Chapter 2
Preliminaries

2.1 Introduction

This chapter is a discourse to the essential definitions, mathematical relations and their properties for understanding the fractional calculus. The chapter also provides a brief overview of fractional-order systems, approximation of fractional order operators, fractional order control and fractional order toolbox for MATLAB. Although the preliminaries elaborated in the chapter is not a complete overview of fractional order calculus but they illustrate the major essentials used in this thesis.

2.2 Mathematical Preliminaries

Fractional calculus and fractional control involve a lot of mathematics. As a result, a brief and essential introduction of the mathematical definitions and formulas is provided here for understanding the proposed work.

2.2.1 Gamma function

The gamma function plays an important role for calculating the Laplace transform of non-integer power of 's'. Usage of the function conveniently defines certain formulas related to fractional calculus [4]. It is defined in different manner depending on its application. The most elementary definition of the gamma function is Euler's integral, given as:

$$\Gamma(\mathrm{z}) = \int_0^{\infty} e^{-u} u^{z-1} du, \quad (z > 0) \tag{2.1}$$

For convergence of the integral, it is restricted to positive values of z such that it can fulfill at least one of the following conditions: $\mathrm{z} \in \mathrm{R}^{+}$, $\mathrm{z} \notin \{\{0\} \cup \mathrm{Z}^{-}\}$, or $\mathrm{Im[z]} \neq 0$. Some of the properties of gamma function are given as:

I. The most significant property is by applying integration by parts in (2.1); resulting in:

$$\Gamma(z+1)=z\Gamma(z) \tag{2.2}$$

II. If $z = n$ (a positive integer), then the gamma function is defined in terms of a factorial function as:

$$\Gamma(n)=(n-1)! \tag{2.3}$$

III. For some particular value of z there exist an exact values of $\Gamma(z)$. Then the gamma function calculated at $z = {}^{1}/_{2}$ is

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi} \tag{2.4}$$

The value of the gamma function for any positive real number is calculated by using the above three properties and it lies between 1and 2 [4].

2.2.2 Mittag-Leffler function

This function was introduced by the Swedish mathematician G. M. Mittag-Leffler (1846-1927). It is a generalized form of the exponential function that plays a significant role in fractional calculus. Two different forms of the Mittag-Leffler function is given below [4].

Definition 1: The one-parameter representation of the Mittag-Leffler function is given as:

$$E_{\varsigma}(t)=\sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(\varsigma k+1)}, \quad (\varsigma>0) \tag{2.5}$$

Definition 2: The two-parameter representation of the Mittag-Leffler function is given as:

$$E_{\varsigma,\tau}(t)=\sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(\varsigma k+\tau)}, \quad (\varsigma,\tau>0) \tag{2.6}$$

Another well-known function is Miller-Ross function alike the Mittag-Leffler function is defined as:

$$E_{\varsigma,\tau}(t)=\sum_{k=0}^{\infty}\frac{\tau^{k}t^{k+\varsigma}}{\Gamma(\varsigma+\mathrm{k}+1)}=\mathrm{t}^{\varsigma}\mathrm{E}_{1,\varsigma+1}(\tau t) \tag{2.7}$$

2.2.3 Riemann-Liouville derivatives

Let us consider the Riemann-Liouville formula defined for the fractional integral as

$${}_{a}^{R}I_{t}^{\upsilon}f(t)=\frac{1}{\Gamma(\upsilon)}\int_{a}^{t}f(t-\tau)^{\upsilon-1}f(\tau)d\tau \tag{2.8}$$

where $\upsilon\in\Re^{+}$ is the order of integral at terminals $(a,\ t)$ of the function $f(t)$ [25].

Extending the equation (2.8) for $\upsilon\leq 0$, the expression for Riemann-Liouville (R-L) fractional order differ-integral modifies to:

$${}_{a}^{R}D_{t}^{\upsilon}f(t)=\frac{1}{\Gamma(\mathrm{m}-\upsilon)}\frac{d^{m}}{dt^{m}}\int_{a}^{t}\frac{f(\tau)}{(t-\tau)^{\upsilon-m+1}}d\tau \tag{2.9}$$

where $\upsilon\in\Re^{+}$ is a fractional order of the differ-integral of a function $f(t)$. For $\upsilon>0, m-1<\upsilon\leq m$ where $m\in\Re$ and for $\upsilon\leq 0, m=0$.

It is observed that, when $\upsilon>0$ the result of the equation (2.9) is equal to the fractional order derivative, for $\upsilon<0$, it gives fractional order integral and for $\upsilon=0$ it becomes the function itself. Thus, the above definition is known as a differ-integral, and the Riemann-Liouville (R-L) fractional order differ-integral is re-written as:

$${}_{a}^{R}D_{t}^{\upsilon}f(t)=\begin{cases}\dfrac{1}{\Gamma(\mathrm{m}-\upsilon)}\dfrac{d^{m}}{dt^{m}}\displaystyle\int_{a}^{t}\frac{f(\tau)}{(t-\tau)^{\upsilon-m+1}}d\tau, & m-1<\upsilon\leq m\in\Re\\ \dfrac{d^{m}}{dt^{m}}f(t) & \upsilon=m\in\Re\end{cases} \tag{2.10}$$

The major property of ${}_{a}^{R}D_{t}^{\upsilon}f(t)$ is the linearity for integer as well as fractional-order differentiation. Other properties of this function are described in [3], [5], [125].

2.2.4 Caputo derivatives

The fractional order about the initial conditions creates difficulty with exploitation of R-L definition in realistic problems. Although, the initial values problem can be fruitfully resolved mathematically but the physical interpretation of these types of initial conditions is still unknown. Solution of this problem was presented by M. Caputo in 1966 [126].

Assuming that $\upsilon > 0, t > a,$ and $\upsilon, a, t \in \Re$, Caputo (C) definition for fractional order differ-integral is given as:

$$
{}_{a}^{C}D_{t}^{\upsilon}f(t) = \begin{cases} \dfrac{1}{\Gamma(\mathrm{m}-\upsilon)} \displaystyle\int_{a}^{t} \dfrac{f(\tau)}{(t-\tau)^{\upsilon-m+1}} d\tau, & m-1 < \upsilon \leq m \in \Re \\ \dfrac{d^{m}}{dt^{m}} f(t) & \upsilon = m \in \Re \end{cases} \tag{2.11}
$$

Although, the Caputo operator ${}_{a}^{C}D_{t}^{\upsilon}f(t)$ is linear but it offers more restriction for function $f(t)$ then the R-L definition. The Caputo definition can only be applied to the function $f(t)$ whose m^{th}-order derivative is absolutely integrable.

2.2.5 Left-sided Grunwald-Letnikov derivatives

A varied form of formula for calculation of fractional order derivatives is presented by Grunwald and Letnikov using the derivative by limit. This definition is the generalization of backward difference for fractional order [5].

Consider a continuous-time function $y = f(t)$. Let the first-order derivative of $f(t)$ is given as:

$$f'(t) = \frac{df}{dt} = \lim_{h \to 0} \frac{f(t-h) - f(t)}{h} \tag{2.12}$$

Similarly, the second-order derivative can be written as:

$$f''(t) = \frac{d^2 f}{dt^2} = \lim_{h \to 0} \frac{f'(t-h) - f'(t)}{h} = \lim_{h \to 0} \frac{f(t-2h) - 2f(t-h) + f(t)}{h^2}$$

Similarly, the formula for n[th]-order derivative can be written as:

$$f^n(t) = \frac{d^n f}{d^n t} = \lim_{h \to 0} \frac{1}{h^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(t - rh), \qquad (n \in \Re) \tag{2.13}$$

The equation (2.13) is the generalized form of non-integer order derivative known as Grunwald-Letnikov definition. The approximation of this derivative is widely used in computer applications.

2.2.6 Properties of fractional order derivatives

Some of the major properties of the fractional order derivative are as:

- For any analytical function $f(t)$ in t, its fractional derivative ${}_aD_t^{\upsilon} f(t)$ will also be an analytical function of z and a.
- If α is an integer (i.e.$a = n$), the process ${}_aD_t^{\upsilon} f(t)$ gives the identical result as classical differentiation of integer order n.
- For a $= 0$; the fractional order differ-integral operator ${}_aD_t^{\upsilon} f(t) = f(t)$.
- Both the fractional differentiation and integration are linear operations:

$${}_0D_t^{\upsilon}\{\mathrm{k}_1 f(t) + k_2 g(t)\} = \mathrm{k}_1\, {}_0D_t^{\upsilon} f(t) + \mathrm{k}_2\, {}_0D_t^{\upsilon} g(t).$$

- The additive index law (semi-group property)

$${}_0D_t^{\upsilon}\, {}_0D_t^{\theta} f(t) = {}_0D_t^{\theta}\, {}_0D_t^{\upsilon} f(t) = {}_0D_t^{\upsilon+\theta} f(t)$$

This condition holds under some logical constraints on the function $f(t)$.

