Chapter 4

A Stefan Problem with Temperature and Time Dependent Thermal Conductivity

4.1 Introduction

It is know that many processes like melting, freezing, sediment mass transport, tumour growth, etc. in the field of science and industry involve moving boundary/boundaries, and these problems are referred as moving boundary problems (or Stefan problems). Initially, the Stefan problems are restricted to heat-transfer problem and the formulations of these problems are developed for constant thermal properties (Crank, 1984). But, the Stefan problems are not only limited to heat-transfer problem with constant thermal properties. Some Stefan problems with different thermal properties and other diffusion controlled transport systems are discussed by Carslaw and Jaeger (1959), Hill (1986), Voller et al. (2004), Zhou and Li-jiang (2015).

From the literature (Cho and Sunderland, 1974; Oliver and Sunderland, 1987; Briozzo et al., 2007; Briozzo and Natale, 2015), it can be seen that moving boundary problems with temperature dependent thermal conductivity have been a fruitful research in the field of heat transfer. In 2017, Briozzo and Natale (2017) considered the temperature-dependent thermal conductivity in study of the supercooled one-phase Stefan problem for a semi-infinite material. Recently, Ceretani et al. (2018) discussed the similarity solutions for a one-phase Stefan problem with temperature-dependent thermal conductivity and a Robin condition at a fixed face. Voller and Falcini (2013) presented a one phase Stefan problem with diffusivity as a function of space and discussed an exact solution for it. In context of time-dependent thermal conductivity, Hussein and Lesnic (2014) discussed the identification of time-dependent thermal conductivity of an orthotropic rectangular conductor. Recently, Huntul and Lesnic (2017) also discuss an inverse problem of determining the time-dependent thermal conductivity and the transient temperature satisfying the heat equation with initial data. Motivated by these works, we consider a one phase Stefan problem with time and temperature dependent thermal conductivity of the form

$$k(T) = k_0\left(1+\beta\left(\frac{T-T_m}{\Delta T}\right)t^{-\frac{\alpha}{2}}\right), \tag{4.1}$$

where T is the temperature distribution, t is the time, ΔT is the reference temperature, β and α are the positive constants.

Due to presence of moving boundary/boundaries or unknown domain, the moving boundary problems are nonlinear in nature even in its simplest form. If thermal conductivity is time and temperature dependent then the problem becomes more complicated to get its exact solution. In general, scaling invariance analysis and similarity variables (Briozzo et al., 2007; Ceretani et al., 2018; Fazio, 2013) play an important role for getting the exact solutions of these problems. In our study, we have also used the appropriate similarity variables to convert the governing system of partial differential equations into another system that includes ordinary differential equations with its conditions. For a particular case, the exact solution of the problem is established. We have also discussed the existence and uniqueness of the obtained exact solution.

Besides the exact solution, shifted Chebyshev tau method based on Chebyshev operational matrix of differentiation and shifted Legendre collocation approach are used to find the approximate solutions of the problem for the general case. In literature, the work related to shifted Chebyshev tau and collocation methods are reported by many researchers and some of them are Guo and Yan (2009), Ghoreishi and Yazdani (2011), Vanani and Aminataei (2011), Atabakzadeh et al. (2013). In Doha et al. (2011a, 2011b), the authors have discussed the shifted Chebyshev tau and collocation methods based on Chebyshev operational matrix of fractional derivatives for solving the linear multi-order fractional differential equations. To solve the different types of boundary value problems, Abd-Elhameed et al. (2015) presented a new operational matrix method with the aid of the Petrov-Galerkin method and collocation method. Zaky et al. (2018) also reported a paper to explain a Legendre spectral-collocation procedure to find the numerical solution of the fractional initial value problems of the distributed order.

4.2 Mathematical Model

In this section, we consider the temperature and time dependent thermal conductivity as given in Eq. (4.1) and a mathematical model of one phase Stefan problem with nonlinear heat conduction is presented for melting process which is as follow:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k(T) \frac{\partial T}{\partial x} \right), \qquad 0 < x < s(t), \tag{4.2}$$

$$T(0,t) = T_0(t), \tag{4.3}$$

$$T(s(t),t) = T_m, \tag{4.4}$$

$$k\,(T_m)\left.\frac{\partial T}{\partial x}\right|_{x=s(t)} = -\rho\, l\,(s(t))^{\alpha}\,\frac{ds}{dt}, \tag{4.5}$$

$$s(0) = 0, \tag{4.6}$$

where $T(x,t)$ is the temperature at the position x and time t, $T_0(t)$ is the time dependent temperature at $x=0$, T_m is the constant phase change temperature $T_0(t) > T_m$, $s(t)$ is the moving interface; c, ρ and l are the specific heat, the density and the latent heat, respectively.

