Chapter 3

Exact and Approximate Solutions for a Freezing Problem having Varying Thermal Coefficients and Convective Boundary Condition

3.1 Introduction

The phase change problems have been investigated since the nineteenth century. These problems involve one or more moving boundaries and encounter in many aspects of engineering and natural sciences. The problems related to heat conduction, especially in freezing of materials, have many applications in industries and materials science (Lunardini, 1991; Gupta, 2003; Evans, 2009). Due to involvement of moving phase front in the mathematical model, phase change problems become nonlinear problems and it is challenging to have the solutions in closed form for these problems even in many simple cases (Crank, 1984; Hill, 1986). Therefore, the establishment of solution of such types of phase change problem (Stefan problem) always attracts the researchers.

One phase Stefan problems in semi-infinite material have been studied widely and their solutions have been proposed analytically and numerically (Meek and Norbury, 1984; Das and Rajeev, 2010; Fazio, 2013; Rajeev, 2014). In the classical literatures of Stefan problem, it was assumed that the thermal coefficients of the phases involved are constants but it did not happen in many of the physical phase change processes. Therefore, variable thermal coefficients have been considered in many researches to include such type of behaviour of the materials (Cho and Sunderland, 1974; Petrova and Tarzia, 1994; Tritscher and Broadbridge, 1994; Natale and Tarzia, 2006; Kumar et al., 2018a). In 1987, Oliver and Sunderland (1987) discussed a two phase Stefan problem having variable specific heat and thermal conductivity which are linear functions of unknown temperature. Ramos et al. (1994) and Briozzo et al. (2007) also studied temperature depending specific heat and thermal conductivity in their models of phase change problems. In 2017, Briozzo and Natale (2017) discussed a closed form solution for a super-cooled Stefan problem in infinite domain having nonlinear thermal conductivity and Dirichlet boundary condition at the fixed boundary. They have presented the exact solution of the problem and also discussed uniqueness of the solution. Kumar et al. (2018b) assumed Dirichlet boundary condition and discussed a Stefan problem which involves non-linear specific heat and thermal conductivity. The Dirichlet boundary condition in the Stefan problem is chosen due to the assumption that the fixed surface of the material is suddenly imposed by a constant temperature from an external source. But, this is not valid in real sense. Therefore, convective types of boundary conditions are considered by many researchers (Natale and Tarzia, 2003; Briozzo and Natale, 2015; Briozzo and Natale, 2016). Recently, Ceretani et al. (2018) discussed a Stefan problem involving variable conductivity and convective boundary conditions at the fixed face. They have also obtained exact solution for the problem in term of the modified error function. Based on these observations, the time-dependent convective boundary condition applied at the fixed boundary, varying specific heat $c(T)$ and thermal conductivity $k(T)$ are considered in our study.

The establishment of the closed form solution to a Stefan problem is always exciting and beneficial for comparison point of view to validate the numerical and approximate solution obtained by various techniques of a complex problem. We have constructed the closed form solution to the considered problem when $\alpha = \beta$ and $p = q = 1$ with the aid of suitable similarity variables. The uniqueness and existence of the obtained solution is also discussed. For $\alpha \neq \beta$ and $p \neq q$, a semi analytical solution for the problem has been deliberated with the aid of similarity variable and tau method. The applications of collocation and tau methods in conjunction with shifted Chebyshev operational matrix of derivatives can be seen in Parand and Razzaghi (2004), Doha et al. (2011a, 2011b), Vanani and Aminataei (2011), Ghoreishi and Yazdani (2011). Recently, Kumar et al. (2018b) applied shifted Chebyshev tau method to a non-linear phase change problem with the first type of boundary condition.

The primary focus of this chapter is to investigate a Stefan problem in one-dimension including temperature depending specific heat and thermal conductivity with convective boundary condition. The similarity solution of the problem is obtained for a special case. A semi-analytical solution for the problem is also explored for general case and comparisons of this solution are made with the calculated exact solution to check its accuracy. The effect of the various parameters on moving phase front is also considered in our study.