The fractional-order derivative can also adjust with the integer order derivative as:

$$\frac{d^n}{dt^n}\left({}_aD_t^{\upsilon} f(t)\right) = {}_aD_t^{\upsilon}\left(\frac{d^n f(t)}{dt^n}\right) = {}_aD_t^{\upsilon+n} f(t),$$

in the clause $t = a$, $f^{(k)}(\mathrm{a}) = 0,\ (k = 0,1,2,........,n-1).$

In addition to the above discussion, many researchers have interpreted the fractional order integral and derivative in varied forms [4], [127].

2.2.7 Laplace transforms of fractional order operator

The Laplace transform is widely used as integral transform in mathematics for many engineering applications. Laplace transform of a function $f(t)$ with real argument $(t \geq 0)$ is represented by $\mathrm{F(s)}$ with complex arguments and given by the integral relation:

$$\mathrm{F(s)} = \int_0^{\infty} f(t)e^{-st}dt \tag{2.14}$$

For many practical applications the Laplace transform is bijective function and it is most useful property. Hence, the Laplace transform pose an important role in fractional order calculus [4]. Some of the repeatedly used Laplace transformations of fractional order operators are examined below.

a) Laplace transformation of Riemann-Liouville integration.

$$L\left[{}_0I_t^{\upsilon} f(t)\right] = s^{-\upsilon}F(s) \tag{2.15}$$

b) Laplace transformation of Riemann-Liouville derivative.

$$L\left[{}_0D_t^{\upsilon} f(t)\right] = s^{\upsilon}F(s) - \sum_{k=0}^{m-1} s^k\, {}_0D_t^{\upsilon-k-1} f(0) \tag{2.16}$$

where $\upsilon \in \Re, m-1 < \upsilon \leq m$ and $m \in \Re$.

c) Laplace transformation of Caputo derivative

$$L\left[{}_{0}^{C}D_{t}^{\upsilon}f(t)\right] = s^{\upsilon}F(\mathrm{s}) - \sum_{k=0}^{m-1} s^{\upsilon-k-1}D^{k}f(0) \qquad (2.17)$$

where $\upsilon \in \Re, m-1 < \upsilon \leq m$ and $m \in \Re$.

The detailed explanation and proof of equation (2.15), (2.16) and (2.17) is available in [4].

2.3 Fractional-order systems and representation

In control theory, a fractional order system is a dynamical system which is derived from the fractional differential equation having derivatives of non-integer order. These types of system are known as fractional dynamics and are described as fractional order derivatives and integrals. The main advantage of fractional derivatives is its capability of describing the memory and hereditary properties of various materials and processes which are negligible in usual integer-order models. The benefits of the fractional derivatives become clear in modeling many electrical and mechanical properties of actual materials. Moreover, Fractional order systems are also very helpful in studying the inconsistent actions of dynamical systems in all fields like; physics, biology, electrochemistry, viscoelasticity and many messy systems [125], [128], [129].

The generalized equation of a continuous-time fractional order dynamic system is given as:

$$H\left(D^{\upsilon_0 \upsilon_1 \upsilon_2 \ldots\ldots\ldots \upsilon_m}\right)\left(y_1, y_2, \ldots\ldots\ldots\ldots, y_l\right) = G\left(D^{\theta_0 \theta_1 \theta_2 \ldots\ldots\ldots \theta_m}\right)\left(u_1, \mathrm{u}_2, \ldots\ldots\ldots\ldots, \mathrm{u}_k\right) \qquad (2.18)$$

where $H(.), G(.)$ are combination laws of fractional order derivative operator and u_k, y_l are time dependent function of input and output.

For continuous-time linear time-invariant single input single output system, the equation is written as:

$$H\left(D^{\upsilon_0 \upsilon_1 \upsilon_2 ....\upsilon_m}\right) y(t) = G\left(D^{\theta_0 \theta_1 \theta_2 ....\theta_m}\right) u(t) \tag{2.19}$$

with $H\left(D^{\upsilon_0 \upsilon_1 \upsilon_2 ....\upsilon_m}\right) = \sum_{k=0}^{n} a_k D^{\upsilon k}; G\left(D^{\theta_0 \theta_1 \theta_2 ....\theta_m}\right) = \sum_{k=0}^{m} b_k D^{\theta k}$ where $a_k, b_k \in \Re$.

Hence, (2.19) can be explicitly written as:

$$(a_n D^{\upsilon_n} + a_{n-1} D^{\upsilon_{n-1}} + ... + a_0 D^{\upsilon_0}) y(t) = (\mathrm{b}_m D^{\theta_n} + b_{n-1} D^{\theta_{n-1}} + ... + b_0 D^{\theta_0}) u(t) \tag{2.20}$$

In equation (2.20), if all the orders of derivation are integer multiples of a base order i.e. $\upsilon_k, \theta_k = k\upsilon, \upsilon \in \Re^+$ then this type system is known as commensurate-order system and equation (2.20) is:

$$\sum_{k=0}^{n} a_k D^{k\upsilon} y(t) = \sum_{k=0}^{m} b_k D^{k\theta} u(t) \tag{2.21}$$

Briefly, the continuous time linear time-invariant systems are classified as:

$$\text{LTI Systems} \begin{cases} \text{Non-integer(fractional)} \begin{cases} \text{Commensurate} \begin{cases} \text{Rational} \\ \text{Irrational} \end{cases} \\ \text{Non-Commensurate} \end{cases} \\ \text{Integer} \end{cases}$$

The transfer function, stability and other aspects of continuous time fractional order system is described in following subsections.

2.3.1 Transfer function of fractional order models

The generalized fractional order transfer function is defined as the ratio of Laplace transform of input and the output with zero initial conditions and is written as:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + ... + b_0 s^{\beta_0}}{a_n s^{\upsilon_n} + a_{n-1} s^{\upsilon_{n-1}} + ... + a_0 s^{\upsilon_0}} \tag{2.22}$$

where $\mathrm{U}(s)$ and $Y(s)$ are the Laplace transform of input and output respectively. In addition to this, all initial conditions are assumed to be zero i.e. $\mathrm{u}(0)=0, \mathrm{y}(0)=0$ and $m, n \in \Re^{+}$.

Suppose, the system is a commensurate order system then the fractional order transfer function is written as:

$$G(s)=\frac{\sum_{k=0}^{m} b_k \left(s^{\upsilon}\right)^{k}}{\sum_{k=0}^{n} a_k \left(s^{\upsilon}\right)^{k}} \tag{2.23}$$

On the basis of order of the system, the LTI systems can be classified as integer order systems or fractional order systems. The Fractional order systems are further categorized into two parts; commensurate and non-commensurate order systems. In commensurate order systems the fractional powers are integer multiples of a fractional number whereas, in non-commensurate order systems no such generalization occurs.

2.3.2 State-space representation of fractional order system

The state-space representation of a linear continuous time fractional order system is presented by Grunwald-Letnikov and given as:

$$ {}_{a}D_{t}^{\vartheta} x(t)=Ex(t)+Fu(t) \tag{2.24}$$

$$y=Gx(t)+Hu(t) \tag{2.25}$$

where $\upsilon \in \Re$ is the order of differential equation, $E \in \Re^{n\times n}, F \in \Re^{n\times m}, G \in \Re^{p\times n}, H \in \Re^{p\times m}$ are the constant matrices, p is number of inputs, m is number of outputs, n is number of state equations [12].

