APPENDIX A

Consider that a wave , which is initially travelling in air (permittivity ε_0 , permeability μ_0 , characteristic impedance Z_0) in a waveguide along z- direction, strikes the sample AB having permittivity ε , permeability μ and characteristic impedance Z .

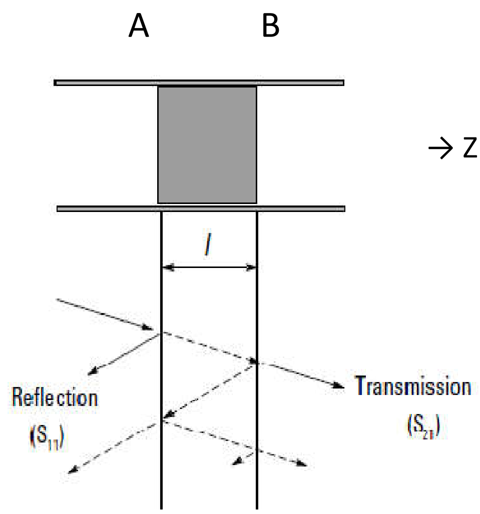

Figure A.1 Sketch showing reflection and transmission of electromagnetic wave propagating in Z direction and striking a dielectric medium of length l [Keysight (2016)].

The wave sees a mismatched load and suffers partial reflection and transmission at the air-sample interface. The reflection coefficient , assuming that the load is of infinite extent, is give as [Tripathi (2015); Baker-Jarvis et al. (1993); Hayt et al. (2012); Nicolson et al.(1979); Sadiku (2007)]

$$\Gamma = \frac{Z - Z_0}{Z + Z_0} = \frac{\mu/\gamma - \mu_0/\gamma_0}{\mu/\gamma + \mu_0/\gamma_0} \qquad \text{...(A.1)}$$

where γ and γ_0 are the propagation constants for the fundamental TE_{10} mode in the waveguide filled with the material and air respectively. μ and μ_0 are the permeability of the sample and air respectively.

If the amplitude of the wave incident at the air-sample interface A is V , the amplitude of the reflected wave would be V Γ_1 where $\Gamma_1 = \Gamma$. The transmitted amplitude at A would be V (1+Γ_1) which would get attenuated by a factor $\xi = e^{-\gamma z}$ as the wave traverses the medium along z direction. If the thickness of the sample is ‘d’ (ie the sample extends from plane A to plane B where AB = d) , the amplitude at B would be V (1+Γ_1) ξ where ξ is the transmission coefficient between faces A and B. At B , which is a material - air interface, the wave sees an impedance mismatch and is, therefore partly reflected with reflection coefficient Γ_2 where $\Gamma_2 = -\Gamma$. Thus we have (i) a reflected wave of amplitude V (1+Γ_1) ξ Γ_2 starting at B travelling towards A and (ii) a transmitted wave of amplitude V (1+Γ_1) ξ (1+ Γ_2) at B coming out of the sample and further travelling in air along z-direction. Now, the wave in (i) travels back in the material through a distance ‘d’ and reaches the plane A with attenuated amplitude V (1+Γ_1) ξ Γ_2 ξ . At plane A, this gives rise to a transmitted wave of amplitude V (1+Γ_1) ξ Γ_2 ξ (1+Γ_2) which comes out of the sample and travels towards opposite (-z) direction and a reflected wave of amplitude V (1+Γ_1) ξ Γ_2 ξ Γ_2 which travels towards B and encounters multiple reflections and transmissions at planes A and B. The total amplitude , V_B , transmitted at B would be given by

$$V_B = V(1+\Gamma_1)\xi\,(1+\Gamma_2) + V(1+\Gamma_1)(\xi\Gamma_2)^2\,\xi\,(1+\Gamma_2) + V(1+\Gamma_1)(\xi\Gamma_2)^4\,\xi\,(1+\Gamma_2) + V(1+\Gamma_1)(\xi\Gamma_2)^6\,\xi\,(1+\Gamma_2) + \cdots \quad \text{... (A.2)}$$