3.2 Mathematical Model of the Problem

In this section, we will take a one phase freezing problem having the temperature depending thermal conductivity $k(T)$ and specific heat $c(T)$. Moreover, it is assumed that the convective boundary condition on the fixed face $x = 0$ is time-dependent as given in (Ceretani et al., 2018). The mathematical model of the governing process consists of non-linear heat conduction which is described as follow:

$$\rho c(T)\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right),\quad 0<x<s(t), \tag{3.1}$$

$$k(T(0,t))\frac{\partial T(0,t)}{\partial x}=\frac{q}{\sqrt{t}}(T(0,t)-T_0), \tag{3.2}$$

$$T(s(t),t)=T_f, \tag{3.3}$$

$$k(T_f)\frac{\partial T(s(t),t)}{\partial x}=\rho l\frac{ds}{dt}, \tag{3.4}$$

$$s(0)=0, \tag{3.5}$$

where $T(x,t)$ denotes the temperature profile at the position x and time t, $T_0<T_f$ is constant temperature imposed in the neighbourhood of the boundary $x=0$, T_f is the freezing temperature, $s(t)$ is the moving phase front, $q>0$ is the heat flux coefficient, ρ and l are the density and the latent heat, respectively.

Motivated by (Kumar et al., 2018b), the temperature-dependent thermal conductivity $k(T)$ is assumed as:

$$k(T)=k_0\left(1+\beta\left(\frac{T-T_0}{T_f-T_0}\right)^p\right) \tag{3.6}$$

and the specific heat capacity $c(T)$ is considered as:

$$c(T)=c_0\left(1+\alpha\left(\frac{T-T_0}{T_f-T_0}\right)^q\right), \tag{3.7}$$

where $c_0>0$, $k_0>0$, $\alpha>0$, $\beta>0$ are the constants and p, q are non-negative integers.

3.3 Solution of the Problem

First, we substitute

$$\theta(x,t)=\frac{T(x,t)-T_0}{T_f-T_0}, \tag{3.8}$$

in the Eqs. (3.1)-(3.5) that produces the following equations:

$$(1+\alpha\theta^q)\frac{\partial\theta}{\partial t}=\alpha_0\frac{\partial}{\partial x}\left((1+\beta\theta^p)\frac{\partial\theta}{\partial x}\right),\quad 0<x<s(t), \tag{3.9}$$

$$k_0(1+\beta\theta^p(0,t))\frac{\partial\theta(0,t)}{\partial x}=\frac{q}{\sqrt{t}}\theta(0,t), \tag{3.10}$$

$$\theta(s(t),t)=1, \tag{3.11}$$

$$\frac{\partial\theta(s(t),t)}{\partial x}=\frac{1}{\alpha_0(1+\beta)Ste}\frac{ds}{dt}, \tag{3.12}$$

$$s(0)=0, \tag{3.13}$$

where $\alpha_0=\dfrac{k_0}{\rho c_0}$ (thermal diffusivity for k_0 and c_0) and $Ste=\dfrac{c_0(T_f-T_0)}{l}$ is the Stefan number.

Here, we take the following similarity variable

$$\theta(x,t)=f(\eta)\text{ with }\eta=\frac{x}{2\sqrt{\alpha_0 t}} \tag{3.14}$$

and it is supposed that phase front moves as

$$s(t)=2\lambda\sqrt{\alpha_0 t}, \tag{3.15}$$

where λ is a positive constant to be determined later.

Now, we substitute Eqs. (3.14) and (3.15) into the Eqs. (3.9)-(3.12) that generates the following non-linear ordinary differential equation:

$$2\eta(1+\alpha f^{q})\frac{df}{d\eta}+\frac{d}{d\eta}\left((1+\beta f^{p})\frac{df}{d\eta}\right)=0, \qquad 0<\eta<\lambda, \tag{3.16}$$

$$(1+\beta f^{p}(0))\, f'(0)-\gamma f(0)=0, \tag{3.17}$$

$$f(\lambda)=1, \tag{3.18}$$

$$f'(\lambda)=\frac{2\lambda}{(1+\beta)Ste}, \tag{3.19}$$

where $\gamma=2Bi>0$ with $Bi=\dfrac{q\sqrt{\alpha_0}}{k_0}$ (generalized Biot number).

Now, we consider the following two cases to discuss the solution of the system given in Eqs. (3.16)-(3.19):

3.3.1 Case: 1

In this case, $\alpha=\beta$ and $p=q=1$ are considered and the exact solution is discussed.