2.3.3 Stability of fractional order system

Stability is a common topic of discussion for any dynamical system and its behavior. In mathematical terminology, stability theory locates the convergence of solutions of differential or difference equations of dynamical systems under specified initial conditions. In a similar manner, the researchers have given much focus on the stability and stabilization of the system represented by fractional order differential equations. In general control theory, it is well defined that a linear time-invariant system will be stable if and only if the roots of its characteristic equation lie on the left half of the s-plane. In another word, the roots of the characteristics polynomial must have negative real part. But in case of fractional-order LTI system, the stability is defined in a different manner discussed in [4]. It is fascinating that a fractional system having roots in the right half of s-plane may be stable and roots in the left half plane may be unstable. Stability of the FOS is discussed by I. Petras [130] which is based on Riemann surface. The behavior and stability of the FOS is presented in the form of power law which also shows the long-term memory on FOS. Here, the stability region of a fractional order system represented by s^{υ} is discussed.

Consider the Laplace transform of fractional order differential equation in equation (2.24); be given as:

$$s^{\upsilon}\,\mathrm{X(s)} = E\,\mathrm{X(s)} + FU\mathrm{(s)} \tag{2.26}$$

Then, the system transfer function is:

$$G(s) = \frac{X(s)}{\mathrm{U}(s)} = \frac{F}{s^{\upsilon} + E} \tag{2.27}$$

Performing the conformal transformation of s as

$$\psi = s^{\upsilon} \tag{2.28}$$

The equation (2.27) is re-written as:

$$G(s) = \frac{F}{\psi + E} \tag{2.29}$$

Using this transformation, the poles of ψ -plane will be analyzed. Once the time domain responses corresponding to the location of new ψ -plane poles are found, their performance in new ψ -plane can be easily described.

For this, the s-plane is mapped along with the time-domain function properties of each point, into the ψ -plane. For ease of the engineers it is assumed that $0 < \upsilon \leq 1$. Then equation (2.28) is written as:

$$\psi = s^{\upsilon} = \left(re^{j\theta}\right)^{\upsilon} = r^{\upsilon} e^{j\theta\upsilon} \tag{2.30}$$

For stability, it is compulsory to map the imaginary axis $s = re^{\pm j\pi/2}$. So the mapping of the axis in ψ -plane is:

$$\psi = r^{\upsilon} e^{\pm j\frac{\upsilon\pi}{2}} \tag{2.31}$$

which represents a pair of lines at $\phi = \pm\frac{\upsilon\pi}{2}$, where ϕ is the angle in ψ -plane.

Here, the right half side of the s-plane is mapped into a segment in the ψ -plane of angle less than $\pm\frac{\upsilon\pi}{2}$ degrees. Similarly, a different situation may be analyzed in the case of $1 < \upsilon < 2$. Both the cases are shown in figure 2.1.

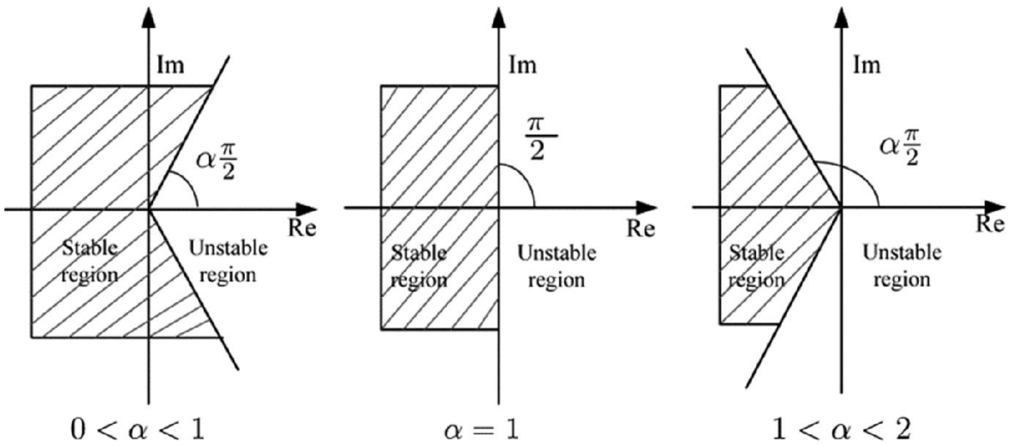

Fig. 2.1. Stability region of the fractional order system

In addition to the above elaboration of stability for fractional order system, a lot more types of stability are discussed in review papers published by M. Riveroetal [22], [23], [124], [131].

2.3.4 Time and frequency domain analysis

One of the major difficulties while working with the fractional order systems is its time and frequency domain analysis. Exact analytical solution for these problems is not possible. However, there is no universal method for time and frequency domain analysis. Most of the researchers use the approximation techniques to convert the fractional order system into an integer order. In this subsection, method for calculation of transient and frequency domains response is illustrated.

2.3.4.1 Transient response

Analytical method is used to determine the time-domain response. The transient response is only depends upon the roots of the characteristics equation and there are six different case for considering this [73]:

i. The response will be monotonically decreasing function if no root lies in the Riemann principal sheet.

ii. If the roots lie in the Riemann principal sheet and their co-ordinates are $\Re(s) < 0, \mathrm{I(s)} = 0$; then the response will also be a monotonically decreasing function.

iii. If the roots in the Riemann principal sheet and $\Re(s) < 0, \mathrm{I(s)} \neq 0$, then the response will be a damped oscillations.

iv. If the roots in the Riemann principal sheet and $\Re(s) = 0, \mathrm{I(s)} \neq 0$, then the response will be an oscillations with constant amplitude.

v. If the roots in the Riemann principal sheet and $\Re(s) > 0, \mathrm{I(s)} \neq 0$, then the response will be an oscillations with increasing amplitude.

vi. If the roots in the Riemann principal sheet and $\Re(s) > 0, \mathrm{I(s)} = 0$, then the response will be a monotonically increasing function.

For a specific case of commensurate-order systems, the impulse response is calculated by:

$$L^{-1}\{\mathrm{H}(\psi\}, \psi = \mathrm{s}^{\upsilon} = L^{-1}\left\{\frac{\sum_{k=0}^{m} a_k \psi^k}{\sum_{k=0}^{n} b_k \psi^k}\right\} = L^{-1}\left\{\sum_{k=0}^{n} \frac{r_k}{\psi - \psi_k}\right\} \tag{2.32}$$

In general form,

$$L^{-1}\left\{\frac{s^{\upsilon-\beta}}{s^{\upsilon} - \psi_k}\right\} = t^{\beta-1} E_{\upsilon,\beta}(\psi_k t^{\upsilon}) \tag{2.33}$$

For calculation of impulse response $g(t)$, put $\upsilon = \beta$ in equation (2.33) that result in:

$$g(t) = \sum_{k=0}^{n} r_k t^{\upsilon-1} E_{\upsilon,\upsilon}(\psi_k t^{\upsilon}) \tag{2.34}$$

And the step response is given by:

$$y(\mathrm{t}) = L^{-1}\left\{\sum_{k=0}^{n} \frac{r_k s^{-1}}{\left(s^{\upsilon} - \psi_k\right)}\right\} \tag{2.35}$$

The different conditions for nature of response are:

i. Monotonically falling if $\left|\arg(\psi_k)\right| \geq \upsilon\pi$.

ii. Oscillatory with falling amplitude if $\frac{\upsilon\pi}{2} < \left|\arg(\psi_k)\right| < \upsilon\pi$.

iii. Oscillatory with unvarying amplitude if $\left|\arg(\psi_k)\right| = \frac{\upsilon\pi}{2}$.

iv. Oscillatory with growing amplitude if $\left|\arg(\psi_k)\right| < \frac{\upsilon\pi}{2}, \left|\arg(\psi_k)\right| \neq 0$.

v. Monotonically growing if $\left|\arg(\psi_k)\right| = 0$.

Time domain response corresponding to the five cases is shown in Figure 2.2.

2.3.4.2 Frequency Domain Response

The frequency response of fractional order transfer function is directly evaluated along the imaginary axis by substituting $s = j\omega$, where $\omega \in (0, \infty)$ [73]. However, frequency response of commensurate order fractional systems is analyzed by obtaining its Bode plots. Hence, the frequency response of any fractional order system is obtained by factorizing the system transfer function and adding the individual contributions of the terms having fractional power υ. The other way for frequency response analysis of fractional order system is to approximate it in integer order but may not show exact response of the system. The detail analysis of different approximation algorithms are discussed in the next section.