By considering the following transformation:

$$\theta(x,t) = \frac{T(x,t)-T_m}{\Delta T}, \tag{4.7}$$

the Eqs. (4.2)-(4.6) become:

$$\frac{\partial \theta}{\partial t} = \alpha_0\,\frac{\partial}{\partial x}\left((1+\beta\theta\, t^{-\alpha/2})\,\frac{\partial \theta}{\partial x}\right), \qquad 0 < x < s(t), \tag{4.8}$$

$$\theta(0,t) = \frac{T_0(t)-T_m}{\Delta T} \approx t^{\frac{\alpha}{2}}, \tag{4.9}$$

$$\theta(s(t),t) = 0, \tag{4.10}$$

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=s(t)} = -\frac{(s(t))^{\alpha}}{\alpha_0 Ste}\,\frac{ds(t)}{dt}, \tag{4.11}$$

$$s(0) = 0, \tag{4.12}$$

where $\alpha_0 = \dfrac{k_0}{\rho\, c} > 0$ (thermal diffusivity for k_0) and $Ste = \dfrac{c\,\Delta T}{l} > 0$ (Stefan number).

4.3 Solution for the Problem

Now, we consider the following similarity variables:

$$\theta(x,t) = f(\eta)\, t^{\alpha/2} \text{ with } \eta = \frac{x}{2\sqrt{\alpha_0\, t}} \tag{4.13}$$

and assuming that the melt front moves as

$$s(t) = 2\lambda\sqrt{\alpha_0\, t}\,, \tag{4.14}$$

where λ is an unknown positive constant.

Substituting Eqs. (4.13) and (4.14) into Eqs. (4.8)-(4.11) which provide the following equations:

$$\frac{d}{d\eta}\left((1+\beta\, f(\eta))\frac{d}{d\eta} f(\eta)\right) + 2\eta \frac{d}{d\eta} f(\eta) - 2\alpha\, f(\eta) = 0, \quad 0<\eta<\lambda, \tag{4.15}$$

$$f(0) = 1, \tag{4.16}$$

$$f(\lambda) = 0, \tag{4.17}$$

$$\frac{d}{d\eta} f(\lambda) = -\frac{1}{Ste}(2\lambda)^{(\alpha+1)}\alpha_0^{\frac{\alpha}{2}}\,. \tag{4.18}$$

In this chapter, we discuss the following two cases for the solutions of Eqs. (4.15)-(4.18) by two approaches:

4.3.1 Case 1: Shifted Chebyshev tau Method

According to Doha et al. (2011a, 2011b), we express the unknown function $f(\eta)$ in terms of the shifted Chebyshev polynomials as:

$$f_N(\eta) \approx \sum_{i=0}^{N} c_i T_{L,i}(\eta) = C^T \phi(\eta), \tag{4.19}$$

where $C^T = [c_0, c_1, c_2, \cdots, c_N]$

and $\phi(\eta) = \left[T_{\lambda,0}(\eta), T_{\lambda,1}(\eta), \ldots, T_{\lambda,N}(\eta)\right]^T$.

As given in Eq. (1.8), the derivatives are approximated as:

$$\frac{df}{d\eta} = D^{(1)}\phi(\eta), \qquad \frac{d^2 f}{d\eta^2} = (D^{(1)})^2 \phi(\eta)\,. \tag{4.20}$$

Using Eqs. (4.19) and (4.20), the residual $R_N(x)$ for Eq. (4.15) is defined as:

$$\begin{aligned} R_N(x) = {} & (C^T (D^{(1)})^2 \phi(\eta)) + \beta (C^T D^{(1)} \phi(\eta))^2 + \beta (C^T \phi(\eta))(C^T (D^{(1)})^2 \phi(\eta)) \\ & + 2\eta (C^T D^{(1)} \phi(\eta)) - 2\alpha\, C^T \phi(\eta)\,. \end{aligned} \tag{4.21}$$

According to Tau method (Doha et al., 2011a, 2011b), we generate $(N-1)$ non-linear algebraic equations by using the condition

$$\left\langle R_N(x), T_{\lambda,i}(x) \right\rangle = \int_0^{\lambda} R_N(x) T_{\lambda,i}(x) dx = 0, \qquad i = 0,1,\ldots,N-2. \tag{4.22}$$

Also, by using Eqs. (4.19) and (4.20) in the Eqs. (4.16)-(4.18), we get

$$C^T \phi(0) = 1,\ C^T \phi(\lambda) = 0 \tag{4.23}$$

and

$$C^T D^{(1)} \phi(\lambda) = -\frac{1}{Ste} (2\lambda)^{(\alpha+1)} \alpha_0^{\frac{\alpha}{2}}\,, \tag{4.24}$$

respectively.

Eq. (4.22) generates $(N-1)$ equations and two more equations are generated by Eq. (4.23). Hence, we have $(N+1)$ equations in $(N+1)$ unknowns that can be easily solved and it gives the unknown coefficients of the vector C. Consequently, $f(\eta)$ given in Eq. (4.19) can be calculated in terms of λ which is still to be determined. In order to get the value of λ, we use the calculated value of $f(\eta)$ in the interface condition given in Eq. (4.24).