As mentioned in (Kumar et al., 2018b), the general solution of Eq. (3.16) in term of error function ($erf(.)$) is given by

$$f(\eta)=\frac{1}{\beta}\left(\left(1+2\beta C_2(\lambda)-C_1(\lambda)\sqrt{\pi}\beta\, erf(\eta)\right)^{1/2}-1\right), \tag{3.20}$$

where $C_1(\lambda)$ and $C_2(\lambda)$ are the following expressions of λ which are determined with the aid of conditions (3.17) and (3.18):

$$C_1(\lambda) = \frac{1}{2\beta}\left(2\gamma + \sqrt{\pi}\gamma^2 erf(\lambda) - C_3(\lambda)\right), \tag{3.21}$$

$$C_2(\lambda) = \frac{1}{4\beta}\left(4\beta + 2\beta^2 + 2\sqrt{\pi}\gamma\, erf(\lambda) + \pi\gamma^2 erf(\lambda)^2 - C_3(\lambda)\sqrt{\pi} erf(\lambda)\right) \tag{3.22}$$

and

$$C_3(\lambda) = \left(4\beta\gamma^2(2+\beta) + (2\gamma + \sqrt{\pi}\gamma^2 erf(\lambda))^2\right)^{1/2}. \tag{3.23}$$

From Eqs. (3.14) and (3.20), we have

$$\theta(x,t) = \frac{1}{\beta}\left(\left(1 + 2\beta C_2(\lambda) - C_1(\lambda)\sqrt{\pi}\beta\ erf(x/2\sqrt{\alpha_0 t})\right)^{1/2} - 1\right). \tag{3.24}$$

In Eq. (3.24), the unknown parameter λ is involved. To determine λ, we substitute Eq. (3.20) into Eq. (3.19) which produces the following transcendental equation:

$$\frac{e^{-\lambda^2}}{2\beta}\left(2\gamma + \sqrt{\pi}\gamma^2 erf(\lambda) - \sqrt{4\beta\gamma^2(2+\beta) + (2\gamma + \sqrt{\pi}\gamma^2 erf(\lambda))^2}\right) + \frac{2\lambda}{Ste} = 0\,. \tag{3.25}$$

Once λ is obtained from the Eq. (3.25), the phase front position $s(t)$ can be found by substituting the value of λ into Eq. (3.15).

3.3.1.1 Existence of Unique Solution

To establish the existence and uniqueness of the above constructed solution discussed in case 1 of the section 3.3, we will show that there is a unique positive value of λ which is the solution of transcendental equation (3.25).

To show the unique value of λ, we define a function $g(\lambda)$ as

$$g(\lambda)=\frac{e^{-\lambda^2}}{2\beta}\left(2\gamma+\sqrt{\pi}\gamma^2 erf(\lambda)-\sqrt{4\beta\gamma^2(2+\beta)+(2\gamma+\sqrt{\pi}\gamma^2 erf(\lambda))^2}\right)+\frac{2\lambda}{Ste}. \quad (3.26)$$

It can be observed easily that $g(\lambda)$ is continuous on $(0,\infty)$ and

$$\lim_{\lambda\to 0^+} g(\lambda)<0, \qquad \lim_{\lambda\to\infty} g(\lambda)=\infty. \quad (3.27)$$

From the intermediate value theorem, it is clear that $g(\lambda)=0$ has at least one solution in $(0,\infty)$. Now, the derivative of $g(\lambda)$ defined in Eq. (3.26) is given by the following expression

$$g'(\lambda)=\frac{e^{-\lambda^2}}{\beta}\left(\sqrt{4\beta\gamma^2(2+\beta)+(2\gamma+\sqrt{\pi}\gamma^2 erf(\lambda))^2}-\left(2\gamma+\sqrt{\pi}\gamma^2 erf(\lambda)\right)\right)$$
$$\left(\frac{\gamma^2 e^{-\lambda^2}}{\sqrt{4\beta\gamma^2(2+\beta)+(2\gamma+\sqrt{\pi}\gamma^2 erf(\lambda))^2}}+\lambda\log(e)\right)+\frac{2}{Ste}. \quad (3.28)$$

Since $\left(2\gamma+\sqrt{\pi}\gamma^2 erf(\lambda)\right)<\sqrt{4\beta\gamma^2(2+\beta)+(2\gamma+\sqrt{\pi}\gamma^2 erf(\lambda))^2}$ therefore, the derivative of the function $g(\lambda)$ i.e., $g'(\lambda)$ is positive on the interval $(0,\infty)$ for positive values of Ste, β and γ. Hence $g(\lambda)$ is strictly increasing function in $(0,\infty)$ and it shows the uniqueness of λ. Clearly, this demonstrates the unique solution to the problem considered in case 1.

3.3.2 Case: 2

In this case, we will find the approximate solution to the problem (3.16)-(3.19) for all non-negative values of α and β.