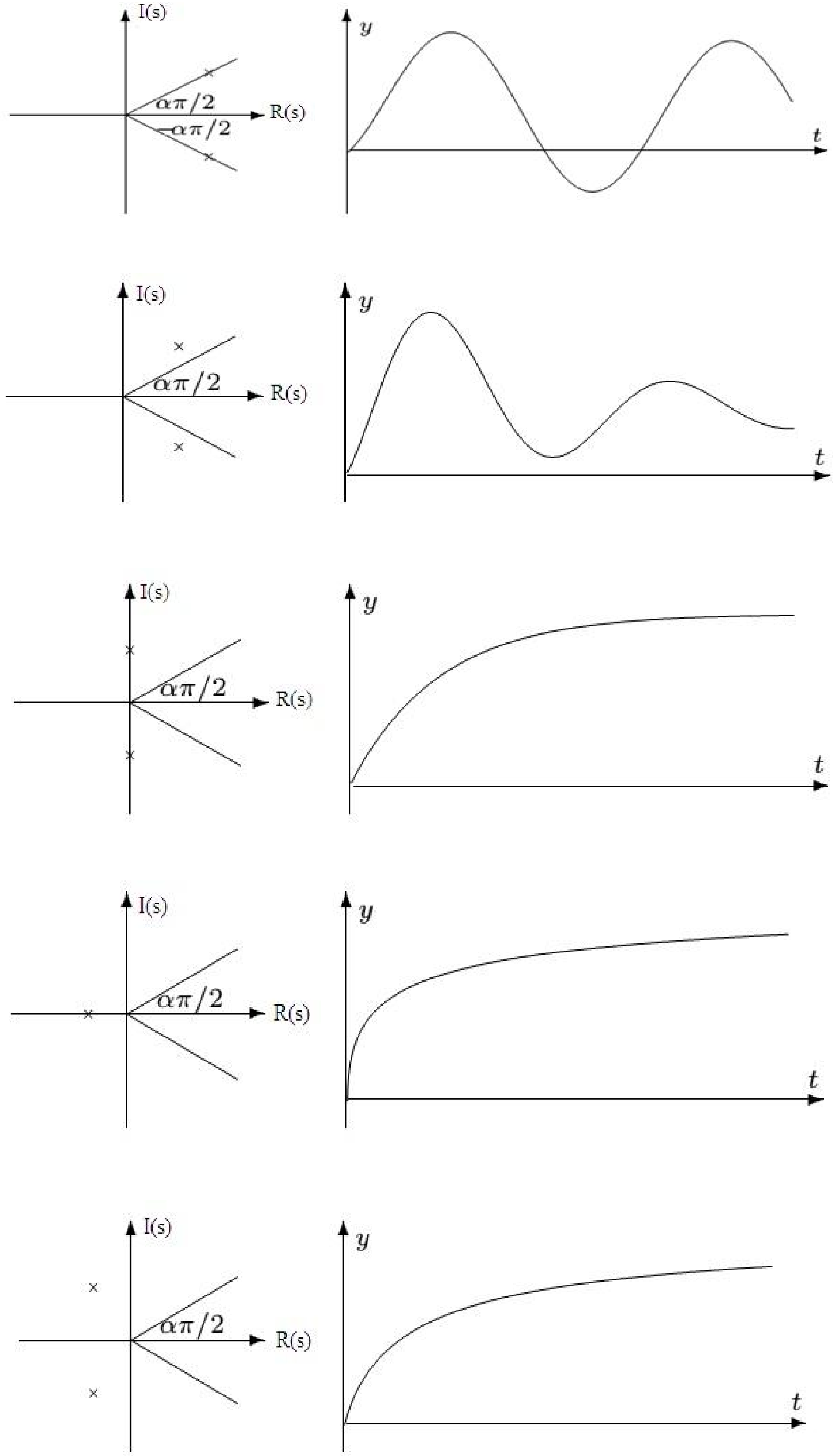

Fig. 2.2. Location of Roots and the corresponding time-domain responses of fractional order system

2.3.4.3 Steady-state Response

The time-domain analysis of any system is incomplete without knowing the steady-state behavior of the systems. In this subsection, the standard definition of steady state error coefficients of different type of fractional order system is stated. From the literature, the standard definition of steady state error coefficients for integer order system is;

i. Position error coefficient

$$K_p = \lim_{s \to 0} G(s) \tag{2.36}$$

ii. Velocity error coefficient

$$K_v = \lim_{s \to 0} sG(s) \tag{2.37}$$

iii. Acceleration error coefficient

$$K_a = \lim_{s \to 0} s^2 G(s) \tag{2.38}$$

Now, consider a generalized form of a fractional order system given by the transfer function:

$$G(s) = \frac{K\left(b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \dots\dots\dots + b_0 s^{\beta_0}\right)}{s^{\sigma}\left(a_n s^{\upsilon_n} + a_{n-1} s^{\upsilon_{n-1}} + \dots\dots\dots\dots\dots + a_0 s^{\upsilon_0}\right)} \tag{2.39}$$

Then, the modified standard definition of steady state error coefficients for fractional order system is given by following relations:

$$K_p = \lim_{s \to 0} \frac{K}{s^{\sigma}} = \lim_{s \to 0} K s^{-\sigma}, e_p = \frac{1}{1 + K_p} \tag{2.40}$$

$$K_v = \lim_{s \to 0} K \frac{s}{s^{\sigma}} = \lim_{s \to 0} K s^{1-\sigma}, e_v = \frac{1}{K_v} \tag{2.41}$$

$$K_a = \lim_{s \to 0} K \frac{s^2}{s^\sigma} = \lim_{s \to 0} K s^{2-\sigma}, e_a = \frac{1}{K_a} \tag{2.42}$$

Table 2.1 Steady-state errors and steady-state error coefficients

S.N.	Steady state (Integer order system)					Steady state(Fractional order system)				
1	σ	K_p, e_p	K_v, e_v	K_a, e_a	Type	σ	K_p, e_p	K_v, e_v	K_a, e_a	Type
2	0	$K, K/(1+K)$	$0,\infty$	$0,\infty$	0	(0,1)	$\infty,0$	$0,\infty$	$0,\infty$	0/1
3	1	$\infty,0$	$K, 1/K$	$0,\infty$	1	(1,2)	$\infty,0$	$\infty,0$	$0,\infty$	1/2
4	2	$\infty,0$	$\infty,0$	$K, 1/K$	2	(2,3)	$\infty,0$	$\infty,0$	$\infty,0$	2/3

The steady-state errors and steady-state error coefficients for both the integer and fractional order systems for different values of σ are summarized in Table 2.1 that notifies about the steady-state error coefficients of fractional order system to be either 0 or ∞. Thus, the fractional and integer order systems are analogous in terms of their performance stated by steady state error and coefficients.

2.4 Approximation of Fractional Order Operators

In common practice the control system may have either the fractional order dynamic system (i.e. the fractional order plant) to control or the fractional-order controller or both. However, implementation of fractional order controller is common because mathematical model of the plant may has obtained using classical sense and has integer order. The fractional order PID controller involves the fractional-order differentiation and integration which offers additional efficiency and robustness to control integer order as well as fractional-order dynamical systems.

The main problem with the fractional order system or controller occurs for its digital implementation. Here approximation of fractional terms into an integer order model by preserving same properties in a suitable frequency range is needed. Many techniques are available in continuous-time and discrete time domain, using transfer function as well as the state space representation [132]. Two of the commonly used approximation algorithms are listed below:

1) **Approximations using continued fraction expansions and interpolation techniques.**

Continued fraction expansion (CFE) is well known technique for estimation of functions which converges much rapidly than power series expansions in larger domain of the complex plane. There are three major techniques which are based on this approximation:

a) General CFE method for approximation of fractional integro-differential operators.
b) Carlson's method.
c) Matsuda's method.

2) **Approximations using curve fitting or identification techniques.**

The three fundamental approximation techniques based on curve fitting technique that are common in practice.

a) Oustaloup Recursive Approximations.
b) Modified Oustaloup Approximations.
c) Chareff's method.

The Oustaloup Recursive Approximations is used in this work and is discussed further.

2.4.1 Oustaloup recursive approximations

The Oustaloup approximation[1], [47], [133] is widely used for integer order approximation where a frequency band of interest is considered. Suppose $[\omega_b, \omega_h]$ is the frequency range to be fit. Then the term s/ω_u is substituted with:

$$C_0 \frac{1+ {}^{s}\!/_{w_b}}{1+ {}^{s}\!/_{w_h}} \tag{2.43}$$

where $\sqrt{w_b w_h} = w_\mu$ and $C_0 = \frac{w_b}{w_\mu} = \frac{w_\mu}{w_h} w_b$.