4.3.2 Case 2: Shifted Legendre Collocation Approach

In this case, we substitute the following transformation:

$$f(\eta) = y(\eta) + \frac{\lambda - \eta}{\lambda}, \tag{4.25}$$

into the Eqs. (4.15)-(4.18) which produce

$$\begin{aligned} y''(\eta) + \beta\left(y'(\eta) - \frac{1}{\lambda}\right)^2 + \beta\left(y(\eta) + \frac{\lambda-\eta}{\lambda}\right) y''(\eta) + 2\eta\left(y'(\eta) - \frac{1}{\lambda}\right) \\ - 2\alpha\left(y(\eta) + \frac{\lambda - \eta}{\lambda}\right) = 0, \end{aligned} \tag{4.26}$$

$$y(0) = 0, \; y(\lambda) = 0, \tag{4.27}$$

$$y'(\lambda) - \frac{1}{\lambda} = -\frac{1}{Ste}(2\lambda)^{(\alpha+1)} \alpha_0^{\frac{\alpha}{2}}. \tag{4.28}$$

Now, we take the following approximation of $y(\eta)$ in terms of the shifted Legendre polynomials:

$$y(\eta) \approx y_N(\eta) \approx \sum_{i=0}^{N} c_i \, \phi_i(\eta) = \mathbf{C}^T \mathbf{\Phi}(\eta), \tag{4.29}$$

where $\mathbf{C}^T = [c_0, c_1, \ldots, c_N]$, $\quad \mathbf{\Phi}(\eta) = [\phi_0(\eta), \phi_1(\eta), \ldots, \phi_N(\eta)]^T$. (4.30)

As mentioned in section 1.3.2, we take the approximation of $y'(\eta)$ and $y''(\eta)$ as

$$y'(\eta) \approx \mathbf{C}^T D\mathbf{\Phi}(\eta) + \mathbf{C}^T\boldsymbol{\delta} \tag{4.31}$$

and

$$y''(\eta) \approx \mathbf{C}^T D^2\mathbf{\Phi}(\eta) + \mathbf{C}^T D\boldsymbol{\delta} + \mathbf{C}^T\boldsymbol{\delta}'. \tag{4.32}$$

Substituting the considered approximations of $y(\eta)$, $y'(\eta)$ and $y''(\eta)$ into the Eq. (4.26), we get the following residual denoted by $R_N(\eta)$ corresponding to the Eq. (4.26):

$$\begin{aligned} R_N(\eta) = {} & \left(\mathbf{C}^T D^2\mathbf{\Phi}(\eta) + \mathbf{C}^T D\boldsymbol{\delta} + \mathbf{C}^T\boldsymbol{\delta}'\right) + \beta\left(\mathbf{C}^T\mathbf{\Phi}(\eta) + \frac{\lambda-\eta}{\lambda}\right)\left(\mathbf{C}^T D^2\mathbf{\Phi}(\eta) + \mathbf{C}^T D\boldsymbol{\delta} + \mathbf{C}^T\boldsymbol{\delta}'\right) \\ & + \beta\left(\mathbf{C}^T D\mathbf{\Phi}(\eta) + \mathbf{C}^T\boldsymbol{\delta} - \frac{1}{\lambda}\right)^2 + 2\eta\left(\mathbf{C}^T D\mathbf{\Phi}(\eta) + \mathbf{C}^T\boldsymbol{\delta} - \frac{1}{\lambda}\right) - 2\alpha\left(\mathbf{C}^T\mathbf{\Phi}(\eta) + \frac{\lambda-\eta}{\lambda}\right). \end{aligned} \tag{4.33}$$

According to the spectral collocation method (Abd-Elhameed et al., 2015), we impose $R_N(\eta)=0$ at the first $(N+1)$ roots of $L^*_{N+1}(\eta)$ which produces $(N+1)$ algebraic equations involving $(N+2)$ unknowns ($c_0, c_1, \cdots, c_N$ and λ). Besides these $(N+1)$ algebraic equations, Eq. (4.28) gives rise to the following additional equation:

$$\mathbf{C}^T D\mathbf{\Phi}(\lambda) + \mathbf{C}^T\boldsymbol{\delta} - \frac{1}{\lambda} = -\frac{1}{Ste}(2\lambda)^{(\alpha+1)}\alpha_0^{\frac{\alpha}{2}}. \tag{4.34}$$

The obtained $(N+2)$ algebraic equations can be solved by an appropriate numerical technique that will provide the $(N+2)$ unknowns. After that, the solution of $y(\eta)$ can be found from Eq. (4.29), and therefore the $\theta(x,t)$ can be obtained by back substitution. Also, the moving phase front $s(t)$ can be achieved with the aid of Eq. (4.14).