As given in Eq. (1.4), first we take an approximation $f_N(\eta)$ of the dependent variable $f(\eta)$ of the Eq. (3.16) as follows:

$$f_N(\eta)=\sum_{i=0}^{N} c_i T_{\lambda,i}(\eta)=C^T\phi(\eta), \tag{3.29}$$

where $C^T=[c_0,c_1,c_2,...,c_N]$ and $\phi(\eta)=\left[T_{\lambda,0}(\eta),T_{\lambda,1}(\eta),...,T_{\lambda,N}(\eta)\right]^T$.

According to the Eqs. (1.6) and (1.8), we consider the following approximations for the derivatives of the function $f(\eta)$:

$$\frac{df_N(\eta)}{d\eta}=C^T D\phi(\eta), \tag{3.30}$$

$$\frac{d^2 f_N(\eta)}{d\eta^2}=C^T D^2\phi(\eta). \tag{3.31}$$

With the help of the approximations mentioned in Eqs. (3.29)-(3.31) for f, f' and f'', we define the following residual $R_N(\eta)$ for Eq. (3.16):

$$\begin{aligned} R_N(\eta)=2\eta(C^T D\phi(\eta))+2\alpha\eta(C^T\phi(\eta))^q(C^T D\phi(\eta))+(C^T D^2\phi(\eta)) \\ +p\beta(C^T\phi(\eta))^{p-1}(C^T D\phi(\eta))^2+\beta(C^T\phi(\eta))^p(C^T D^2\phi(\eta)). \end{aligned} \tag{3.32}$$

According to Tau method (Doha et al., 2011a, 2011b), we have the following condition:

$$\left\langle R_N(\eta),T_{\lambda,i}(\eta)\right\rangle=\int_0^{\lambda} R_N(\eta)T_{\lambda,i}(\eta)d\eta=0, \qquad i=0,1,...,N-2, \tag{3.33}$$

and from Eq. (3.33), we can generate a system of $(N-1)$ algebraic equations.

Besides above $(N-1)$ equations, we have three more equations obtained by putting the Eqs. (3.29)-(3.31) into the Eqs. (3.17)-(3.19):

$$C^T D\phi(0)+\beta(C^T\phi(0))^p(C^T D\phi(0))-\gamma(C^T\phi(0))=0, \tag{3.34}$$

$$C^T \phi(\lambda) = 1 \tag{3.35}$$

and

$$C^T D\phi(\lambda) = \frac{2\lambda}{(1+\beta)Ste}. \tag{3.36}$$

Now, the unknown coefficients of the vector C and λ can be found by solving the system of $(N+2)$ algebraic equations deliberated from Eqs. (3.33)-(3.36). From Eqs. (3.14) and (3.29), the temperature distribution $\theta(x,t)$ can be determined. Moreover, the location of moving interface $s(t)$ can be found with the aid of calculated λ and Eq. (3.15).

3.4 Comparisons and Results

In this section, we discuss the approximate results for accuracy of the proposed approximate solution of the considered problem and dependence of moving interface on various parameters involved in the model of the problem. Here, all the calculations are reported through tables and figures by taking the following operational matrices of derivatives D of order three, four and five, respectively:

$$D = \frac{2}{\lambda}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix},\ D = \frac{2}{\lambda}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 3 & 0 & 6 & 0 \end{pmatrix} \text{ and } D = \frac{2}{\lambda}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 3 & 0 & 6 & 0 & 0 \\ 0 & 8 & 0 & 8 & 0 \end{pmatrix}. \tag{3.37}$$

Table 3.1 displays the comparisons for the approximate values of the temperature distribution θ_A for N = 2, 3, 4 and exact values of temperature θ_E for the different Biot numbers (Bi) at $\alpha = \beta = 5,\ \alpha_0 = 1,\ t = 1$ and $Ste = 0.5$. The numerical results of approximate position of moving phase front $s_A(t)$ for N = 2, 3, 4 and exact location of moving phase front $s_E(t)$ are portrayed in Table 3.2 at $\alpha = \beta = 5,\ \alpha_0 = 1$ and $Ste = 0.5$. These tables demonstrates that the proposed approximate solutions i.e., θ_A and $s_A(t)$