The technique is implemented for approximation of a function in a particular form $H(s) = s^\upsilon$, where $\upsilon \in \Re^+$.

The Oustaloup's approximation model to a fractional order differentiator s^υ is written as:

$$G_I(s) = (C_0)^\upsilon \prod_{k=-N}^{N} \frac{1+s/\omega'_k}{1+s/\omega_k} \tag{2.44}$$

where $\omega'_k = \omega_b \left(\frac{\omega_b}{\omega_u}\right)^{\frac{k+N+\frac{1}{2}+\frac{\upsilon}{2}}{2N+1}}$ and $\omega_k = \omega_b \left(\frac{\omega_b}{\omega_u}\right)^{\frac{k+N+\frac{1}{2}-\frac{\upsilon}{2}}{2N+1}}$ are the zeros and poles of rank k respectively and $(2N+1)$ is the total number of zeros or poles.

Matlab function for Oustaloup's approximation is given as ***ousta_fod()*** and defined as:

```
function G=ousta_fod(gam,N,wb,wh)
k=1:N;
wu=sqrt(wh/wb);
wkp=wb*wu.^((2*k-1-gam)/N);
wk=wb*wu.^((2*k-1+gam)/N);
G=zpk(-wkp,-wk,wh^gam);
G=tf(G);
```

where *gam* is order of derivative, *N* is order of the filter, *(wb,wh)* is desired frequency range [47].

In the case of FOTF with multiple fractional powered terms, each fractional term is approximated individually and combined to obtain the final approximant.

2.4.2 Modified Oustaloup's filter

In many practical applications it is found that the Oustaloup approximation does not provide exactly fit function in the entire frequency range of interest$[\omega_b, \omega_h]$. Here a modified version of the Oustaloup recursive filter is proposed for fractional-order derivative. That performs better than the previous version for all the systems. The modified Oustaloup's approximation model for a fractional order differentiator s^{υ} is given as:

$$s^{\upsilon} \approx \left(\frac{d\omega_h}{b}\right)^{\upsilon} \left(\frac{ds^2 + b\omega_h s}{d(1-\upsilon)s^2 + b\omega_h s + d\upsilon}\right) \prod_{k=-N}^{N} \frac{s+\omega'_k}{s+\omega_k} \tag{2.45}$$

The above modified filter is stable for $\upsilon \in (0,1)$. Numerous experimental studies and theoretical analysis suggest the numerical values of $b=10$ and $d=9$ to achieve excellent approximation.

The MATLAB function for this algorithm is given as ***newfod()*** and defined as[47]:

```
function G=new_fod(r,N,wb,wh,b,d)
if nargin==4,
b=10; d=9; end
k=1:N;
wu=sqrt(wh/wb);
K=(d*wh/b)^r;
wkp=wb*wu.^((2*k-1-r)/N);
wk=wb*wu.^((2*k-1+r)/N);
G=zpk(-wkp',-wk',K)*tf([d,b*wh,0],[d*(1-r),b*wh,d*r]);
```

2.5 Fractional Order Control

Various practical dynamical systems are better characterized using fractional order dynamic models. These non-integer dynamic models are based on fractional calculus or fractional order differentiation or integration. Fractional calculus has incredible potential to enhance the way we think, perceive, model, and control the natural things around us. Refusing the benefits of fractional calculus is like denying from the existence of zero, fractional, or irrational numbers. Although, the fractional calculus was being extensively used for modeling and analysis purposes till the mid of twentieth century but still researchers face various problems in implementation of fractional calculus for control application. A major problem in designing a fractional controller is the frequency characteristics of the system. Frequency characteristic is advantageous in terms of robustness of any system to change in change system parameter or any uncertainty.

Manabe [21] pioneered the work in this direction in 1961. He suggested that the fractional integrator can be used as a substitute for control purpose and presented the frequency and transient response of fractional-order integral and its application to control systems [42], [43], [134]. Later on interest of researchers towards designing the fractional order controller increased and many research papers were published in design and implementation of fractional order controller [124], [135]. Different types of fractional order control strategies like; optimal control [75], [136], nonlinear fractional order controller [79], [137], fractional order sliding mode control [4], [81] etc. are presented for integer as well as fractional order dynamic systems [124], [135]. Different forms of fractional controller proposed in the literature has been shown in Table 1 in previous chapter. Among all, the fractional order PID controller is the most powerful, flexible, robust and frequently used for any system.

2.5.1 Fractional order PID controller

The fractional order PID controller was initially developed by A. Oustaloup in the form of CRONE (Commande Robuste d'Ordre Non-Entier, meaning Non-integer-order Robust Control) controller [138]. He applied this CRONE controller in different areas of control application and established the superior performance of FOPID controller over the conventional PID controller. He also presented three difference generations of CRONE control techniques in 1993 [6], [7], [46]. A generalized form of fractional order PID controller is presented by Podlubny in 1994 which is also known as $PI^{\lambda}D^{\mu}$ controller[47]. In this sequence, Podlubny presented many research papers related to different applications of fractional order differentiation, integration and $PI^{\lambda}D^{\mu}$ controller [3], [48]–[51].

In addition to this, various techniques are available for designing the FOPID controller [74-77]. The FOPID controller is found better for many practical systems like unmanned aerial vehicle (UAV) in [54], velocity control of servo motor, control of DC-motor with elastic shaft, terminal voltage control of the automatic voltage regulator and many more [9], [50], [55]–[61], [63]–[71], [140].

2.5.2 Transfer function of FOPID controller

FOPID controller is the generalized form of conventional PID controller and represented as $PI^{\lambda}D^{\mu}$ [3]. The fractional order control action in the form of integro-differential equation is given by:

$$u(t) = K_p e(t) + K_i D^{-\lambda} e(t) + K_d D^{\mu} e(t) \qquad (\lambda, \mu > 0) \qquad (2.46)$$

By applying Laplace transform in the above equation one can easily calculate the transfer function of the FOPID controller as:

$$G_{FOPID} = K_p + \frac{K_i}{s^{\lambda}} + K_d s^{\mu} \qquad (\lambda, \mu > 0) \tag{2.47}$$

where for λ and μ are the fractional power of integral and differential control respectively. All the classical controllers can be realized with a different set of values of λ and μ in FOPID controller which is shown below.

$$\begin{cases} \lambda = 1, \mu = 1; & PID\ controller; \quad G_{PID} = K_p + \dfrac{K_i}{s} + K_d s \\ \lambda = 1, \mu = 0; & PI\ controller; \quad G_{PI} = K_p + \dfrac{K_i}{s} \\ \lambda = 0, \mu = 1; & PD\ controller; \quad G_{PD} = K_p + K_d s \end{cases} \tag{2.48}$$

This can also be realized in a two-dimensional plane depicted in Figure 2.3[4].

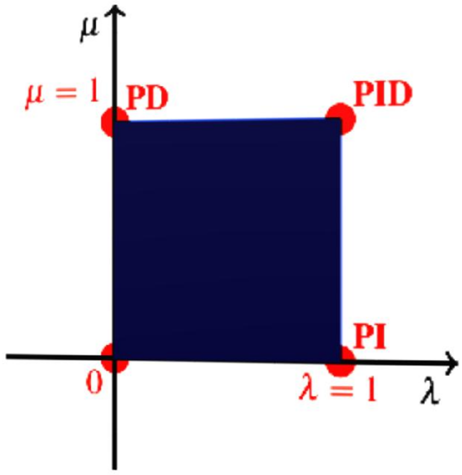

Fig. 2.3. The plane of FOPID controller

The fractional order controller provides more flexibility in tuning the gain and phase characteristics of the system. This flexibility is due to the presence of two additional tuning knobs as compare to the conventional PID controller i.e. fractional power of differential (λ) and integral (π). These additional tuning knobs are very useful for getting better controller which enhances the stability and robustness of the system. A generalized form of closed-loop system with FOPID controller has been shown in Figure 2.4.