4.4 Results and Discussion

In this section, we discuss the accurateness of our obtained results as well as dependence of temperature distribution and phase front on various parameters. By using the similarity transformation (given in Eqs. (4.13) and (4.14)), the analytical solution of Eqs. (4.15)-(4.18) is calculated for the constant thermal conductivity i.e., $\beta = 0$ which is given as:

$$\theta(x,t) = \left(\frac{2^{1+\alpha} e^{-\frac{x^2}{4\alpha_0 t}} \Gamma\left(\frac{2+\alpha}{2}\right) H_{(-1-\alpha)}(\lambda)\, {}_1F_1\left(\frac{1+\alpha}{2}; \frac{1}{2}; \frac{x^2}{4\alpha_0 t}\right)}{2^{1+\alpha} \Gamma\left(\frac{2+\alpha}{2}\right) H_{(-1-\alpha)}(\lambda) - \sqrt{\pi}\, {}_1F_1\left(\frac{1+\alpha}{2}; \frac{1}{2}; \lambda^2\right)} \right) t^{\frac{\alpha}{2}} - \left(\frac{2^{1+\alpha} e^{-\frac{x^2}{4\alpha_0 t}} \Gamma\left(\frac{2+\alpha}{2}\right) H_{(-1-\alpha)}\left(\frac{x}{2\sqrt{\alpha_0 t}}\right) {}_1F_1\left(\frac{1+\alpha}{2}; \frac{1}{2}; \lambda^2\right)}{2^{1+\alpha} \Gamma\left(\frac{2+\alpha}{2}\right) H_{(-1-\alpha)}(\lambda) - \sqrt{\pi}\, {}_1F_1\left(\frac{1+\alpha}{2}; \frac{1}{2}; \lambda^2\right)} \right) t^{\frac{\alpha}{2}}, \tag{4.35}$$

where $H_{(-n)}(x)$ is the Hermite function and ${}_1F_1$ is the hypergeometric function.

The location of phase front is given by

$$s(t) = 2\lambda\sqrt{\alpha_0 t}\,, \tag{4.36}$$

where λ is a constant which can be determined by following transcendental equation:

$$h(\lambda) = \frac{2e^{-\lambda^2}(1+\alpha)\lambda\, \Gamma\left(\frac{2+\alpha}{2}\right) H_{(-1-\alpha)}(\lambda)\, {}_1F_1\left(1+\frac{1+\alpha}{2}; \frac{3}{2}; \lambda^2\right)}{2^{1+\alpha} \Gamma\left(\frac{2+\alpha}{2}\right) H_{(-1-\alpha)}(\lambda) - \sqrt{\pi}\, {}_1F_1\left(\frac{1+\alpha}{2}; \frac{1}{2}; \lambda^2\right)} - \frac{2e^{-\lambda^2}(-1-\alpha)\Gamma\left(\frac{2+\alpha}{2}\right) H_{(-2-\alpha)}(\lambda)\, {}_1F_1\left(\frac{1+\alpha}{2}; \frac{1}{2}; \lambda^2\right)}{2^{1+\alpha} \Gamma\left(\frac{2+\alpha}{2}\right) H_{(-1-\alpha)}(\lambda) - \sqrt{\pi}\, {}_1F_1\left(\frac{1+\alpha}{2}; \frac{1}{2}; \lambda^2\right)} + \frac{\alpha_0^{\frac{\alpha}{2}}}{Ste} \lambda^{\alpha+1} = 0. \tag{4.37}$$

From Eq. (4.37), it is clear that for all $\lambda > 0, dh/d\lambda > 0$ for positive values of α, α_0 and *Ste*. Moreover, $h(\lambda) \to -\infty$ if $\lambda \to 0$ and $h(\lambda) \to \infty$ as $\lambda \to \infty$ when α, α_0 and Ste are positive. Therefore, there exists one and only one positive value of λ as the solution of Eq. (4.37). With the help of Eq. (4.36) and the obtained value of λ from Eq. (4.37), the location of phase front $s(t)$ can be determined.

Now, we present comparisons among approximate solutions obtained by shifted Legendre collocation approach, shifted Chebyshev tau method and analytical solution of the considered problem.

To show the accuracy of the solution by shifted Chebyshev tau approach, the comparisons of temperature distribution and interface location at $\beta = 0$ are depicted in Tables 4.1 and 4.2, respectively. Table 4.1 shows the obtained approximate values of temperature distribution θ_T for N = 3, 4, 5 and its exact value θ_E at $Ste = 0.2,$ $\alpha_0 = 1,\ t = 1.5$ and $\beta = 0$. Table 4.2 depicts the values of approximate position of phase front $s_T(t)$ for N = 3, 4, 5 on different time and its exact values $s_E(t)$ at $Ste = 0.2,\ \alpha_0 = 1$ and $\beta = 0$. From these tables, it is clear that our approximate results are near to exact value and accuracy increases as the order of operational matrix of differentiation increases.

In Table 4.3, θ_E, $[\theta_T]_{N=2}$ and $[\theta_C]_N$ symbolize the different values of temperature profile obtained by analytical approach, shifted Chebyshev tau method at N = 2 and shifted Legendre collocation approach at N (N = 2, 3), respectively at the fixed value of $\beta = 0,\ t = 1.5,\ \alpha = 0.5$ and $\alpha_0 = 0.5$. In Table 4.4, $s_E(t)$, $[s_T(t)]_N$ and $[s_C(t)]_N$ denote the various values of the position of moving phase front calculated by analytical approach, shifted Chebyshev tau method at N = 2 and shifted Legendre spectral collocation approach at N (N = 2, 3), respectively at the fixed value of $\beta = 0$, $\alpha = 0.5$ and $\alpha_0 = 0.5$. From both the tables, it is clear that solution of the considered problem achieved by shifted Chebyshev tau method and shifted Legendre spectral collocation approach are sufficiently near to analytical solution. But, for the considered problem, the shifted Legendre collocation approach provides more accurate results at N = 2 than shifted Chebyshev tau method at N = 2 as well as N = 3.