$$= V(1-\Gamma^2)\xi + V(1-\Gamma^2)\xi\,(\xi\Gamma)^2 + V(1-\Gamma^2)\xi\,(\xi\Gamma)^4 + V(1-\Gamma^2)\xi\,(\xi\Gamma)^6 + \cdots$$

$$= \frac{V(1-\Gamma^2)\xi}{1-\Gamma^2\xi^2} \quad \text{...(A.3)}$$

Therefore , we get

$$S_{21} = \frac{V_B}{V} = \frac{(1-\Gamma^2)\xi}{1-\Gamma^2\xi^2} \quad \text{...(A.4)}$$

This is Equation (3) of Nicolson et al. (1970)

Simiarly, the total amplitude , V_A , reflected at A would be given by

$$V_A = V\Gamma_1 + V(1+\Gamma_1)\xi\,\Gamma_2\xi(1+\Gamma_2) + V(1+\Gamma_1)(\xi\Gamma_2)^3\,\xi\,(1+\Gamma_2) + V(1+\Gamma_1)(\xi\Gamma_2)^5\,\xi\,(1+\Gamma_2) + V(1+\Gamma_1)(\xi\Gamma_2)^7\,\xi\,(1+\Gamma_2) + \cdots \quad \text{...(A.5)}$$

$$= V\Gamma_1 + \frac{V(1+\Gamma_1)\xi(1+\Gamma_2)\,\xi\Gamma_2}{1-(\xi\Gamma_2)^2} \quad \text{...(A.6)}$$

$$= V\Gamma_1 + \frac{V(1-\Gamma^2)\xi^2(-\Gamma_1)}{1-(\xi\Gamma_1)^2} \quad \text{...(A.7)}$$

$$= V\Gamma\,\frac{1-\xi^2}{1-(\Gamma\xi)^2} \quad \text{...(A.8)}$$

Thus we get

$$S_{11} = \frac{V_A}{V} = \frac{(1-\xi^2)\Gamma}{1-\Gamma^2\xi^2} \quad \text{...(A.9)}$$

By defining quantities V_1 and V_2 as $V_1 = S_{21}+S_{11}$ and $V_2 = S_{21}-S_{11}$, we get

$$V_1 = S_{21} + S_{11} = \frac{\xi+\Gamma}{1+\Gamma\xi} \quad \text{... (A.10)}$$

$$V_2 = S_{21} - S_{11} = \frac{\xi-\Gamma}{1-\Gamma\xi} \quad \text{... (A.11)}$$

$$V_1V_2 = \frac{\xi^2-\Gamma^2}{1-\Gamma^2\xi^2} \quad \text{...(A.12)}$$

$$V_1 - V_2 = 2\,\Gamma\,\frac{1-\xi^2}{1-\Gamma^2\xi^2} \quad \text{...(A.13)}$$

Similarly, we define a quantity X as

$$X = \frac{1-V_1V_2}{V_1-V_2} \quad \text{...(A.14)}$$

Which , by using Eq and Eq , can be written as

$$X = \frac{1+\Gamma^2}{2\Gamma} \quad \text{...(A.15)}$$

Also, X can be expressed in terms of S_{11} and S_{21} as follows. We have

$$V_1 V_2 = S_{21}^2 - S_{11}^2 \quad ...(A.16)$$

$$V_1 - V_2 = 2S_{11} \quad ...(A.17)$$

which yield

$$X = \frac{S_{11}^2 - S_{21}^2 + 1}{2S_{11}} \quad ...(A.18)$$

Thus Equation (A.15) give s

$$\Gamma = X \pm \sqrt{(X^2 - 1)} \quad ...(A.19)$$

with the appropriate sign being chosen so that $|\Gamma| < 1$

and

$$\xi = \frac{S_{21} + S_{11} - \Gamma}{1 - (S_{21} + S_{11})\Gamma} \quad ...(A.20)$$