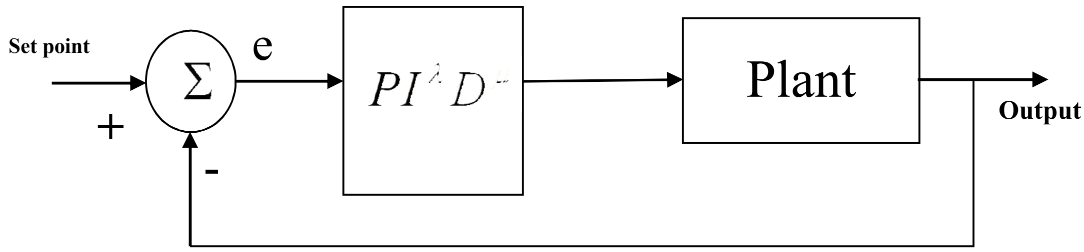

Fig. 2.4. Generalized form of closed-loop system with FOPID controller

2.5.3 Advantages of FOPID controller over conventional PID controller

In most of the control application, the FOPID controller is found better than the traditional PID controller. Some advantages are listed below:

- There are five different tuning knobs available in FOPID controller which provides better tuning than the traditional PID controller.
- It is effortless to achieve the iso-damping property from FOPID controller than the conventional PID controller.
- The FOPID controller performs better than the convention PID controller in case of a system with long time delay. [108], [120].
- In case of higher order systems, the conventional PID controller does not perform well, whereas the FOPID controller provides superior results [115], [141]
- FOPID controller is more robust and stable than the conventional PID controller [66], [101], [142].
- FOPID controller is also able to control the system with nonlinearities which are tough task with the conventional PID controller [86], [87].
- FOPID controller attains better results for nonminimum phase system.

2.5.4 Complexity with FOPID controller design

In spite of having numerous advantages over conventional PID controller, the researchers have to face two major difficulties while designing the FOPID controller which are:

- Digital implementation of FOPID controller
- Measuring the optimum values of controller parameters in minimum time.

In earlier discussion, many approximation techniques are pointed out for approximation of fractional terms into integer order model that solve the problem of digital implementation of FOPID controller.

Reason for second complexity in FOPID controller is the presence of total five control parameters (i.e. two extra tuning knobs as compare to conventional PID); so, measuring the optimum values of all five controller parameters simultaneously in minimum time becomes a difficult job. To rectify the complexity about the five tuning knobs, there are many techniques available which will be discussed in detail in next section.

2.6 Optimization techniques of FOPID controller

As it is earlier mentioned that the FOPID controller has two addition tuning knobs than the conventional PID controller. Hence, it is tough to tune FOPID controller [104], [140], [143]. There are many tuning strategies like; Ziegler-Nichols based tuning method, particle swarm optimization, Genetic algorithm, neural network, etc.[58], [92], [94], [140], [144]–[149], are available to fine tune the parameters of FOPID controller. These tuning methods can be broadly classified into three different categories namely Rule-based methods, Analytical methods and Numerical methods. Different standard fitness function has been used for optimization of controller parameters in the various techniques present in the literature. A brief survey of these techniques has been presented in Chapter 1.

2.7 Fitness Function

The prime task of any optimization technique is to find the values of the variables having optimum value of the fitness or objective function. The fitness function indicates how much each variable contributes to the value of fitness function to be optimum for the problem. Depending upon the desire of the researcher different fitness or objective function are considered for the same problem. Different types of fitness function considered in this work are discussed below:

2.7.1 Integral absolute error

$$IAE = \int_0^T |e(t)|\, dt \tag{2.49}$$

2.7.2 Integral time-weighted absolute error

$$ITAE = \int_0^T t\,|e(t)|\, dt \tag{2.50}$$

2.7.3 Integral squared error

$$ISE = \int_0^T e^2(t)dt \tag{2.51}$$

2.7.4 Integral time-weighted squared error

$$ITSE = \int_0^T te^2(t)dt \tag{2.52}$$

where $e(t)$ is the error, t is the time period, and T is total simulation time. Here error $e(t)$ at time t is calculated as:

$$e(t) = 1 - step\left(G_C\right)\Big|_t \tag{2.53}$$

where G_C is the transfer function of the system with controller.

2.8 Fractional order toolbox in MATLAB

The fractional calculus is extensively used in research in various engineering disciplines, demanding a Sharp tools for modeling, simulations, analysis and implementation of fractional order systems and control. These tools are very essential for the computation of fractional integration/differentiation, approximation of fractional terms, and simulation of the fractional order systems and controllers. Many tools and functions are available in literature as listed in Table 2.2, which are helpful for working in fractional domain.

Table 2.2 List of tools and software for fractional order calculus and control application.

Sl. No.	Matlab Toolbox and functions	Author	Remark
1	**CRONE**	A. Oustaloup et al.	First Toolbox for non-integer control and identification [150]–[152]
2	**NINTEGER**	D.Valerio and J. Costa	GUI for Controller Design [153], [154]
3	**FOMCON**	A.Tepljakov et al.	Helpful for beginners in fractional order modelling and control [114], [124]
4	**fopid**	N. Lachhab et al.	Not Available for download [124]
5	**fotf**	Xue et al.	Overload many functions [124]
6	**Sysquake interactive software tool**	E. Pisoni et al.	Analysis and design of fractional PID Controller [6], [124]
7	**M-L functions**	R. L. Magin	MATLAB functions developed for numerically computing the Mittag-Leffler function [124]
8	**NILT**	Lubomir	Numerical Inversion of Laplace Transform [124]

9	**dfod**	I. Petras	Approximation of fractional order differentiators and integrators in discrete domain [124]
10	**IRID**	Chen, Li, Sheng et al.	Approximation of different type of fractional functions and filters [124]
11	**fderiv**	F. M. bayat	Fractional derivative of order α for the given function using Grunwald-Letnikov (G-L) definition [9], [124]
12	**glfdiff**	Xue et al.	Calculation the α^{th} derivative of a given function using G-L definition [124]
13	**fit**	T. M. Marinov et al.	Numerical computation of fractional order integration and differentiation using Riemann-Liouville definition [124]
14	**DFOC**	I. Petras	Digital version of FOPID controller [59], [124], [155]
15	**Sysquake FO PID**	Pisoni et al.	The only toolbox in Sysquake software environment [59], [74], [123], [124]
16	**FOCP**	Tricaud et al.	Formulation of the Fractional Optimal Control Problems (FOCP) into integer order format using a rational approximation of the fractional derivative [15]
17	**FSST**	D. Sierociuk	A simulation toolkit for the fractional order discrete state-space system [84]

The detail investigation about the tool for fractional calculus and control is present in many review articles [74], [124], [135]. As detail description of all the toolboxes and software is not possible here, only the FOMCON Toolbox used in this work is illustrated in next subsection.

2.8.1 FOMCON toolbox in MATLAB

As the name suggest, it is a fractional-order system modeling and control (FOMCON) toolbox for MATLAB platform developed by A. Tepljakov et al. in 2011. The core of the toolbox is derived from an existing mini toolbox FOTF ("Fractional-order Transfer Functions"), the source code for which is provided in literature [57], [139], [156], [157]. It is also connected to other existing fractional calculus based toolboxes, like CRONE [152] and Ninteger [154], and this relation is shown in Figure 2.5.

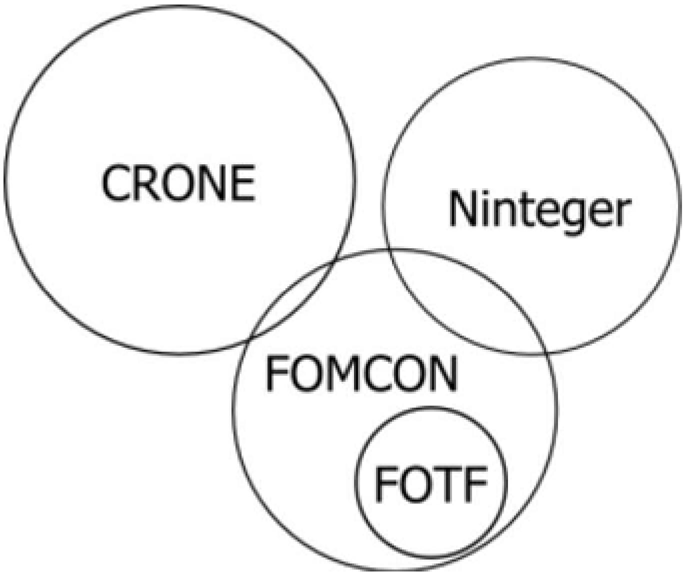

Fig. 2.5. Fractional-calculus based toolbox relations

FOMCON provides very useful and convenient tools for the researchers working with fractional order system and control [114], [158]. In addition, it also provides convenient functions, e.g. the polynomial string parser, building graphical user interfaces (GUIs) to improve the general workflow. There are certain limitations in the existing toolboxes. Thus, the basic practicality of the toolbox is extended with advanced features, such as fractional-order system identification and $PI^{\lambda}D^{\mu}$ design. The availability of graphical user interfaces and advanced functionalities, makes it suitable for both beginners and the expert. Due to the availability of the source code the FOMCON toolbox is ported to other computational platforms such as Scilab or Octave with fewer limitations and/or restrictions. An overview of the basic structure of the toolbox is presented in the following subsection.