When $\beta \neq 0$, the obtained results are presented through Figs. 4.1-4.5 for the study of the dependence of temperature distribution and location of phase front on various parameters. Fig. 4.1 demonstrates the variations of temperature distribution for different value of α ($\alpha = 2.0, 1.0, 0.2$) at fixed values of $\beta = 0.5$, $Ste = 0.2$ and $\alpha_0 = 1.0$. Fig. 4.2 depicts the variations of temperature distribution for different value of β ($\beta = 1.5, 1.0, 0.3$) at fixed values of $\alpha = 0.5$, $Ste = 0.2$ and $\alpha_0 = 1.0$. From these figures, it is clear that temperature at $x = 0$ is highest and decreases continuously to zero. It is also seen that the rate of change of temperature decreases as the parameters α and/or β decrease.

In Fig. 4.3, the dependence of phase front on time for different α (i.e., exponent power of time) is presented at the fixed value of $\beta = 0.5$, $Ste = 0.2$ and $\alpha_0 = 1.0$. From this figure, it can be seen that the movement of phase front increases with the increment in the value of α ($\alpha = 0.2, 1.0, 2.0$). Consequently, the melting process becomes fast as we increase the value of α. Fig. 4.4 shows the trajectory of phase front for different β ($\beta = 0.3, 1.0, 1.5$) at the fixed value of $\alpha = 0.5$, $Ste = 0.2$ and $\alpha_0 = 1.0$. Fig. 4.5 demonstrates the trajectory of phase front for different Stefan numbers ($Ste = 0.2, 1.0, 2.0$) at the fixed value of $\alpha = 0.5$, $\beta = 0.5$ and $\alpha_0 = 1.0$. From Figs. 4.4

and 4.5, it is clear that the movement of phase front increases as the value of β and/or Ste increases. Hence, the melting process becomes fast if we increase the parameter β and/or Stefan number (Ste).

α	x	θ_E	$[\theta_T]_{N=3}$	$[\theta_T]_{N=4}$	$[\theta_T]_{N=5}$
$\alpha = 0.2$	0.0	1.041380	1.041380	1.041400	1.041380
	0.1	0.902061	0.901730	0.902037	0.902058
	0.2	0.763806	0.764121	0.763788	0.763800
	0.3	0.626976	0.628553	0.626980	0.626972
	0.4	0.491923	0.495025	0.491940	0.491924
	0.5	0.358984	0.363539	0.358993	0.358990
$\alpha = 1.0$	0.0	1.224740	1.224740	1.224740	1.224740
	0.1	1.067990	1.068140	1.068010	1.067990
	0.2	0.915301	0.915460	0.915387	0.915302
	0.3	0.766670	0.766697	0.766810	0.766672
	0.4	0.622060	0.621854	0.622227	0.622061
	0.5	0.481424	0.480930	0.481581	0.481423
$\alpha = 2.0$	0.0	1.500000	1.500000	1.500000	1.500000
	0.1	1.306340	1.308470	1.306320	1.306340
	0.2	1.122010	1.123950	1.121940	1.122020
	0.3	0.946368	0.946428	0.946245	0.946378
	0.4	0.778749	0.775913	0.778586	0.778753
	0.5	0.618506	0.612402	0.618333	0.618503

Table 4.1. Comparison of the exact temperature θ_E with approximate values of temperature $[\theta_T]_N$ for N = 3, 4, 5 at $\beta = 0$.

α	$Time(t)$	$s_E(t)$	$[s_T(t)]_{N=3}$	$[s_T(t)]_{N=4}$	$[s_T(t)]_{N=5}$
$\alpha = 0.2$	0.0	0.000000	0.000000	0.000000	0.000000
	0.2	0.285007	0.286641	0.285005	0.285007
	0.4	0.403061	0.405372	0.403057	0.403061
	0.6	0.493646	0.496477	0.493642	0.493646
	0.8	0.570014	0.573282	0.570009	0.570014
	1.0	0.637295	0.640949	0.637290	0.637295
$\alpha = 1.0$	0.0	0.000000	0.000000	0.000000	0.000000
	0.2	0.316075	0.315801	0.316072	0.316075
	0.4	0.446998	0.446610	0.446993	0.446998
	0.6	0.547458	0.546984	0.547453	0.547458
	0.8	0.632150	0.631602	0.632144	0.632150
	1.0	0.706765	0.706153	0.706758	0.706765
$\alpha = 2.0$	0.0	0.000000	0.000000	0.000000	0.000000
	0.2	0.339161	0.336535	0.339164	0.339161
	0.4	0.479646	0.475933	0.479650	0.479646
	0.6	0.587444	0.582896	0.587449	0.587444
	0.8	0.678321	0.673071	0.678328	0.678322
	1.0	0.758386	0.752516	0.758394	0.758387

Table 4.2. Comparison of the exact values of moving boundary $s_E(t)$ and approximate values of moving boundary $[s_T(t)]_N$ for N = 3, 4, 5 at $\beta = 0$.