γ	x	θ_E	$\theta_A\big\vert_{N=2}$	$\theta_A\big\vert_{N=3}$	$\theta_A\big\vert_{N=4}$
$\gamma = 0.5$	0.0 0.2 0.4 0.6 0.8 1.0	0.989950 0.998213 1.006260 1.013940 1.021130 1.027730	0.989973 0.998284 1.006330 1.014980 1.022310 1.029610	0.989950 0.998213 1.006260 1.013920 1.021040 1.027470	0.989950 0.998213 1.006260 1.013930 1.021120 1.027710
$\gamma = 5$	0.0 0.2 0.4 0.6 0.8 1.0	0.662276 0.735700 0.802467 0.862675 0.916418 0.963834	0.661211 0.734761 0.801858 0.862501 0.916691 0.964427	0.662287 0.735701 0.802422 0.862590 0.916342 0.963816	0.662274 0.735718 0.802486 0.862670 0.916471 0.963823
$\gamma = 10$	0.0 0.2 0.4 0.6 0.8 1.0	0.437309 0.561909 0.666834 0.756874 0.834537 0.901387	0.436625 0.563429 0.671098 0.762024 0.838601 0.903220	0.437256 0.562629 0.667771 0.757103 0.833970 0.900638	0.437307 0.561906 0.666832 0.756873 0.834370 0.901387

Table 3.1. Comparison of the exact values of temperature θ_E and approximate values of temperature θ_A at $p = q = 1$ and $Ste = 0.5$.

γ	Time (t)	$s_E(t)$	$s_A(t)\big\vert_{N=2}$	$s_A(t)\big\vert_{N=3}$	$s_A(t)\big\vert_{N=4}$
$\gamma = 0.5$	0.2 0.4 0.6 0.8 1.0	0.109047 0.154215 0.188875 0.218094 0.243836	0.109053 0.154224 0.188885 0.218106 0.243850	0.109047 0.154215 0.188875 0.218094 0.243836	0.109047 0.154215 0.188875 0.218094 0.243836
$\gamma = 5$	0.2 0.4 0.6 0.8 1.0	0.524790 0.742165 0.908963 1.049580 1.173470	0.523404 0.740205 0.906562 1.046810 1.170370	0.524752 0.742111 0.908897 1.049560 1.173380	0.524791 0.742161 0.908960 1.049570 1.173460
$\gamma = 10$	0.2 0.4 0.6 0.8 1.0	0.612142 0.865699 1.060260 1.224280 1.368790	0.613977 0.868294 1.063440 1.227950 1.372890	0.612358 0.865102 1.060910 1.224050 1.368150	0.612142 0.865699 1.060260 1.224280 1.368790

Table 3.2. Comparison of the exact values of moving boundary $s_E(t)$ and approximate values of moving boundary $s_A(t)$ at $p = q = 1$ and $Ste = 0.5$.

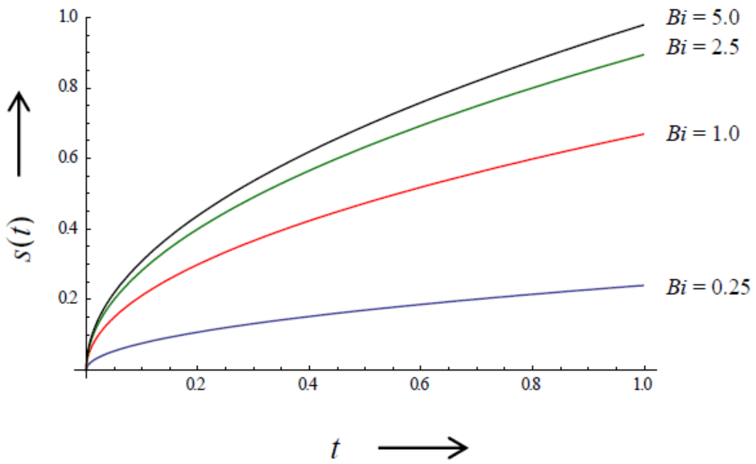

Fig.3.1. Plot of tracking of phase front $s(t)$ vs. t for different Bi at $p = q = 2$.

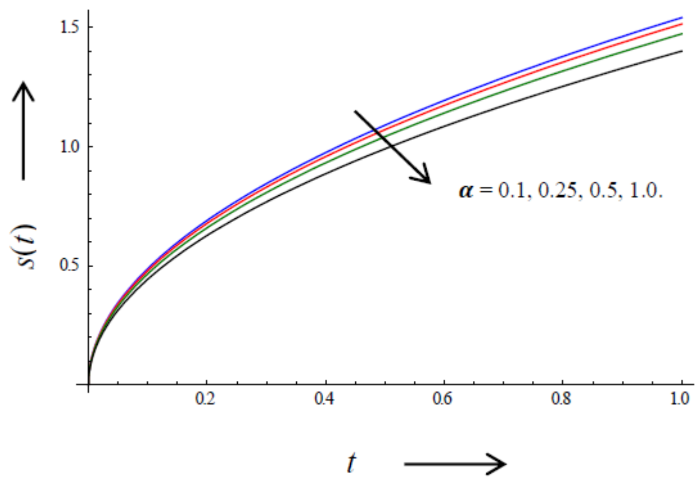

Fig.3.2. Plot of tracking of phase front $s(t)$ vs. t for different α at $p = q = 2$.