The key features which make the toolbox most popular among the researchers are given as:

- Extremely appropriate for the beginners in fractional control.
- It contains features of almost all the basic toolbox for fractional order control.
- Has the capability to generate accurate and quick practical results.
- It provides a set of tools and commons for researchers in the fractional-order control field.
- Provides many graphical user interfaces (GUIs) for fractional order model identification and fractional PID controller design and optimization.
- Also, provides a Simulink blockset for fractional order controller.
- Digital implementation of fractional order controller is also possible with this toolbox.

2.8.1.1 Toolbox structure

The toolbox pose a modular structure and currently consists of the following modules:

- Main module (for fractional order system analysis)
- Identification module (for fractional order system identification in both time and frequency domains)
- Control module (for FOPID design, tuning, optimization tools and some additional features).

All these modules are interconnected and can be accessed through GUIs as shown in Figure 2.6.

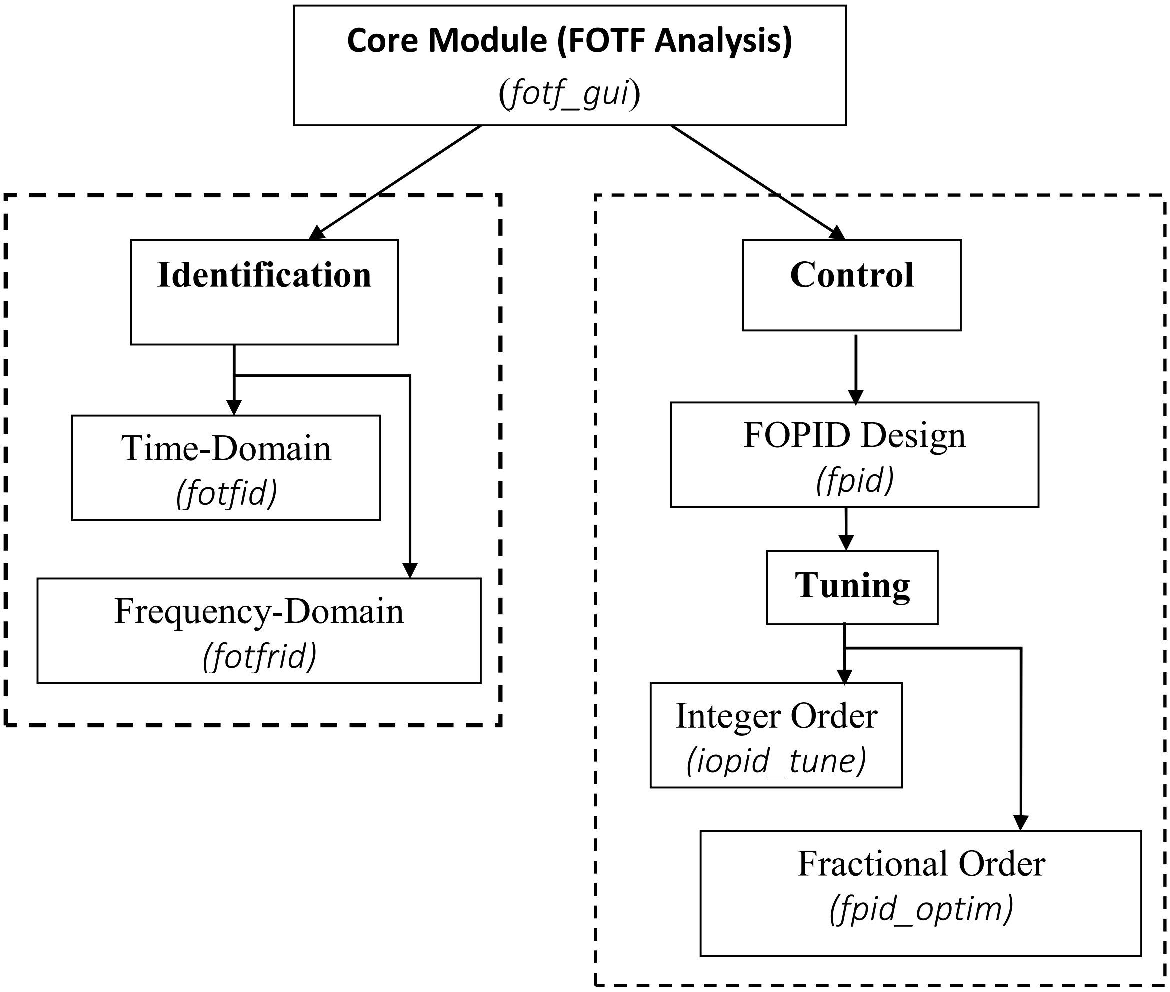

Fig. 2.6. FOMCON toolbox module overview

2.8.1.2 GUIs

Many graphical user interfaces (GUIs) are available in FOMCON toolbox. Few of them are frequently used for system identification and controller optimization:

i. *fotfid* (used for time-domain identification tool for fractional order system shown in Figure 2.7)

ii. *iopid_tune* (used for integer order PID tuning for a fractional order system shown in Figure 2.8)

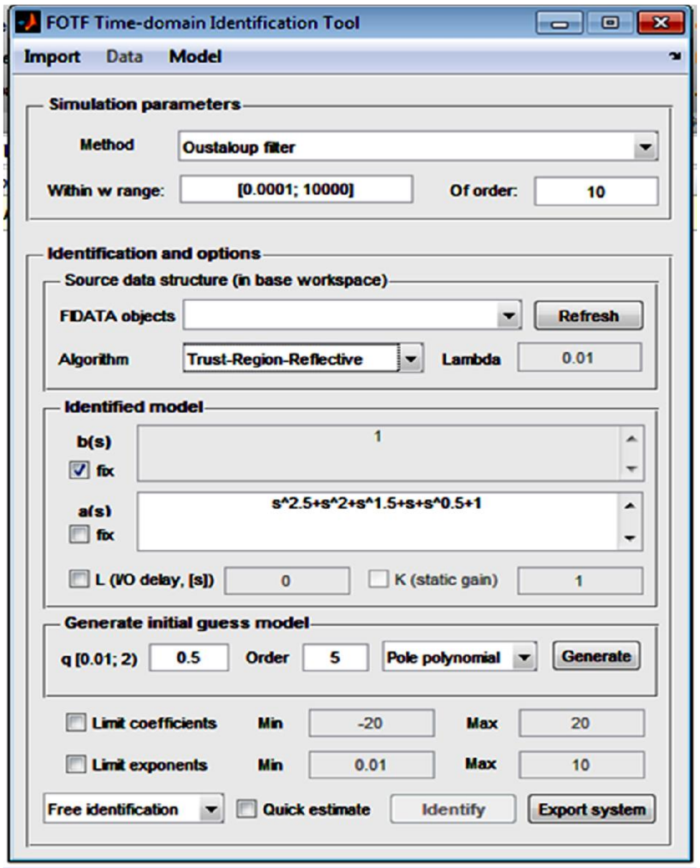

Fig. 2.7. GUI for *fotfid*

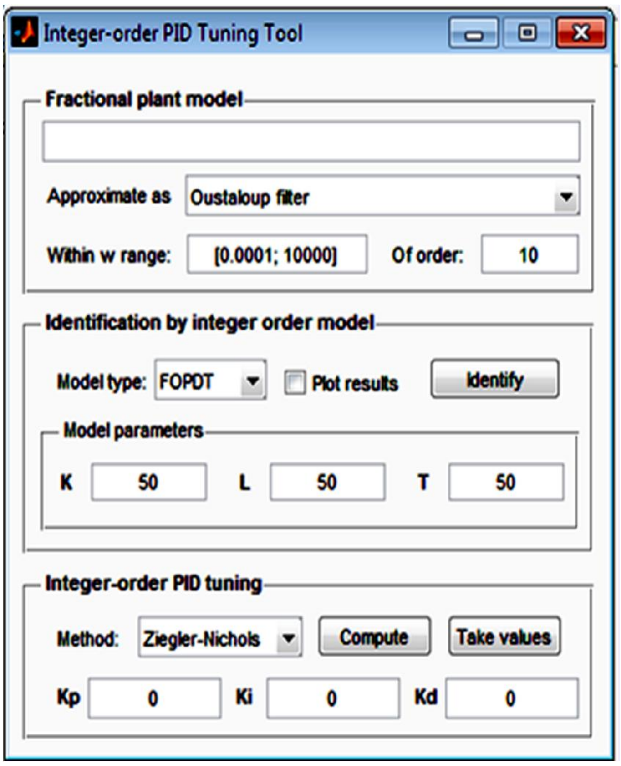

Fig. 2.8. GUI for *iopid_tune*

iii. *fpid_optim* (used for optimization of FOPID controller for both integer order as well as fractional order system shown in Figure 2.9)