Ste.	x	θ_E	$[\theta_T]_{N=2}$	$[\theta_T]_{N=3}$	$[\theta_C]_{N=2}$
0.2	0.0	1.10668191	1.10668191	1.10668191	1.10668191
	0.1	0.91415820	0.91392700	0.91422146	0.91415464
	0.2	0.72594753	0.72653440	0.72611851	0.72594259
	0.3	0.54263008	0.54450410	0.54285597	0.54262892
	0.4	0.36472988	0.36783611	0.36491673	0.36473294
	0.5	0.19270713	0.19653042	0.19278369	0.19271015
0.5	0.0	1.10668191	1.10668191	1.10668191	1.10668191
	0.1	0.95859394	0.95816852	0.95871262	0.95858753
	0.2	0.81467143	0.81486741	0.81505125	0.81465786
	0.3	0.67534942	0.67677861	0.67597491	0.67533598
	0.4	0.54101160	0.54390210	0.54176070	0.54100447
	0.5	0.41198492	0.41623788	0.41268574	0.41198538
1.0	0.0	1.10668191	1.10668191	1.10668191	1.10668191
	0.1	0.98007921	0.97963797	0.98024616	0.98007227
	0.2	0.85757062	0.85763569	0.85816979	0.85755308
	0.3	0.73952098	0.74067505	0.74061005	0.73949777
	0.4	0.62624616	0.62875607	0.62772420	0.62622308
	0.5	0.51800864	0.52187874	0.51966948	0.51798904

Table 4.3. Comparisons of different values of exact temperature $[\theta_E]$ and approximate temperatures ($[\theta_T], [\theta_C]$).

Ste.	t	$s_E(t)$	$[s_T(t)]_{N=2}$	$[s_T(t)]_{N=3}$	$[s_C(t)]_{N=2}$
0.2	0.2	0.22516554	0.22595122	0.22516226	0.22516553
	0.4	0.31843217	0.31954328	0.31842752	0.31843214
	0.6	0.38999816	0.39135900	0.38999247	0.38999814
	0.8	0.45033109	0.45190245	0.45032452	0.45033106
	1.0	0.50348547	0.50524229	0.50347812	0.50348543
0.5	0.2	0.31227230	0.31396045	0.31224193	0.31227237
	0.4	0.44161972	0.44400712	0.44157678	0.44161983
	0.6	0.54087149	0.54379545	0.54081889	0.54087162
	0.8	0.62454460	0.62792090	0.62448387	0.62454475
	1.0	0.69826209	0.70203691	0.69819419	0.69826226
1.0	0.2	0.39267096	0.39504690	0.39253083	0.39267278
	0.4	0.55532060	0.55868068	0.55512242	0.55532317
	0.6	0.68012606	0.68424130	0.67988334	0.68012921
	0.8	0.78534193	0.79009380	0.78506166	0.78534557
	1.0	0.87803897	0.88335172	0.87772562	0.87804304

Table 4.4. Comparisons of different values of exact position of moving boundary $s_E(t)$ and approximate position of moving boundary $([s_T(t)], [s_C(t)])$.

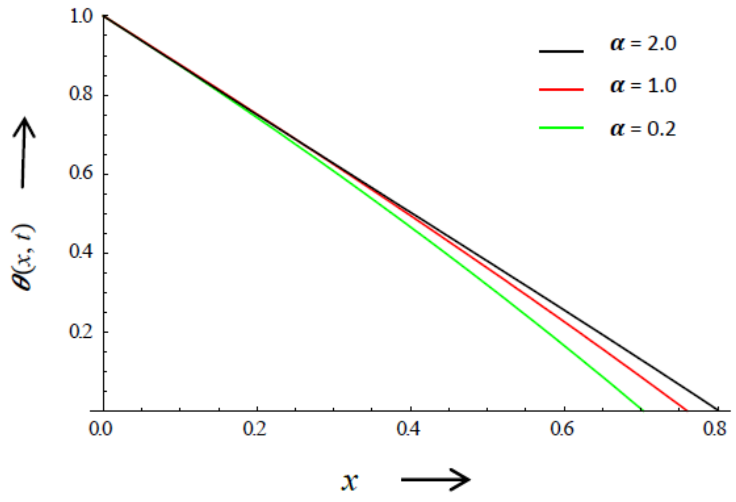

Fig.4.1. Plot of temperature profile for different values of α at $\beta = 0.5$, $Ste = 0.2$ and $\alpha_0 = 1.0$.

