_1$, $\alpha_2$, $Bi$ and $Pe$ rapidly influence the melting process, i.e. the melting process becomes fast as we raise the values of $\alpha_1$, $Bi$ and $Pe$ whereas this process becomes slow with the increment in the value of $\alpha_2$.

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