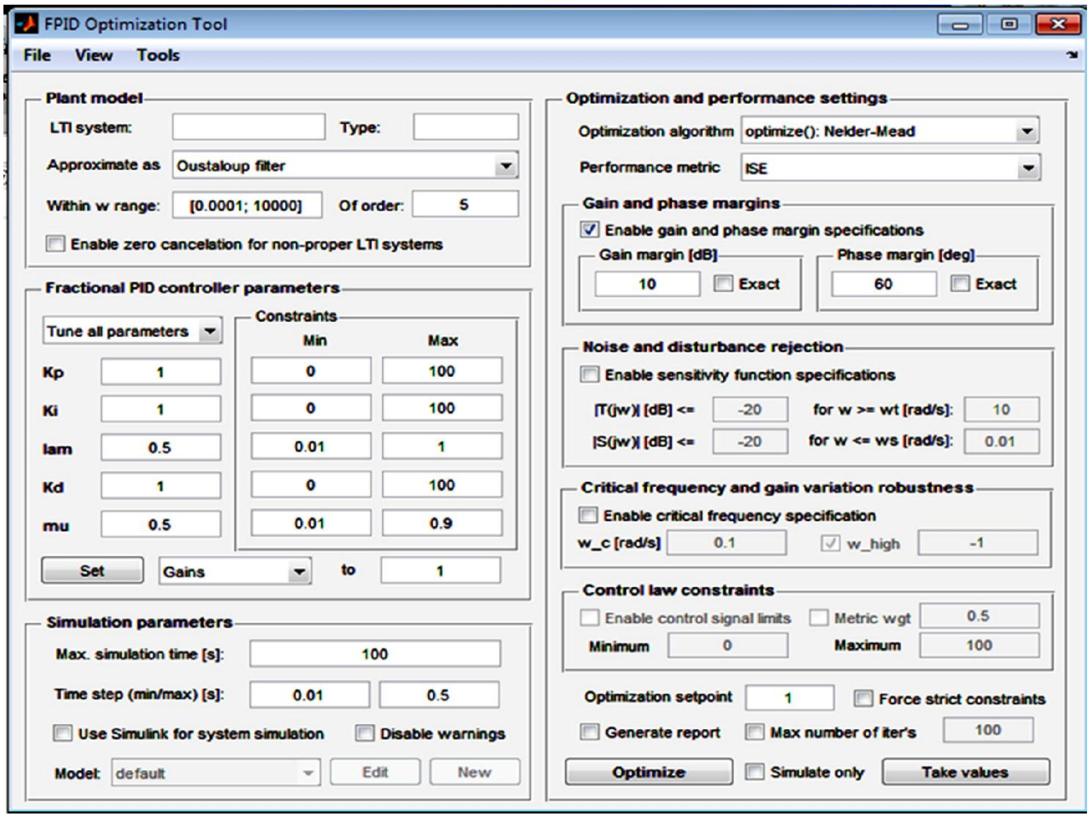

Fig. 2.9. GUI fot *pid_optim*

2.8.1.3 Simulink block

The FOMCON toolbox is also offered with a Simulink blockset which assist the researchers to perform complex modeling tasks. The conventional approach is used for block construction where it is needed. The Simulink blocks which can be currently realize in FOMCON toolbox are listed below and also shown in Figure 2.10.

- General fractional-order operator
- Fractional order integrator and differentiator
- Fractional order transfer function (Continuous and Discrete both)
- $PI^{\lambda}D^{\mu}$ controller (Continuous and Discrete both)
- TID controller

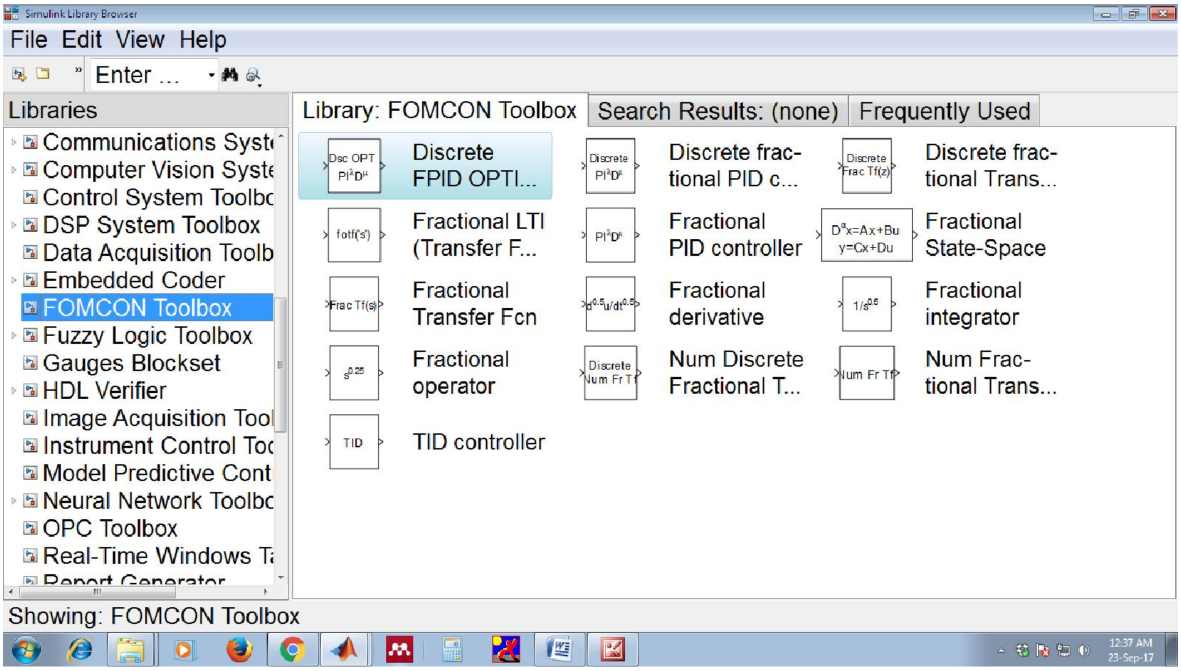

Fig. 2.10. Simulation Blocks available in FOMCON toolbox

2.8.1.4 Dependency on other toolboxes

Although, the FOMCON relies on the MATLAB platform as a very useful toolbox for the researchers, but it also depends on other MATLAB products to provide such advanced features. Few of them are as below;

- Control System toolbox (Essential requirement for most of the features)
- Optimization toolbox (Required for time-domain identification, integer-order PID tuning as well as in a part of fractional-order PID tuning)
- optimize() function [6];
- Various Ninteger toolbox (For frequency domain identification functions)
- Various other tools are directly used as per the BSD license.
- It is also possible to export fractional-order systems to the CRONE toolbox format but installation of the object-oriented CRONE toolbox is needed to avail this feature.

2.8.1.5 Availability of the toolbox

This toolbox is freely downloaded from the link http://fomcon.net/fomcon-toolbox/download/. More truthful discussion and solution of the questions raised by the users are also entertained online.

2.9 Summery

The main focus of this chapter was to discuss about the essential preliminaries for this dissertation. The chapter enlighten about the representation, stability, approximation of fractional order systems and various advantages of fractional order system and control. In addition to this, various fractional order control schemes available in the literature and MATLAB toolbox used for this dissertation are also covered. Based on these preliminaries and basic knowledge of fractional calculus and fractional order control, the proposed tuning algorithms for this dissertation will be discussed in the next chapter.