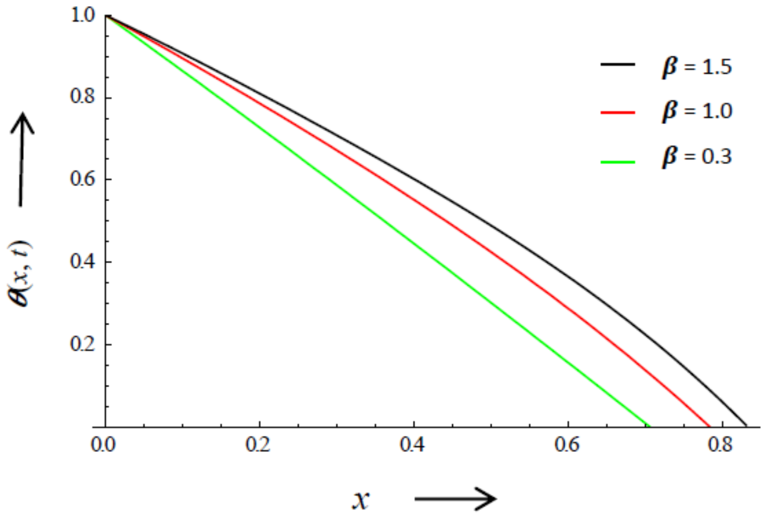

Fig.4.2. Plot of temperature profile for different values of β at $\alpha = 0.5$, $Ste = 0.2$ and $\alpha_0 = 1.0$.

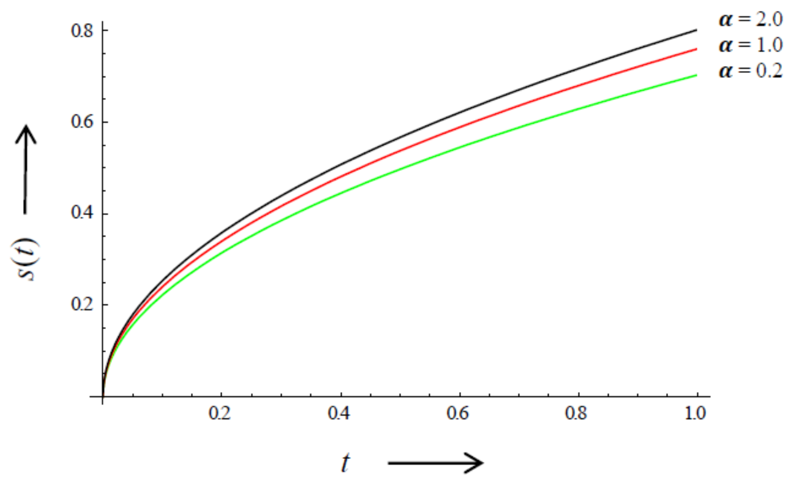

Fig.4.3. Plot of moving interface for different values of α at $\beta = 0.5$, $Ste = 0.2$ and $\alpha_0 = 1$.

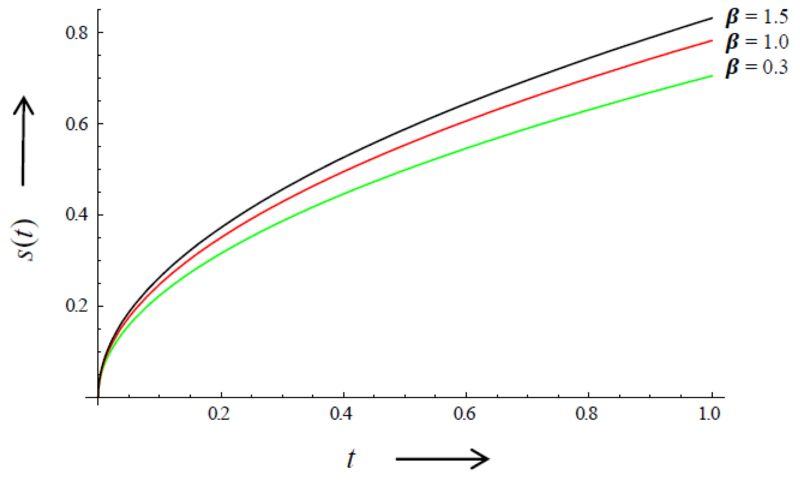

Fig.4.4. Plot of moving interface for different values of β at $\alpha = 0.5$, $Ste = 0.2$ and $\alpha_0 = 1$.

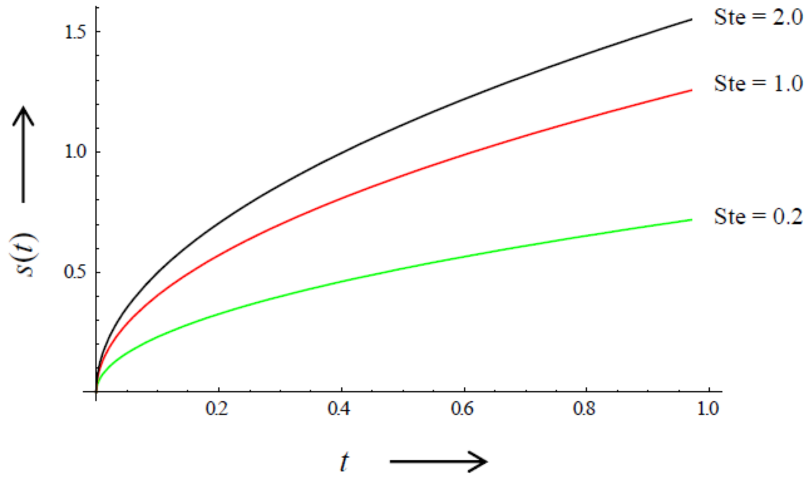

Fig.4.5. Plot of moving interface for different values of Ste at $\alpha = 0.5$, $\beta = 0.5$ and $\alpha_0 = 1$.