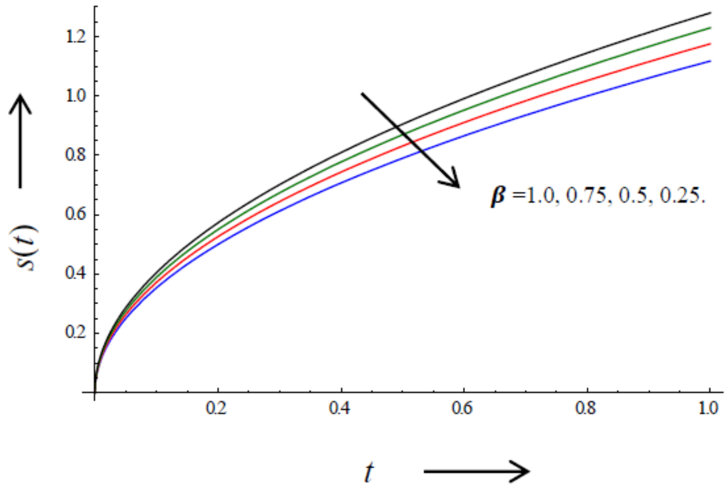

Fig.3.3. Plot of tracking of phase front $s(t)$ vs. t for different β at $p = q = 2$.

are close to its exact solutions in case 1. When we increase the order of operational matrix of differentiation, then the growth in the accuracy of the proposed solution is also noticed from Tables 3.1 and 3.2. Hence, this approximate technique will be a beneficial tool to explore the non-linear phase change problems.

Now, we discuss the effect of parameters α and β on moving phase front $s(t)$ at p = q = 2 which are shown in Figs. 3.1-3.3 by considering the approximate technique discussed in case 2. Fig. 3.1 shows the dependence of $s(t)$ on Biot number (Bi) for $\alpha = 0.5$, $\beta = 1.0$, $Ste = 0.5$ and $\alpha_0 = 1.0$. It is observed from this figure that the tracking of phase front $s(t)$ upswings if we increase the value of Biot number. Moreover, the evolution in the movement of $s(t)$ is detected when the parameter Bi is raised. The dependence of the location of phase front $s(t)$ on α is displayed in Fig. 3.2 at $\beta = 0.5$, $\gamma = 2.0$, $\alpha_0 = 1.0$ and $Ste = 2.5$. Fig. 3.3 indicates the effects of β on trajectory of $s(t)$ at $\alpha = 0.5$, $\gamma = 2.0$, $\alpha_0 = 1.0$ and $Ste = 1.5$. It can be observed from

Fig. 3.1 that the tracking of moving interface enhances with the decrease in α. However, opposite to this trend is observed in Fig. 3.3 that is the tracking of $s(t)$ increases when β increases. Beside these observations, it is also found that the Biot number has more impact on the movement of phase front than α and β.

3.5 Conclusion

The present chapter is devoted to a freezing problem having variable heat capacity and thermal conductivity in a semi-infinite domain. A Robin boundary condition on the boundary $x = 0$ is also taken into the account. For a particular case ($\alpha = \beta$, $p = q = 1$), the similarity solution for the problem is discussed with its uniqueness and existence. An approximate solution to the problem based on spectral tau method is also proposed for small values of α and β. In our study, it is established that the moving phase front $s(t)$ is equal to the constant multiple of $\sqrt{t}$ in the considered problem which is similar to the previous results given in (Crank, 1984; Hill, 1986).

From comparisons and discussion section, it is noticed that the applied approximate technique is reliable, easy and efficient to apply on the phase change problems after the similarity reduction. It is also apparent that when we increase the order of matrix of differentiation, the obtained result tends to the exact solution rapidly. One more important point is observed for $p = q = 2$ that is the freezing process becomes slow if we increase the value of α. However, the freezing process becomes fast when we increase the values of either Biot number (Bi) or β or both. It is also observed that the effect of Biot number on the freezing process is more than α or β.

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