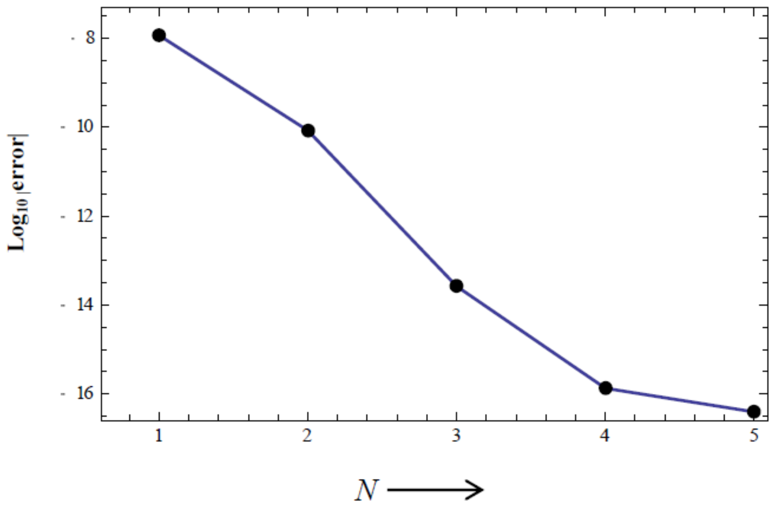

Fig.4.6. Plot of convergence of approximate $f(\eta)$.

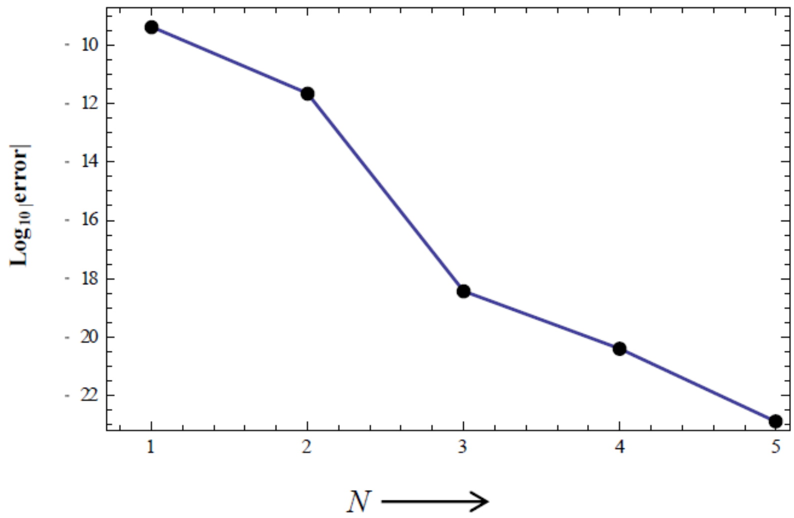

Fig.4.7. Plot of convergence of approximate $s(t)$.

The correctness of the solution by shifted Legendre collocation approach of $f(\eta)$ and $s(t)$ with increasing the number of terms or N are depicted in Figs. 4.6 and 4.7. In Fig. 4.6, we displays the graph of $\log_{10}$|error| of the calculated solution of $f(\eta)$ for different shifted Legendre polynomials of degree (N+2) at the value of $\alpha = 1.0$, $\beta = 0$, $\eta = 0.5$, $\alpha_0 = 1.0$ and $Ste = 1.5$. Fig. 4.7 shows the plot of $\log_{10}$|error| of the obtained moving interface locations $s(t)$ for different approximating polynomials of degree (N+2) at the value of $\alpha = 1.0$, $\beta = 0$, $\alpha_0 = 1.0$ and $Ste = 1.5$. Figs. 4.6 and 4.7 demonstrate that the proposed solutions converge rapidly as we increase the degree of approximating polynomials (shifted Legendre polynomials) or N.

4.5 Conclusion

In this work**,** a special type of one phase Stefan problem with time and temperature dependent thermal conductivity is explored and its approximate solutions are discussed by shifted Chebyshev tau method based on Chebyshev operational matrix of differentiation and shifted Legendre collocation method. In order to check accuracy of our obtained results, an exact solution of the problem is also discussed for a particular case i.e., $\beta = 0$. From this study, it is seen that the proposed algorithms for the solution of Stefan problems are simple and accurate. It is observed that the solution of the problem by shifted Legendre collocation method is better in term of accuracy than shifted Chebyshev tau method. Moreover, it is found that the accurateness of results obtained by shifted Legendre collocation method increases as we increase the number of approximating terms or *N*.

From this study, it is establish that the rate of change of temperature increases as the power of time (i.e., α) and/or β increases and movement of moving interface increases if we increase the value of power of time (i.e., α) or β or Ste . Consequently, the increment in the value of parameters α or β or Ste increases the rate of melting process. It is also observed that the variation of Stefan number is more pronounced than the parameters α and β in the movement of interface.

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